Never Give a Non-Mathematician an Even Break!

Ron Gould

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The Tools: You have three cards, one red on both sides, one blue on both sides, and one blue on one side and red on the other side.
The Game: You place the three cards into a hat and ask someone to blindly select one card and only look at one side of the card. They show you that side of the card and you offer to bet even money you can tell them the color of the other side of the card.
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Is this a fair bet?
Suppose the side you saw is **red**.

Then, most people say this is a fair bet since:

The card is either **red-red**

or the card is **red-blue**

and the man has a 50% chance of guessing correctly.
The Scam - I mean the real solution

Despite the last argument
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▶ You will always guess the same color as that shown and be correct 2/3 of the time!
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- Moral of the story -
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One hundred randomly selected coins are placed in a row. The sum of the values of these coins is $T$ which is odd.

**GAME:** Player 1 selects one coin from either end of the row. Player 2 selects one coin from either end of the row. Play continues, alternating turns, until all coins have been selected.

**Problem:** Show that there is a strategy so that

**Player 1 always wins!** (i.e. get most of the money)
Sketch of proof
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2. Count the value of the coins in one color to determine which color has the most (> \(T/2\)) WLOG say white.
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5. Player 1 now selects the white coin uncovered by the choice Player 2 just made.

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1. Think of the coins as partitioned into two groups.
2. Count the value of the coins in one color to determine which color has the most \((> \frac{T}{2})\) WLOG say white.
3. Player 1 selects the end coin that is white.
4. Player 2 now has only red coins on either end to select from.
5. Player 1 now selects the white coin uncovered by the choice Player 2 just made.
6. Player 2 again has only reds to choose from. This continues until the game ends with Player 1 having all the white coins and the win.
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The Game: Three friends, **Matt** (the mathematician) **Phil** (the physicist) and **Ed** (the engineer) decide to play paintball.

**Matt** - limited experience: 50% shooter.
**Phil** - plays more: 75% shooter.
**Ed** - plays constantly: 100% shooter.
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- **Ed** will get the third shot (100%).
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- **Matt** will get the first shot (50%).
- **Phil** will get the second shot (75%).
- **Ed** will get the third shot (100%).
- Should it be needed, another round then follows among any survivors.
What strategy should Matt use?
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- Surrender and go to a bar and wait for the winner to join him there later!
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- If he shoots (and hits) Ed, then there is a 75% chance Phil shoots him.
- If he shoots Phil, then Ed immediately will shoot him!
- Surrender and go to a bar and wait for the winner to join him there later!
- Matt’s best move is to **purposely miss!**
Why is this a good move?

Now Phil’s shot

Phil kills Ed

Matt kills Ed

Phil misses

Ed kills Phil

Matt misses

Matt wins

1/2

1/2

1

Ed kills Matt

Ed wins

$P(\text{Ed wins}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = .125$

Figure: The strategy tree diagram.
Why is this a good move?

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Use the fact we are repeating turns.

The subtree $D$ contains this repeated pattern. Let $p =$ probability $Phil$ wins from this given position (as shown with the arrows)

$$p = \frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{4} \times p.$$ 

Solving: $p = \frac{3}{7}$.

Now $P(Phil \ wins) = \frac{3}{4} \times p = \frac{9}{28} = .321$.

$$P(Matt \ wins) = 1 - P(Phil \ wins) - P(Ed \ wins) = 1 - .321 - .125 = .554.$$
Never give a non-mathematician and even break.
Suppose you and a partner want to perform the following 5 card trick.
A normal deck is shuffled and you are dealt 5 cards (without your partner seeing). Then you:
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A normal deck is shuffled and you are dealt 5 cards (without your partner seeing). Then you:
Now your partner enters the scene and identifies the face down card exactly!
How is this trick done?
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1. We need to convey two pieces of information:
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2. The suit and the rank of the down card!
How the suit information is encrypted

Given any 5 cards from a standard deck, the

**The Pigeon Hole Principle:**
There must be at least 2 cards of the same suit!

Thus, we can use one card from that suit placed in a predetermined position to signal the suit. Say for now - the left most position. Now how do we handle the rank?
3 ”unused cards” - Use the relative ranks to have a low (L), medium (M) and high (H) card among the 3 remaining cards.

The lexicographic orderings of these three produce:

L M H (call this 1)
L H M (call this 2)
M L H (call this 3)
M H L (call this 4)
H L M (call this 5)
H M L (call this 6)
The solution

8♣ 10♥ J♠ 5♥

2 4 6 8 10 J Q K A

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The solution

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The Moral of the Story

Never give a non-mathematician an even break.
This game was invented by Walter Penny in 1969.

