Motivic indecomposable summands of $\text{PGL}_1(A)$-homogeneous varieties

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Ramification in Algebra and Geometry at Emory
Conventions

Let $F$ be a field and $p$ be a prime. Consider

- $G$ a semisimple affine algebraic group of inner type
- $X$ a projective $G$-homogeneous variety
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The category of Grothendieck Chow motives with coefficients in $\mathbb{F}_p$ will be denoted $\text{CM}(F; \mathbb{F}_p)$. 
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Theorem (Chernousov, Merkurjev, 2006)

The motive of $X$ decomposes in an "unique" way as a direct sum of indecomposable motives in $\text{CM}(F; \mathbb{F}_p)$.
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The motive of $X$ decomposes in an "unique" way as a direct sum of indecomposable motives in $\text{CM}(F; \mathbb{F}_p)$.

The set of indecomposable $p$-motives of $G$ is the set of isomorphism classes of twists of indecomposable motives of projective $G$-homogeneous varieties in $\text{CM}(F; \mathbb{F}_p)$. This set is denoted $\mathcal{X}_G$. 
Statement of the problem

Problem

If $G$ and $G'$ are two semisimple affine algebraic groups of inner type, can we compare $\mathcal{X}_G$ and $\mathcal{X}_{G'}$?

We give the following answer to this question, if $G$ is isomorphic to $\text{PGL}_1(\mathbb{A})$ and $G'$ to $\text{PGL}_1(\mathbb{A}')$, where $\mathbb{A}$, $\mathbb{A}'$ are central simple algebras.

Theorem (Motivic dichotomy of $\text{PGL}_1$, D. C., 2011)

Let $A$ and $A'$ be two central simple algebras over $F$. Then either $\mathcal{X}_{\text{PGL}_1}(A) \cap \mathcal{X}_{\text{PGL}_1}(A')$ is reduced to the Tate motives or $\mathcal{X}_{\text{PGL}_1}(A) = \mathcal{X}_{\text{PGL}_1}(A')$.

Main ingredients of the proof:

1. The theory of upper motives of Karpenko
2. The index reduction formula of Merkurjev, Panin and Wadsworth
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Problem

If \(G\) and \(G'\) are two semisimple affine algebraic groups of inner type, can we compare \(X_G\) and \(X_{G'}\)?

We give the following answer to this question, if \(G\) is isomorphic to \(\text{PGL}_1(A)\) and \(G'\) to \(\text{PGL}_1(A')\), where \(A, A'\) are central simple algebras.

Theorem (Motivic dichotomy of \(\text{PGL}_1\), D. C., 2011)

Let \(A\) and \(A'\) be two central simple algebras over \(F\). Then either

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X_{\text{PGL}_1(A)} \cap X_{\text{PGL}_1(A')} \text{ is reduced to the Tate motives or}
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X_{\text{PGL}_1(A)} = X_{\text{PGL}_1(A')}. 
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Main ingredients of the proof:

1. The theory of upper motives of Karpenko
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Let $X$ be a projective $G$-homogeneous variety ($G$ still of inner type).

**Definition**

The isomorphism class of the indecomposable summand $U_X$ of $X \in \text{CM}(F; \mathbb{F}_p)$ satisfying $\text{Ch}^0(U_X) \neq 0$ is the *upper* motive of $X$. 

The set of twists of upper motives of projective $G$-homogeneous varieties is the set of upper $p$-motives of $G$, denoted $U^p_G$.

**Theorem (Karpenko, 2009)**

If $G$ be a semisimple affine algebraic group of inner type, $U_G = X_G$.

**Reformulation of the problem**

If $A$ and $A'$ are central simple algebras over a field $F$, can we compare $U_{\text{PGL}_1}(A)$ and $U_{\text{PGL}_1}(A')$?
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Review of the theory of upper motives

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**Theorem (Karpenko, 2009)**

*If $G$ be a semisimple affine algebraic group of inner type, $U_G = \mathfrak{x}_G$.***
Review of the theory of upper motives

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**Theorem (Karpenko, 2009)**

*If* $G$ *be a semisimple affine algebraic group of inner type,* $U_G = \mathcal{X}_G$.

