OPEN PROBLEM SESSION FROM THE CONFERENCE "RAMIFICATION IN ALGEBRA AND GEOMETRY"

This is a summary of the session on open problems held on 19 May 2011 during the conference *Ramification in algebra and geometry* at Emory. The moderator was Alexander Duncan and the notetakers were Anthony Ruozzi and Uzi Vishne.

Participants were requested to focus on more geometric questions and to de-emphasize division-algebra-centric themes already surveyed in [ABGV11]. This text is a lightly-edited transcript, as opposed to a more formal write-up which would include many more references, etc. Through June 2011, please send any corrections or amendments to Asher Auel and Skip Garibaldi.

1. Index Reduction Formulas

Let G and G' be not necessarily distinct, semisimple algebraic groups. Let X be a projective homogeneous G-variety and similarly for X', G'. Under what circumstances is $X(k(X'))$ nonempty?

Known cases.

(1) When $G = G'$, the problem is already nontrivial in some cases. For example, taking $G = G' = \mathrm{SO}(q)$ for a quadratic form q and X and X' to correspond to different maximal parabolic subgroups (i.e., orthogonal Grassmannians), one is led to questions of the possible splitting patterns of quadratic forms. Many results are known in this case, due to many people (starting with Manfred Knebusch).

(2) When G has inner type A, the problem is solved by the index reduction formulas from [MPW96] and [MPW98].

(3) When X and X' are quadrics (so G and G' have types B or D), many results are known, see for example the talks by Detlev Hoffmann and James O'Shea at this conference.

(4) The case where G is a classical group and X' is a Severi-Brauer variety is discussed in several papers by Nikita Karpenko.

2. What does cyclicity mean geometrically?

What property of a Severi-Brauer variety X encodes the condition that the corresponding division algebra is cyclic? Is this condition that the Brauer-Severi variety contains a polygon (Saltman)? That is, over an algebraic closure, $X_{\bar{k}}$ contains n lines whose intersections form a polygon where n is the index of the cyclic algebra.

Date: Version of May 26, 2011.

3. Splitting

Let A be a central simple k-algebra. *Is there a genus* 1 *curve* C *such that* $k(C)$ *splits* A*?*

Known cases.

(1) If $\deg(A) = 3$, the answer is yes by [Swe95].
(2) If $\deg(A) = 2$, then A is a quaternion algebra and it is split using the function field of a genus 0 curve, namely the Severi-Brauer variety $\mathrm{SB}(A)$. The answer is yes because you can construct a C of genus 1 that is a cover of $\mathrm{SB}(A)$ ramified in the right number of points.

Ideas. Try to construct a curve using the tangent bundle?

4. Local-Global Principles

Let F be the field of fractions of a dimension 2 complete local domain. Consider a G-torsor X over F for a connected linear algebraic group G. *If* X_{F_v} *is a trivial torsor over all residue fields* F_v *of discrete valuations* v*, is* X *trivial? Is it true if we further suppose that* G *is rational?* Probably one needs the local domain to be regular.

The motivation for this problem is that one knows the answer is yes in a similar situation, namely when F is a number field and G is semisimple simply connected.

Known cases.

(1) Known when all residue fields are algebraically closed of characteristic 0.
(2) For a complete regular domain, it holds for PGL_n (problem if char divides n?).

Ideas. Start with the case where G is an algebraic torus and look for counter-examples. Can then bridge into counter-examples for semisimple groups?

5. u-Invariant

Continue with F as in the previous section and write k for the residue field. *Does* $u(F) = 4u_s(k)$*?* Here u_s denotes the *strict* u-invariant.

Known cases.

(1) The answer is yes if $u(k) = 1$, i.e., if every element of k is a square.
(2) The answer is yes if k is finite.
(3) The answer is yes if $R = A[[t]]$ for A a complete DVR (with residue field k, obviously) such that $u(k) = u_s(k)$.

Related problem. Let r be the the smallest non-negative integer such that the exponent of a central simple F-algebra necessarily divides the index^r.[1] *Is* $\widetilde{u}(F) \leq 2^{r+1}$*?*

Note that for an algebraically closed field, $1 = u(F) = \widetilde{u}(F) < 2$, so the answer is yes.

Related problem. More generally, what numbers are possible as the u-invariant of a field? For elementary reasons, 3, 5, and 7 are impossible. On the positive side, 1 occurs (for $\mathbb{C}$), every even number occurs [Mer92], and so does $2^r + 1$ for $r \geq 3$ (Izhboldin, Vishik [Vis10]). *Is there a field with u-invariant* 11*?*

In a similar vein, is it true that for every *finitely-generated* field[2] with finite u-invariant, the u-invariant is necessarily a power of 2?

6. Indecomposable Division Algebras

Find a minimal n such that there exists an indecomposable division algebra of period 2 and index 8 over $\mathbb{C}(t_1, ..., t_n)$*. Try to do the same for period 3 index 9.*

By De Jong, index equals exponent for every division algebra over $\mathbb{C}(t_1, t_2)$, hence $n \geq 3$.

Ideas. A guess for period 2 index 8: 4. There is an explicitly constructed algebra in [ART79] that gives an upper bound of around 10.

Related problem. Let

$$\mathcal{F} : \mathbf{Fields/k} \to \mathbf{Group}$$

be the functor that maps a field extension L to its Brauer group $\mathrm{Br}(L)$. *If A is a central simple algebra over L over index* 2^r *and exponent 2, find a lower bound for the essential dimension* $\mathrm{ed}^{\mathcal{F}}(A)$*.* This problem is related to the question of symbol length $\mu(2, 2^n)$ surveyed in §3 of [ABGV11].

Known cases.

