J-invariant and triality

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Notation

$p$ prime number
$k$ field, \( \text{char}(k) \neq p \)

$G$ absolutely almost simple linear algebraic group over $k$.
$G_0$ corresponding split group.
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We assume \( \xi \in Z^1(k, G_0) \).
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We assume $\xi \in Z^1(k, G_0)$.

We fix $T_0 \subset B_0 \subset G_0$
$T_0$ split maximal torus ; $B_0$ Borel subgroup
$\mathfrak{x}_0 = G_0/B_0, \quad \mathfrak{x} = \xi \mathfrak{x}_0$

Example

$G = O^+(\varphi)$ for some $\varphi : V \to k$, $\dim V = 2n$.
$\mathfrak{x} \leftrightarrow V_1 \subset V_2 \subset \cdots \subset V_{n-1}$,
    totally isotropic of dimension $1, 2, \ldots, n - 1$
J-invariant of the group $G$

$$\text{Ch}^*(\mathcal{X}) \xrightarrow{\text{res}} \text{Ch}^*(\mathcal{X}_0)$$
J-invariant of the group $G$

\[ \text{Ch}^*(\mathcal{X}) \xrightarrow{\text{res}} \text{Ch}^*(\mathcal{X}_0) \xrightarrow{\pi^*} \text{Ch}^*(G_0), \]

where $\pi : G_0 \longrightarrow \mathcal{X}_0 = G_0/B_0$.

V. Kac (1985) \hspace{1cm} \text{Ch}^*(G_0) \cong \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}),

for some integers $r, k_1, \ldots, k_r$, known for each $G_0$. 
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Petrov-Semenov-Zainoulline (2008)

$$\text{coker}(\pi_* \circ \text{res}) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{j_1}}, \ldots, x_r^{p^{j_r}}),$$

for some $j_1, \ldots, j_r$, \quad $0 \leq j_i \leq k_i$

**Definition:** \quad $J_p(G) = (j_1, \ldots, j_r)$
More notation

\[ A \text{ central simple } k \text{ algebra} \]
\[ \text{deg}(A) = 2n \]
\[ \sigma : A \to A \text{ involution} \]
\[ \text{of orthogonal type} \]
\[ \iff (A, \sigma) \otimes_k k_s \cong (M_{2n}(k_s), t) \]
A central simple $k$ algebra
$\deg(A) = 2n$
$\sigma : A \rightarrow A$ involution
of orthogonal type

$\text{Aut}^+(A, \sigma) = \text{PGO}^+(A, \sigma) = \xi\text{PGO}^+_{2n}$

We assume $\text{disc}(\sigma) = 1 \in k^*/k^{*2}$. 

$\Leftrightarrow (A, \sigma) \otimes_k k_s \simeq (M_{2n}(k_s), t)$
More notation

A central simple \( k \) algebra
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\begin{align*}
\text{deg}(A) &= 2n \\
\sigma : A &\rightarrow A \text{ involution} \\
\text{of orthogonal type}
\end{align*}
\]

\[
\iff (A, \sigma) \otimes_k k_s \cong (M_{2n}(k_s), t)
\]

\[
\text{Aut}^+(A, \sigma) = \text{PGO}^+(A, \sigma) = \xi \text{PGO}_{2n}^+ \quad \text{Adjoint of inner type } D_n
\]

We assume \( \text{disc}(\sigma) = 1 \in k^*/k^{*2} \).

\[
\begin{array}{c}
\omega_1 \\
\vdots \\
\omega_{n-1} \\
\omega_n
\end{array}
\]
More notation

A central simple $k$ algebra
$\deg(A) = 2n$
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$\mathrm{Aut}^+ (A, \sigma) = \mathrm{PGO}^+ (A, \sigma) = \xi \mathrm{PGO}_{2n}^+$ Adjoint of inner type $D_n$
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\[
\begin{align*}
\omega_1 & \quad \cdots \quad \omega_{n-1} \\
\omega_n
\end{align*}
\]

Tits algebras

$(A, \sigma)$

$C(A, \sigma) = C_+ \times C_-$
\[ \text{deg}(A) = 8 \quad \text{PGO}^+(A, \sigma) \text{ of type } D_4 \]

\[ (A, \sigma) \rightarrow \begin{array} {c} \bullet \end{array} \rightarrow \begin{array} {c} \bullet \end{array} \rightarrow \begin{array} {c} \bullet \end{array} \]

\( C_- \)

