Essential dimension of separable subalgebras

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1 Essential dimension

2 The Problem

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Recollection on essential dimension

- **Intuitively:** Essential dimension = number of independent parameters needed to define algebraic objects of some kind
- **Formally:** \( \mathcal{A} : \text{Fields}_F \to \text{Sets} \) a functor.
  For \( K \in \text{Fields}_F, a \in \mathcal{A}(K) \)

  \[ \text{ed } a := \min \{ \text{trdeg}_F K_0 \mid \text{K_0 a field of definition of a} \} \]

  \[ \text{ed } \mathcal{A} := \sup_{a \in \text{Im}(\mathcal{A}(K_0) \to \mathcal{A}(K))} \text{ed } a \]

- If \( G \) is an algebraic group over \( F \): \( \text{ed } G := \text{ed } H^1(-, G), \quad H^1(K, G) \cong \{ G\text{-torsors over } K \}/\text{iso}. \)
Recollection on essential dimension

- If $G$ is an algebraic group over $F$: $\text{ed } G := \text{ed } H^1(\_, G)$,
  \[ H^1(K, G) \cong \{ G\text{-torsors over } K \}/\text{iso}. \]

- Other interpretations for some $G$:
  - $H^1(\_, \text{PGL}_n) \cong \text{CSA}_n \cong \text{SB-Var}_{n-1}$ central simple algebras of degree $n$ (or $n-1$-dimensional Severi Brauer varieties).
  - $H^1(\_, \text{O}_n) \cong \text{Quad}_n$ non-degenerate quadratic forms of dimension $n$
  - $H^1(\_, \text{S}_n) \cong \text{ét}_n \cong \text{Gal}_{S_n}$ étale algebras of dimension $n$ (or Galois $S_n$-algebras).
  - etc.

- Thus essential dimension of algebraic groups can be used to study the “complexity” of interesting algebraic objects.
1. Essential dimension
2. The Problem
3. Solution of a special case
4. Split versus non-split groups
Throughout this talk: $G = N_{\text{GL}_1(A)}(\text{GL}_1(B))$, where

- $A$ is a central simple $F$-algebra
- $B$ is a separable $F$-subalgebra of $A$

$$B = B_1 \times \cdots \times B_r,$$

where each $B_i$ is simple, and $Z(B) = K_1 \times \cdots \times K_r$ is étale.

- $\text{GL}_1(A)$ is the algebraic group of invertible elements of $A$:

$$\text{GL}_1(A)(R) = A^*_R,$$

where $A_R = A \otimes_F R$.

- Problem: Compute $\text{ed } G$!
Motivation: Part 1

**Interpretation of the functor $H^1(K, G)$**

\[ H^1(K, G) \cong \{ B' \subseteq A_K \text{ sep. subalg. of the “same type” as } B \}/\text{conj.} \]

- $B_1, B_2 \subseteq A$ are **conjugate** if $\exists a \in A^\times : aB_1a^{-1} = B_2$.
- type of $B \subseteq A$ is a discrete invariant under conjugation:
  
  Write $B_{\text{sep}} = B_1 \times \cdots \times B_m, \quad B_i \cong M_{d_i}(F_{\text{sep}})$,

  $C_{A_{\text{sep}}}(B_{\text{sep}}) = C_1 \times \cdots \times C_m, \quad C_i \cong M_{r_i}(F_{\text{sep}})$

  s.t. $Z(B_i) = Z(C_i)$.

- type of $B$: multiset $t_B = [(d_1, r_1), \ldots, (d_m, r_m)]$ (repetitions allowed, order does not matter).