(1) For $n = 2$, the answer is 4.
(2) For $n = 3$, there is an upper bound of 8(?) produced by Tignol.

7. Upper motives of Severi-Brauer varieties

Let k be a field and consider a division algebra D over k of index p^n. Choose a field extension E/F such that $D \otimes E$ is still a division algebra. Let $X = \mathrm{SB}(p^m, D)$ for $m < n$, the generalized Severi-Brauer variety of right ideals

[1] Some participants suggested to name this invariant r of the field the "exponent-index exponent". It was called the *Brauer dimension* in [ABGV11, §4].

[2] This term means: finitely generated as a field over its prime field.

of D of reduced dimension p^m. The upper motive[3] $U(X)$ is indecomposable. *Is $U(X) \otimes E$ indecomposable?*

One can ask the same question if we take $p = 2$ and replace X by a quadric.

Known cases. The answer is "yes" if $m = 0$ (usual Severi-Brauer variety), because in these cases the motive of X is itself indecomposable, so $U(X) = \mathcal{M}(X)$. Note that the hypothesis that $D \otimes E$ is division comes into play here, because if $D \otimes E$ is not division then $\mathcal{M}(X) \otimes E$ is decomposable by [Bro05].

Charles de Clercq shows in [dC11] that the answer is "yes" when $m = 1$ or $m = p = 2$.

8. Motives of quadrics and orthogonal Grassmannians

Vishik's generic discrete invariant and the related (coarser) elementary discrete invariant, J-invariant, and motivic decomposition type—see, e.g., [Vis04], [Vis05], and [Vis10]—give information about various cycles on quadrics and orthogonal Grassmannians. *What possible values can it take?*

For example, the Steenrod operations give some restrictions on the J-invariant of a quadric $q = 0$, and for $\dim q \leq 18$ these are the only restrictions. What about for forms of larger dimension?

Related problem. The same question can be asked about the J-invariant defined in [PSZ08], which gives related information about the Borel variety of a simple algebraic group. Just as for quadrics, one has some very general restrictions coming from topology [Kac85] and from Steenrod operations; these restrictions are described in [PSZ08]. But in contrast with what is known for small-dimensional quadrics, there can be additional restrictions, such as for type E_6 [GPS10] and for type D_4 [QMSZ11].

9. Limits of Azumaya Algebras

R a commutative ring. Is there a nice description of a category the contains all Azumaya algebras/R of degree n that is also closed under flat families ("limits")? The description should also be functorial.

References

[ABGV11] A. Auel, E. Brussel, S. Garibaldi, and U. Vishne, *Open problems on central simple algebras*, Transformation Groups **16** (2011), no. 1, 219–264.

[ART79] S.A. Amitsur, L.H. Rowen, and J.-P. Tignol, *Division algebras of degree* 4 *and* 8 *with involution*, Israel J. Math. **33** (1979), 133–148.

[Bro05] P. Brosnan, *On motivic decompositions arising from the method of Białynicki-Birula*, Invent. Math. **161** (2005), no. 1, 91–111.

[dC11] C. de Clercq, *Motivic rigidity of generalized Severi-Brauer varieties*, arxiv:1105.4981, May 2011.

[3] In the category of Chow motives with $\mathbb{F}_p$-coefficients, as described in [EKM08]. The *upper motive* of a smooth projective irreducible variety X is the unique summand $U(X)$ of $\mathcal{M}(X)$ in the category of Chow motives with $\mathbb{F}_p$-coefficients such that $\mathrm{Ch}^0(U(X)) \neq 0$. This concept was explicitly introduced in [Kar09].

[EKM08] R.S. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, Colloquium Publications, vol. 56, Amer. Math. Soc., 2008.

[GPS10] S. Garibaldi, V. Petrov, and N. Semenov, *Shells of twisted flag varieties and non-decomposibility of the Rost invariant*, arxiv:1012.2451, December 2010.

[Kac85] V.G. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), no. 1, 69–79.

[Kar09] N.A. Karpenko, *Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties*, http://www.math.uni-bielefeld.de/LAG/man/333.html, April 2009.

[Mer92] A. Merkurjev, *Simple algebras and quadratic forms*, Math. USSR Izv. **38** (1992), no. 1, 215–221.

[MPW96] A.S. Merkurjev, I. Panin, and A.R. Wadsworth, *Index reduction formulas for twisted flag varieties, I*, *K*-Theory **10** (1996), 517–596.

[MPW98] ______, *Index reduction formulas for twisted flag varieties, II*, *K*-Theory **14** (1998), no. 2, 101–196.

[PSZ08] V. Petrov, N. Semenov, and K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Ec. Norm. Sup. (4) **41** (2008), 1023–1053.

[QMSZ11] A. Quéguiner-Mathieu, N. Semenov, and K. Zainoulline, *The J-invariant and the Tits algebras of a linear algebraic group*, arxiv:1104.1096v1, April 2011.

[Swe95] Paul Kenneth Swets, *Global sections of higher powers of the twisting sheaf on a Brauer-Severi variety*, Ph.D. thesis, University of Texas at Austin, 1995.

[Vis04] A. Vishik, *Motives of quadrics with applications to the theory of quadratic forms*, Geometric methods in the algebraic theory of quadratic forms, Lecture Notes in Math., vol. 1835, Springer, Berlin, 2004, pp. 25–101.

[Vis05] ______, *On the Chow groups of quadratic Grassmannians*, Documenta Math. **10** (2005), 111–130.

[Vis10] ______, *Fields of u-invariant* $2^r + 1$, Algebra, arithmetic and geometry: in honour of Yu.I. Manin (Y. Tschinkel, ed.), vol. II, Birkhäuser, 2010, pp. 661–685.