**Reformulation of the problem**

*If* $A$ *and* $A'$ *are central simple algebras over a field* $F$, *can we compare* $U_{\text{PGL}_1(A)}$ *and* $U_{\text{PGL}_1(A')}$?
Reduction to $\text{SB}(p^k; D)$

- Any projective $\text{PGL}_1(A)$-homogeneous variety is isomorphic to a variety of flags of right ideals $X(d_1, ..., d_k; A)$ of reduced dimension $d_1, ..., d_k$ in $A$. 

The upper motive of $X(d_1, ..., d_k; A)$ in $\text{CM}(F; F_p)$ is the upper motive of a generalized Severi-Brauer variety $\text{SB}(p^k; D)$, where $D$ is a division algebra Brauer equivalent to the $p$-primary component of $A$.

Reformulation of the problem

If $D$ and $D'$ are two $p$-primary division algebras over a field $F$, when do we have $U_{\text{SB}(p^k; D)} = U_{\text{SB}(p^k; D')}$ in $\text{CM}(F; F_p)$?

Theorem (Amitsur, 1955)

Let $D$ and $D'$ be two $p$-primary division algebras. Then $U_{\text{SB}(D)} = U_{\text{SB}(D')}$ if and only if $D$ and $D'$ generate the same subgroup of $\text{Br}(F)$. 

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**Reformulation of the problem**

*If $D$ and $D'$ are two $p$-primary division algebras over a field $F$, when do we have $U_{\text{SB}(p^k; D)} = U_{\text{SB}(p^k'; D')}$ in $\text{CM}(F; \mathbb{F}_p)$?*

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Theorem (D. C., 2011)

Let $D$ and $D'$ be two division algebras of degree $p^n$. The following assertions are equivalent:

1. for some $0 \leq k < n$, there are two rational maps $\mathrm{SB}(p^k; D) \leftrightarrow \mathrm{SB}(p^k; D')$;
2. the classes of $D$ and $D'$ generate the same subgroup of $\mathrm{Br}(F)$;
3. for any $0 \leq k < n$, there are two rational maps $\mathrm{SB}(p^k; D) \leftrightarrow \mathrm{SB}(p^k; D')$. 

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Theorem (D. C., 2011)

Let $D$ and $D'$ be two $p$-primary division algebras. If $0 \leq k < \deg(D)$ and $0 \leq k' < \deg(D')$,

$U_{SB}(p^k; D) = U_{SB}(p^{k'}; D')$ in $\text{CM}(F; \mathbb{F}_p)$ $\iff$ $k = k'$ and $<[D]> =<[D']>$

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Corollary

If $X = X(d_1, ..., d_k; A)$ and $X' = X(d'_1, ..., d'_k; A')$ are two anisotropic varieties of flags of right ideals in $A$ and $A'$,

$$U_X = U_{X'} \text{ in } \text{CM}(F; \mathbb{F}_p) \iff v_p(\gcd(d_1, ..., d_k)) = v_p(\gcd(d'_1, ..., d'_k))$$

and $<[A_p]> =<[A'_p]>$. 

Theorem (Motivic dichotomy of $\text{PGL}_1$, D. C., 2011)

Let $A$ and $A'$ be two central simple algebras over $F$. Then either $X_{\text{PGL}_1}(A) \cap X_{\text{PGL}_1}(A')$ is reduced to the Tate motives or $X_{\text{PGL}_1}(A) = X_{\text{PGL}_1}(A')$. 

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What happens if the ring a coefficient is not $\mathbb{F}_p$?

**Theorem (D. C. 2010)**

Let $G$ be a semisimple algebraic group of inner type and $X$ a projective $G$-homogeneous variety. Then for any finite field $\mathbb{F}_q$ of characteristic $p$, the motivic decomposition of $X \in \text{CM}(F; \mathbb{F}_p)$ lifts to the motivic decomposition of $X \in \text{CM}(F; \mathbb{F}_q)$. 