**Special cases**

- $B$ étale: $t_B = [(1, r_1), \ldots, (1, r_m)]$ (Krashen)
- $B$ central simple: $t_B = [(d, r)]$

- Relation: $\sum_{i=1}^m d_ir_i = \deg A$. 
Motivation: Part 1

Interpretation of the functor $H^1(K, G)$

$$H^1(K, G) \cong \{ B' \subseteq A_K \text{ sep. subalg. of the “same type” as } B \}/\text{conj.}$$

- Thus \textit{ed}\ $G$ measures the complexity of certain separable subalgebras of $A$ (and the $A_K$‘s)!

- When $t_B$ satisfies ($d_i = d_j \Rightarrow r_i = r_j)$:

  conjugacy classes = isomorphism classes

  for type $t_B$ subalgebras.
Motivation: Part 2

\[ H^1(K, G) \cong \{ B' \subseteq A_K \text{ sep. subalg. of type } t_B \}/\text{conj.} \]

For some choice of $B$ and $A$ the value $\text{ed } G$ coincides with:

- $\text{ed } \text{PGL}_n = \text{ed } \text{CSA}_n$
- $\text{ed } S_n = \text{ed } \text{Ét}_n$
- More generally: essential dim. of \textit{forms of any separable algebra}.

In general computing $\text{ed } G$ is very \textbf{hard}, but also very \textbf{interesting}. We will solve an interesting special case.
Motivation: Part 2

\[ H^1(K, G) \cong \{ B' \subseteq A_K \text{ separ. subalg. of type } t_B \}/\text{conj.} \]

For some choice of \( B \) and \( A \) the value \( \text{ed } G \) coincides with:
- \( \text{ed } \text{PGL}_n = \text{ed } \text{CSA}_n \)
  - Choose \( B \) central simple of degree \( n \), \( n^2 \text{ ind } A \mid \text{deg } A \)
  - e.g. \( B = M_n(F) \hookrightarrow M_n(F) \otimes_F M_n(F), \ a \mapsto a \otimes 1. \)
    Here \( G \cong (\text{GL}_n \times \text{GL}_n)/\{(t, t^{-1}) \mid t \in \mathbb{G}_m\}. \)
- \( \text{ed } S_n = \text{ed } \hat{\text{Et}}_n \)
- More generally: essential dim. of forms of any separable algebra.

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- \( \text{ed } \text{PGL}_n = \text{ed } \text{CSA}_n \)
- \( \text{ed } S_n = \text{ed } \text{ét}_n \)
  
  - Choose \( B \) étale of dimension \( n \), \( t_B = [(1, r), \ldots, (1, r)] \), \( \text{ind } A | r \)
  
  - e.g. \( F^n \hookrightarrow M_n(F) \otimes M_r(F), (\lambda_1, \ldots, \lambda_n) \mapsto \text{Diag}(\lambda_1, \ldots, \lambda_n) \otimes 1 \)

Here \( G \cong (\text{GL}_r)^n \rtimes S_n \).

More generally: essential dim. of forms of any separable algebra.

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\[ H^1(K, G) \cong \{ B' \subset A_K \text{ sep. subalg. of type } t_B \}/\text{conj.} \]

For some choice of \( B \) and \( A \) the value \( \text{ed } G \) coincides with:

- \( \text{ed PGL}_n = \text{ed CSA}_n \)
  - \( \text{ed CSA}_n \) is completely open (even if \( n \) is a prime)
  - \( \text{ed}_p \text{ CSA}_n \) is solved for \( \nu_p(n) \leq 1 \) and \( \nu_p(n) = 2 \) (Merkurjev), \( \text{char } F \neq p \)
- \( \text{ed } S_n = \text{ed } \text{Ét}_n \)
- More generally: essential dim. of forms of any separable algebra.