\( C_+ \)
Triality

\[
\text{deg}(A) = 8 \quad \text{PGO}^+(A, \sigma) \text{ of type } D_4
\]

\[
\begin{array}{c}
(A, \sigma) \quad \quad \quad (B, \tau) \\
\quad \quad \quad \quad \\
\quad \quad \quad \quad \\
(C, \gamma)
\end{array}
\]
Triality

\[ \text{deg}(A) = 8 \quad \text{PGO}^+(A, \sigma) \text{ of type } D_4 \]

\[
\begin{align*}
(A, \sigma) & \rightarrow (B, \tau) \\
& \rightarrow (C, \gamma)
\end{align*}
\]

Fact: \[
\begin{align*}
\mathcal{C}(A, \sigma) &= (B, \tau) \times (C, \gamma) \\
\mathcal{C}(B, \tau) &= (A, \sigma) \times (C, \gamma) \\
\mathcal{C}(C, \gamma) &= (A, \sigma) \times (B, \tau)
\end{align*}
\]

Trialitarian triple \( \mathcal{T} = ((A, \sigma), (B, \tau), (C, \gamma)) \)
How to order the generators of $\text{Ch}^*(G_0)$?

- $G$ is not adjoint of type $D_n$, with $n$ even.
  Order by degree $d_1 < d_2 < \cdots < d_r$.
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- $G$ adjoint of type $D_{2m}$, that is $G_0 = \text{PGO}^+_{4m}$,
  $\text{CH}^1(G_0) \cong \Lambda_w/\Lambda_r \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
How to order the generators of $\text{Ch}^*(G_0)$?

- $G$ is not adjoint of type $D_n$, with $n$ even.
  Order by degree $d_1 < d_2 < \cdots < d_r$.

- $G$ adjoint of type $D_{2m}$, that is $G_0 = PGO_{4m}^+$, $m \geq 3$
  $\text{CH}^1(G_0) \simeq \Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \{0, \bar{\omega}_1, \bar{\omega}_{n-1}, \bar{\omega}_n\}$

\[\begin{array}{c}
\omega_1 \quad \cdots \quad \omega_{n-1} \\
\omega_n
\end{array}\]

$x_1 = \bar{\omega}_1$, $x_2 = \bar{\omega}_n$ or $\bar{\omega}_{n-1}$, then order by degree.

Notation: $J(A, \sigma) = J_2(PGO^+(A, \sigma))$. 
Triality

\[ \text{deg}(A) = 8 \]

\[ \begin{aligned}
&\omega_1 \\
&\quad \quad \quad \quad \downarrow \\
&\omega_3 \\
&\quad \quad \quad \quad \downarrow \\
&\omega_4 \\
\end{aligned} \]
Triality

$\text{deg}(A) = 8$

$\text{PGO}^+(A,\sigma) \cong \text{PGO}^+(B,\tau) \cong \text{PGO}^+(C,\gamma)$
Triality

$$\text{deg}(A) = 8$$

$$PGO^+(A, \sigma) \simeq PGO^+(B, \tau) \simeq PGO^+(C, \gamma)$$

**Definition**

$$J(A, \sigma) = J_2(PGO^+(A, \sigma))$$, computed with $$x_1 = \bar{\omega}_1, A_{\omega_1} = A.$$
Triality

$$\text{deg}(A) = 8$$

$$\begin{align*}
(A, \sigma) & \quad \longrightarrow \\
(B, \tau) & \quad \longrightarrow \\
(C, \gamma) &
\end{align*}$$

$$\text{PGO}^+(A, \sigma) \cong \text{PGO}^+(B, \tau) \cong \text{PGO}^+(C, \gamma)$$

**Definition**

$$J(A, \sigma) = J_2(\text{PGO}^+(A, \sigma))$$, computed with $$x_1 = \bar{w}_1$$, $$A_{\omega_1} = A$$.

We have $$\text{Ch}^*(\text{PGO}^+_8) = \mathbb{F}_2[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^2)$$.

$$J(A, \sigma) = (j_1, j_2, j_3), \quad \text{with } 0 \leq j_1, j_2 \leq 2 \quad \text{and} \quad 0 \leq j_3 \leq 1.$$  

$$J(B, \tau), J(C, \gamma) \in \{(j_1, j_2, j_3), (j_2, j_1, j_3)\}$$
The value of $J(A, \sigma)$ when $\deg(A) = 8$

$(A, \sigma) = (\text{End}_D(M), \text{ad}_h)$, for some $(M, h)$ hermitian over $(D, -)$

Index: $\text{ind}(A) = \deg(D) = 2^{i_A}, \quad 0 \leq i_A \leq 3$. 
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The involution $\sigma$ is isotropic iff the hermitian form $h$ is isotropic hyperbolic

Theorem (QSZ)

Let $\mathcal{T} = ((A, \sigma), (B, \tau), (C, \gamma))$ be a trialitarian triple, with $\text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C)$.

Let $j = \min\{i_A, 2\}$ and $j' = \min\{i_B, 2\}$

Then

$J(A, \sigma) = (j, j', j_3)$,

$J(B, \tau) = J(C, \gamma) = (j', j, j_3)$.

