Unramified cohomology and Chow groups

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Let $k$ be a field and $G = Gal(k^s/k)$.

Let $X/k$ be a smooth projective geometrically integral variety.

$Z^i(X) = \bigoplus_{x \in X(i)} \mathbb{Z}$

$CH^i(X) = Z^i(X)/\sim_{\text{rat}}$

We have a natural map

$$CH^i(X) \xrightarrow{\phi_i} CH^i(\bar{X})^G,$$

where $\bar{X} = X \times_k k^s$. 
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- $i = 1$. Then $CH^1(X) \cong \text{Pic} X$ and we have an exact sequence

$$0 \to \text{Pic} X \xrightarrow{\phi_1} (\text{Pic} \bar{X})^G \to \text{Br} k \to \text{Br} X.$$

If $\text{Br} k = 0$ (example: $k$ is a finite field), then $\phi_1$ is surjective.
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If $\text{Br } k = 0$ (example: $k$ is a finite field), then $\phi_1$ is surjective.

3. $i = \dim X$ and $k = \mathbb{F}$ is a finite field.

We have a surjection:

$$CH_0(X) \twoheadrightarrow CH_0(\bar{X})^G.$$
Examples

- $i = 0$. $\phi_0 : CH^0(X) \rightarrow CH^0(\tilde{X})^G$ is bijective;
- $i = 1$. Then $CH^1(X) \simeq \text{Pic } X$ and we have an exact sequence

$$0 \rightarrow \text{Pic } X \xrightarrow{\phi_1} \left( \text{Pic } \tilde{X} \right)^G \rightarrow \text{Br } k \rightarrow \text{Br } X.$$ 

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This follows from :

- $X$ has a zero-cycle of degree 1 (Lang-Weil estimates);
- $A_0(X) \twoheadrightarrow \text{Alb}_X(\overline{\mathbb{F}})$ (Kato-Saito);
- $A_0(\tilde{X}) \simeq \text{Alb}_X(\overline{\mathbb{F}})$ (Milne, Rojtman).
Question (T. Geisser):

For \( X/\mathbb{F} \), do we have a surjection

\[
\phi_2 : CH^2(X) \to CH^2(\tilde{X})^G
\]
Using techniques of $K$-theory, one shows:

**Theorem (Kahn, Colliot-Thélène and Kahn)**

Let $\mathbb{F}$ be a finite field, $\text{char} \mathbb{F} = p$. Let $X/\mathbb{F}$ be a smooth projective geometrically rational variety. We have the following complex

$$0 \to CH^2(X) \xrightarrow{\phi^2} CH^2(\bar{X})^G \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to 0$$

which is exact after tensorisation by $\mathbb{Z}[1/p]$. 

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Using a method of Colliot-Thélène and Ojanguren, we will produce, for infinitely many primes $p$, a geometrically rational variety $X/\mathbb{F}_p$ with $H^3_{\text{nr}}(X, \mathbb{Z}/2) \neq 0$. 
Strategy

First step.
Write $F = \mathbb{F}_p(x, y)$ and consider the quadric $Q \subset \mathbb{P}^4_F$, $p \neq 2$ defined by

$$x_0^2 - ax_1^2 - fx_2^2 + afx_3^2 - g_1g_2x_4^2 = 0$$

with $a \in \mathbb{F}_p; f, g_1, g_2 \in F$. 

Theorem (Arason)

$$\ker \left[ H^3(F, \mathbb{Z}/2) \rightarrow H^3(F(Q), \mathbb{Z}/2) \right] = \mathbb{Z}/2(a, f, g_1, g_2).$$
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$$Q(\overline{\mathbb{F}}_p(x, y)) \neq \emptyset \implies Q \text{ is } \overline{\mathbb{F}}_p(x, y)\text{-rational.}$$
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Find necessarily conditions on $a, f, g_1, g_2$ such that, for $Q$ defined by

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- $(a, f, g_1) \in H^3_{nr}(F(Q), \mathbb{Z}/2)$, where

$$H^3_{nr}(F(Q), \mathbb{Z}/2) = \bigcap_{A \text{ dvr Frac}(A) = F(Q)} \text{Ker}[H^3(F(Q), \mathbb{Z}/2) \xrightarrow{\partial A} H^2(k_A, \mathbb{Z}/2)].$$
Third step.

- Find $a \in \mathbb{Z}$ and $f, g_1, g_2 \in \mathbb{Z}(x, y)$ such that for infinitely many $p$ their reductions modulo $p$

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- By Hironaka, find $X/\mathbb{Q}$ smooth and projective with a morphism $X \to \mathbb{P}^2_{\mathbb{Q}}$ whose generic fibre is defined by

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For infinitely many \( p \), \( X \) has a reduction \( X_p \) over \( \mathbb{F}_p \) which is smooth over \( \mathbb{F}_p \) and we have a nonzero element

\[ (\bar{a}, \bar{f}, \bar{g}_1) \]

in \( H^3_{nr}(X_p, \mathbb{Z}/2) \).
Conclusion

For $p \geq 13$ the following choice works:

$$a \in \mathbb{F}_p^* \setminus \mathbb{F}_p^{*2}$$

$$f, g_1, g_2 \in \mathbb{F}_p(\mathbb{P}^2_{\mathbb{F}_p})$$ with homogeneous coordinates $(x : y : z)$:

$$f = \frac{x}{y}$$

$$g_1 = \frac{\prod_j (x + y + 2z + h_j)}{y^8}$$

$$g_2 = \frac{\prod_j (3x + 3y + z + h_j)}{z^8}$$

where $h_j, j = 1, \ldots, 8$, are the linear forms $e_x x + e_y y + e_z z$ with $e_x, e_y, e_z \in \{0, 1\}$. 