In general computing \( \text{ed } G \) is very hard, but also very interesting. We will solve an interesting special case.
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For some choice of \( B \) and \( A \) the value \( \text{ed } G \) coincides with:

- \( \text{ed } \text{PGL}_n = \text{ed } \text{CSA}_n \)
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- \( \text{ed } S_n = \text{ed } \text{Ét}_n \)
  - \( \text{ed } \text{Ét}_n \) is open for \( n \geq 8 \) (\( n \geq 7 \), \( \text{char } F \geq 0 \); \( n \geq 6 \), \( \text{char } F = 2 \))
  - \( \text{ed } \text{Ét}_7 = 4 \) when \( \text{char } F = 0 \) (Duncan)
  - \( \text{ed}_p S_n = \lfloor \frac{n}{p} \rfloor \), \( \text{char } F \neq p \) (Serre). Open in \( \text{char } F = p \).

- More generally: essential dim. of forms of any separable algebra.

In general computing \( \text{ed } G \) is very hard, but also very interesting. We will solve an interesting special case.
Outline

1. Essential dimension
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4. Split versus non-split groups
The special case

We consider the case, where:

- $A$ is a division algebra
- $\deg A = p^n$ ($n \geq 0$)
- (No assumptions on $F$ or char $F$!)

Note:

- $A$ division $\Rightarrow t_B = [(d, r), \ldots, (d, r)]$ (constant)
- $dr \mid \deg A = p^n$. Set $d = p^a, r = p^b$ (with $a + b \leq n$).

The main theorem

Under the above assumptions and notations:

$$
ed G = \operatorname{ed}_p G$$
$$= \dim_F A - \dim G$$
$$= p^{2n} - p^{n+a-b} - p^{n-a+b} + p^{n-a-b}.$$
About the proof

- Computing $\dim G$ (and hence $\dim_F A - \dim G$) is easy.
- $G \subseteq \text{GL}_1(A)$, $\text{ed} \ \text{GL}_1(A) = 0$

$$\Rightarrow \text{ed} \ G \leq \dim_F A - \dim G$$

(in general $G \subseteq H \Rightarrow \text{ed} \ G + \dim G \leq \text{ed} \ H + \dim H$).

- Since $\text{ed}_p \ G \leq \text{ed} \ G$, it remains to show:

$$\text{ed}_p \ G \geq \dim_F A - \dim G.$$

- This bound uses an index formula of Brosnan-Reichstein-Vistoli which relies on the incompressibility of certain Severi-Brauer varieties due to Karpenko.
Subgroups of $\text{GL}_1(A)$

- $A$ a division algebra, $\deg A = p^n$.
- There is a whole bunch of subgroups $S \subseteq \text{GL}_1(A)$ satisfying
  \[ \text{ed} \ S = \dim_F A - \dim S. \]

- Except $N_{\text{GL}_1(A)}(\text{GL}_1(B))$ these include the groups
  \[ S = \text{Sim}(A, \sigma) \text{ and } S = \text{Iso}(A, \sigma), \]
  $\rightarrow$ conjugacy classes of involutions

  ($\sigma$ a involution of the first kind).
A look at cohomology

- $S \subseteq \text{GL}_1(A)$, $A$ a division algebra, $\deg A = p^n$
- Set $H := \text{Int}(S) \subseteq \text{Aut}_F(A)$, $C := S \cap Z(\text{GL}_1(A)) \subseteq Z(\text{GL}_1(A)) \simeq \mathbb{G}_m$.
- Consider the following diagram with exact rows: Case $C \simeq \mathbb{G}_m$

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathbb{G}_m & \rightarrow & S & \rightarrow & H & \rightarrow & 1 \\
\uparrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{G}_m & \rightarrow & \text{GL}_1(A) & \rightarrow & \text{Int} \rightarrow & \text{Aut}_F(A) & \rightarrow & 1
\end{array}
\]

- Induced diagram in cohomology:

\[
\begin{array}{ccc}
H^1(K, H) & \xrightarrow{\delta} & H^2(K, \mathbb{G}_m) = \text{Br}(K) \\
\downarrow & & \downarrow \\
H^1(K, \text{Aut}_F(A)) & \rightarrow & H^2(K, \mathbb{G}_m) = \text{Br}(K)
\end{array}
\]
A look at cohomology

