The cyclicity problem for the projective Schur group

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Joint work with Eli Aljadeff

May 2011
Definitions

Let $k$ be any field. A finite dimensional $k$-central simple algebra $A$ is a projective Schur algebra over $k$ if it is spanned over $k$ as a vector space by a subgroup $\Gamma$ of $A^\times$ which is finite modulo $k^\times$. Notation: $A = k(\Gamma)$. 

Example: Symbol algebras $(a, b)_n$: Assume $\xi_n, a, b \in k$, put $\Gamma = \langle x, y | x^n = a, y^n = b, [x, y] = \xi_n \rangle$, $\Gamma / k^\times \sim = \mathbb{Z}_n \times \mathbb{Z}_n$. The projective Schur group of $k$ is the subgroup of $Br(k)$ generated by (and in fact consists of) all classes that may be represented by a projective Schur algebra over $k$. By Merkurjev-Suslin theorem if $k$ contain all roots of 1 then $PS(k) = Br(k)$. Equality holds also if $k$ is a global field. It was conjectured by Nelis and Van Oystaeyen in 1991 that $PS(k) = Br(k)$ for all fields $k$, but in 1994 Aljadeff and Sonn gave a counter example (e.g. $k = \mathbb{Q}(x)$).
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Theorem: (G & A)

Every projective Schur algebra is Brauer equivalent to a tensor product of cyclic algebras. Or in other words, the projective Schur group is generated by cyclic algebras.
If $\text{char}(k) > 0$

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Of course this is false in $\text{char}(k) = 0$. 
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- Notation: Given any field extension $F/k$ we denote by $\text{Rad}_k(F)$ the subgroup of $F^\times$ generated by all units of $F$ which are of finite order modulo $k^\times$. 
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- A field extension $F/k$ is said to be radical if $F = k(\text{Rad}_k(F))$. 

Is every projective Schur algebra Brauer equivalent to such an algebra? Aljadeff, Sonn and Del-Rio: yes! (Even with $G$ abelian) (A-S for $\text{char}(k) > 0$ and $A-D$ for $\text{char}(k) = 0$)
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- A $k$-central simple algebra $A$ is radical if $A$ is a crossed product of the form $(F/k, G, \alpha)$ where $F/k$ is radical and $\alpha \in \text{Rad}_k(F)$.
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If $\text{char}(k) = 0$

Let $A = k(\Gamma)$ be a $k$-projective Schur algebra.

- By Aljadeff and Del-Rio we can assume $A = (k(H)/k, G, \alpha)$ where $H \subseteq \text{Rad}_k(F)$ is finite modulo $k^\times$, $G$ is abelian and $\alpha \in H$. 
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- We can assume $\alpha \in H_p$ where $H_p$ is the subgroup of $H$ generated by all elements of $H$ with $p$-power order modulo $k^\times$. 
continue:

- $A = k(\Gamma) = \bigoplus_{\sigma \in G} k(H) u_{\sigma}$.
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The group \(\Gamma\) is center-by-finite, therefore by a theorem of Schur \(\Gamma'\) is finite. This implies that for all \(\sigma, \tau \in G\) the commutator \([u_\sigma, u_\tau]\) is a root of unity (and by assumption) of \(p\)-power order.
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"Find" a radical abelian algebra which is Brauer equivalent to \(A\) with suitable representatives \(\{u_\sigma \mid \sigma \in G\}\) such that the commutators \([u_\sigma, u_\tau] = 1\) for all \(\sigma, \tau \in G\).