**The Game:** Ask your opponent to select any pattern of heads (H) and tails (T) of length three they wish. Then you select a different pattern. Flip pennies repeatedly until one of the two patterns occurs on three consecutive tosses. The person with that pattern wins an even money bet.
Is this a fair game?

Many people think the game is unfair because the second person only gets to choose their pattern after the first person has selected. This means they do not have all the choices the first person had.
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Those are the conditions - but that is not the advantage!
Suppose, for ease of argument, your opponent selected $HHH$. 
Suppose, for ease of argument, your opponent selected \text{HHH}. Then you will select \text{THH}.
Case Analysis

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P = 1/8 that first three flips are heads
Now suppose this is the first HHH in the sequence.

\[ \begin{array}{cccc}
  H & H & H & \\
  \ldots & \_ & \_ & \_ \\
\end{array} \]
Case Analysis

Now suppose this is the first HHH in the sequence.

T

H

H

H

...  

Then THH had to occur first!

Thus, 7/8 of the time THH wins!
An analysis of all the other possibilities (more involved tree structures and infinite geometric series) shows that the following rule always wins with probability at least $2/3$.

Let the opponent select a pattern of three. Now move their first two choices to your last two choices and fill the first position with whatever choice does not form a palindrome!
The moral of the story

Never give a non-mathematician an even break.
Suppose two standard decks of cards are individually shuffled and placed on a table. Now, the top card of each deck is turned over to determine if there is an **EXACT MATCH**. This process is continued throughout the remaining cards.

Given an even money bet - would you bet there was an exact match before the cards run out?
One view of this problem is that when the card from deck 1 is turned over, there is a $\frac{1}{52}$ chance the card in deck 2 will match it. Thus, the expected number of matches when going through the entire deck would be:

$$np = (52) \frac{1}{52} = 1.$$ 

Thus, the actual length of the deck does not seem to matter.
Can we squeeze more out of this?

That rough answer seems to say it would seem a wise bet to say there would be a match.

BUT HOW WISE???.

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Another way to look at this problem, and one that brings far more to the table, is to think of the 2nd deck as a permutation of the first deck. Now, the question we are asking is how many permutations of the deck have at least one position unchanged (a fixed point)?

This unchanged position gives us an exact match!

A permutation with no fixed points is called a derangement.
At this point, students would be familiar with De Morgan’s Laws. So the step to the IEP is a natural one.

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Fill the other $52 - i$ positions in $(52 - i)!$ possible ways. Then, applying the IEP we obtain:
Counting derangements via IEP:

\[ D_{52} = 52! \sum_{i=0}^{i=52} \frac{(-1)^i}{i!} \]

But we know from Calculus that:

\[ \lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e} \approx 0.3679. \]

Hence, \( P(\text{noderangement}) \approx 0.3679 \) and thus \( P(\text{match}) \approx 0.63 \). Thus, it is a very good even money bet!
The Game of NIM

**Game:** Given $t \geq 2$ piles of chips, two players proceed as follows:

Player 1 removes some number of coins from exactly one pile.
Player 2 then removes some number of coins from exactly one pile.
Players alternate turns until no chips remain. The first player that cannot remove some chips loses.
Is there a strategy for optimal play?

To find the strategy, it really helps to start at the end of the game. When no chips remain we call this the **ZERO POSITION**. Your goal as a player is to put your opponent in the zero position. How can we achieve this?
The key is to think of each pile of chips as a and to look at the base 2 representation of this number.

Say we have three piles of 8, 5 and 6 chips.
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Base 2 view

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

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Base 2 view

8: 1 0 0 0
5: 0 1 0 1
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A: 1 0 1 1
(not a zero position)
What to do?

Player 1 would like to put the other player into a zero position.

We think of a zero position as $0 \ 0 \ 0 \ 0$

So to get our parity counts above to a zero position we must remove chips from one pile so that all the columns have an even number of ones:

This can always be done!
One such move

Looking at our example:

8: 1 0 0 0
5: 0 1 0 1
6: 0 1 1 0

A: 1 0 1 1
(not a zero position)

Removing 5 chips from the pile of 8 will produce what we need!
3: 0 0 1 1
5: 0 1 0 1
6: 0 1 1 0

A: 0 0 0 0
(zero position)
Player 2 is now in a hopeless position!
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Any move made by Player 2 produces a non-zero position!
Player 2 is now in a hopeless position!
Any move made by Player 2 produces a non-zero position!
Player 1 will then repeat the strategy of moving to a zero position!
Player 2 is now in a hopeless position!
Any move made by Player 2 produces a none zero position!
Player 1 will then repeat the strategy of moving to a zero position!
Eventually player 1 will reduce to the ultimate zero position - no chips, and win the game