- $S \subseteq \text{GL}_1(A)$, $A$ a division algebra, $\deg A = p^n$
- Set $H := \text{Int}(S) \subseteq \text{Aut}_F(A)$, $C := S \cap Z(\text{GL}_1(A)) \subseteq Z(\text{GL}_1(A)) \simeq \mathbb{G}_m$.
- Consider the following diagram with exact rows: Case $C \simeq \mu_r$

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \mu_r & \rightarrow & S & \rightarrow & H & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{G}_m & \rightarrow & \text{GL}_1(A) & \rightarrow & \text{Int} & \rightarrow & \text{Aut}_F(A) & \rightarrow & 1
\end{array}
$$

- Induced diagram in cohomology:

$$
\begin{array}{cccccc}
H^1(K, H) & \overset{\delta}{\rightarrow} & H^2(K, \mu_r) = \text{Br}_r(K) \\
\downarrow & & \downarrow & & \\
H^1(K, \text{Aut}_F(A)) & \rightarrow & H^2(K, \mathbb{G}_m) = \text{Br}(K)
\end{array}
$$
A look at cohomology

- $S \subseteq \text{GL}_1(A)$, $A$ a division algebra, $\text{deg } A = p^n$

$$H^1(K, H) \xrightarrow{\delta} \text{Br}_? (K)$$

$$\downarrow$$

$$H^1(K, \text{Aut}_F(A)) \xrightarrow{} \text{Br}(K)$$

- Brosnan-Reichstein-Vistoli (using Karpenko’s incompressibility result): $\text{ed}_p S \geq \max_t \text{ind } \delta(t) - \dim S$. 
A look at cohomology

- $S \subseteq \text{GL}_1(A)$, $A$ a division algebra, $\deg A = p^n$

$$
\begin{array}{c}
H^1(K, H) \xrightarrow{\delta} \text{Br}_? (K) \\
\downarrow \quad \quad \quad \downarrow \\
H^1(K, \text{Aut}_F(A)) \longrightarrow \text{Br}(K)
\end{array}
$$

- Brosnan-Reichstein-Vistoli (using Karpenko's incompressibility result): $\text{ed}_p S \geq \max_t \text{ind} \delta(t) - \dim S$.

- $H^1(K, \text{Aut}_F(A))$ classifies forms of $A_K$.

- The map $H^1(K, \text{Aut}_F(A)) \rightarrow \text{Br}(K)$ takes the isomorphism class of $A'$ to the Brauer class of $A' \otimes_F A^{op}$.

- If the image of $H^1(K, H) \rightarrow H^1(K, \text{Aut}_F(A))$ contains $[A']$ with $A' \otimes_F A^{op}$ division, then $\text{ed}_p S \geq \dim_F A - \dim S$ follows.
Cohomology continued

- $S \subseteq \text{GL}_1(A)$, $A$ division, $\deg A = p^n$, $H = \text{Int}(S)$.
- In order to prove $\text{ed} S = \text{ed}_{p} S = \dim_F A - \dim S$ it suffices to find $K \in \text{Fields}_F$ and $t \in H^1(K, H)$ such that the algebra $A'$ representing the image of $t$ in $H^1(K, \text{Aut}_F(A))$ makes $A' \otimes_F A^{\text{op}}$ division.
Cohomology continued

- $S \subseteq \text{GL}_1(A)$, $A$ division, $\text{deg} \ A = p^n$, $H = \text{Int}(S)$.
- In order to prove $\text{ed} \ S = \text{ed}_p S = \dim_F A - \dim S$, it suffices to find $K \in \text{Fields}_F$ and $t \in H^1(K, H)$ such that the algebra $A'$ representing the image of $t$ in $H^1(K, \text{Aut}_F(A))$ makes $A' \otimes_F A^{\text{op}}$ division.
- I did this for the groups $H = \text{Aut}_F(A, B)$ and $H = \text{Aut}_F(A, \sigma)$.  
  $\implies$ main theorem
Cohomology continued