Moreover, $j_3 = \begin{cases} 0 & \text{if } \mathcal{T} \text{ is isotropic} \\ 1 & \text{if } \mathcal{T} \text{ is anisotropic} \end{cases}$
Proof

Based on the main theorem in QSZ, which consists of inequalities:

\[
\begin{align*}
    j_1 & \leq i_A, \\
    j_2 & \leq i_B, i_C
\end{align*}
\]

\[j_1, j_2 \quad \leftrightarrow \quad \text{indices of Tits algebras of } G.\]
Proof

Based on the main theorem in QSZ, which consists of inequalities:

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\begin{align*}
& j_1, j_2 \\
\end{align*}
\]

\[
\leftrightarrow \quad \text{indices of Tits algebras of } G.
\]

Example

\[
J(A, \sigma) = (j_1, j_2, j_3); \quad i_J = \min\{i_A, i_B, i_C\};
\]

We have:

\[
1 \quad \begin{cases} 
  j_1 \leq i_A \\
  j_2 \leq i_B, i_C 
\end{cases}
\]

\[
2 \quad \begin{cases} 
  i_J > 0 \quad \Rightarrow \quad j_1 > 0 \text{ and } j_2 > 0 \\
  i_J > 1 \quad \Rightarrow \quad j_1 > 1 \text{ and } j_2 > 1 
\end{cases}
\]
Possible values

Corollary 1
The values $p_1, 2, 0 q, p_2, 1, 0 q$ and $p_2, 2, 0 q$ are impossible.

All other values $p_j, 1, j, 2, j, 3 q$ with $0 \leq j_1, j_2 \leq 2$ and $0 \leq j_3 \leq 1$ do occur.

Theorem (Tits-Allen 1968)
$p E, \rho q$ central simple algebra with orthogonal involution, of degree $0 \mod 4$
If $\rho$ is hyperbolic, then $C \rho E, \rho q$ has a split component.
Possible values

Corollary

1. The values \((1, 2, 0), (2, 1, 0)\) and \((2, 2, 0)\) are impossible.
2. All other values \((j_1, j_2, j_3)\) with \(0 \leq j_1, j_2 \leq 2\) and \(0 \leq j_3 \leq 1\) do occur.
Possible values

Corollary

1. The values $(1, 2, 0), (2, 1, 0)$ and $(2, 2, 0)$ are impossible.
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Theorem (Tits-Allen 1968)

$(E, \rho)$ central simple algebra with orthogonal involution, of degree $\equiv 0 \pmod{4}$
If $\rho$ is hyperbolic, then $C(E, \rho)$ has a split component.
Examples of trialitarian triples

$Q_1, Q_2, Q_3, Q_4$ quaternion algebras, with

$Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4$ split.

$Q_1 \otimes Q_2 \sim Q_3 \otimes Q_4 \sim D$;
Examples of trialitarian triples

$Q_1, Q_2, Q_3, Q_4$ quaternion algebras, with

\[ Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4 \text{ split.} \]

$Q_1 \otimes Q_2 \sim Q_3 \otimes Q_4 \sim D$; Choose $\rho$ orthogonal involution of $D$.

\[
\begin{cases}
(Q_1, -) \otimes (Q_2, -) = \text{Ad}_{h_{12}} & h_{12}, h_{34} \text{ hermitian over } (D, \rho) \\
(Q_3, -) \otimes (Q_4, -) = \text{Ad}_{h_{34}}
\end{cases}
\]

**Definition (Dejaiffe)**

$(Q_1, -) \otimes (Q_2, -) \boxplus (Q_3, -) \otimes (Q_4, -) := \text{Ad}_{h_{12} \oplus h_{34}}$
Examples of trialitarian triples

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\end{cases}
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$h_{12}, h_{34}$ hermitian over $(D, \rho)$

Definition (Dejaiffe)

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(Q_1, -) \otimes (Q_2, -) \boxplus \lambda (Q_3, -) \otimes (Q_4, -) := \text{Ad}_{h_{12} \oplus \lambda h_{34}}
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Examples of trialitarian triples

$Q_1, Q_2, Q_3, Q_4$ quaternion algebras, with

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Proposition

$$(C, \sigma) = (C_+, \sigma_+) \times (C_-, \sigma_-), \text{ where}$$

$$\begin{cases} (C_+, \sigma_+) = (Q_1, -) \otimes (Q_3, -) \boxplus_{\lambda} (Q_2, -) \otimes (Q_4, -), \\ (C_-, \sigma_-) = (Q_1, -) \otimes (Q_4, -) \boxplus_{\lambda} (Q_2, -) \otimes (Q_3, -). \end{cases}$$