- \( S \subseteq \text{GL}_1(A) \), \( A \) division, \( \deg A = p^n \), \( H = \text{Int}(S) \).
- In order to prove \( \text{ed} S = \text{ed}_p S = \dim_F A - \dim S \) it suffices to find \( K \in \text{Fields}_F \) and \( t \in H^1(K, H) \) such that the algebra \( A' \) representing the image of \( t \) in \( H^1(K, \text{Aut}_F(A)) \) makes \( A' \otimes_F A^{\text{op}} \) division.
- I did this for the groups \( H = \text{Aut}_F(A, B) \) and \( H = \text{Aut}_F(A, \sigma) \).

A more general method, using incompressibility of quadratic Weil transfers of Severi-Brauer varieties (Karpenko), applies for certain subgroups \( S \subseteq R_{L/F}(\text{GL}_1(A)) \), where:

- \( L/F \) is a quadratic separable field extension,
- \( A \) is a central simple \( L \)-algebra

Example: \( S = GU(A, \tau) \), where \( \tau \) is a unitary involution
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Split reductive groups

- $S$ a (smooth) algebraic group, $S_{\text{alg}}$ reductive
- Call $S$ split, if $S$ contains a split maximal torus and $S/S^0$ is constant.

**Examples**

- $\text{GL}_n, \text{SL}_n, S_n$ etc. are split, $S_{\text{alg}}$ is split etc.
- $\text{GL}_1(A), \text{SL}_1(A)$ are split if and only if $A$ is split.
- $G = N_{\text{GL}_1(A)}(\text{GL}_1(B))$ is split if and only if both $A$ and $B$ are split, i.e. (direct products of) matrix algebras $M_{n_i}(F)$.

**Until recently** (for ed $S$): mostly split groups considered.
A surprising phenomenon

- **Non-split groups**: $\text{SL}_1(A)$ (Favi-Lequeu), twisted $p$-groups and tori (L-MacDonald-Meyer-Reichstein).
- In every case the split case has been solved earlier, $\text{ed} \text{SL}_n = 0 = \text{ed} \mathbb{G}_m^n$; finite $p$-groups: Karpenko-Merkurjev.

**Natural question:**
Is essential dimension computation **easier** for **split** groups??
A surprising phenomenon

- **Non-split groups**: $\text{SL}_1(A)$ (Favi-Lequeu), twisted $p$-groups and tori (L-MacDonald-Meyer-Reichstein).

- In every case the split case has been solved earlier, $\text{ed} \, \text{SL}_n = 0 = \text{ed} \, \mathbb{G}_m^n$; finite $p$-groups: Karpenko-Merkurjev.

**Natural question:**

Is essential dimension computation **easier** for **split** groups??

- We observed for $G = N_{\text{GL}_1(A)}(\text{GL}_1(B))$: Computing $\text{ed} \, G$ is **very hard** when $G$ is **split**, easier in very **non-split** case ($A$ division)!!

→ **first examples** of groups with $\text{ed} \, G$ known, $\text{ed} \, G_{\text{alg}}$ unknown.
Summary

- **Problem:** Compute $\text{ed} \, N_{GL_1(A)}(GL_1(B))$, where $B$ is a separable subalgebra of a central simple algebra $A$.
  → complexity of *conjugacy-classes of separable subalgebras* of certain type.
- Problem is **interesting**, but in general **very hard**, especially in the **split case**.
- We **solved** the problem in the **division algebra case** (assuming $\deg A = p^n$).
  → first groups, where $\text{ed} \, G$ is known, $\text{ed} \, G_{\text{alg}}$ unknown.
- Same **method** applies for other groups like $\text{Sim}(A, \sigma)$, $\text{Iso}(A, \sigma)$.