Division algebras over function fields of surfaces-d’après Saltman

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Introduction

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**Theorem (Artin – Wedderburn)**

A central simple algebra $A$ is isomorphic to the matrix algebra $M_n(D)$ for some $n \geq 1$ and central division algebra $D$ over $K$.

Let $Br(K)$ be the Brauer group of central simple algebras over $K$.

Every element of $Br(K)$ is represented by a central division algebras over $K$. 
Let $A$ be a central simple algebra over $K$. Then we know that the dimension of $A$ as a vector space over $K$ is a square $n^2$. The integer $n$ is called the **degree** of $A$. If $A = \mathbb{M}_m(D)$ with $D$ a central division algebra, then the degree of $D$ is called the **index** of $A$. For a division algebra degree = index. The order of $A$ in the $Br(K)$ is called the **period** of $A$. We know that period divides index and the prime divisors of degree and index are same. In fact index = period $r$ for some $r \geq 1$. 

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In fact index = period$^r$ for some $r \geq 1$. 
Question. Does there exist a natural number $r$ depending only on the given field $K$ such that $\text{index}(A) \mid \text{period}(A)^r$ for all central simple algebras over $K$?
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This question obviously has a negative answer if we take $K = \mathbb{C}(t_1, \cdots, t_n, \cdots)$. 

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Of course this field is not an interesting field.
Let us see some examples where the question 1 has an affirmative answer.
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If $K$ is a finite field or an algebraically closed field, then there are no non-trivial central division algebras.

Suppose $K$ is a local field or a global field, the class field theory tells us that, index = period for every central simple algebra over $K$. 
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**Question.** Let $K$ be a field. Suppose there exists a natural number $r$ such that $\text{index} \mid \text{period}^r$ for every central simple algebra over $K$. Does there exists a natural number $t$ such that $\text{index} \mid \text{period}^t$ for every central simple algebra over $K(t)$.
Saltman’s result

**Theorem (Saltman (1997))**

Let $k$ be a $p$-adic field and $K$ be a function field of a curve over $k$. Let $A$ be a central simple algebra of period $n$. Suppose that $n$ is coprime to $p$. Then $\text{index}(A) | n^2$.
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There are examples (due to Rowen-Tignol-Sivatski) of central simple algebras over $k(t)$, $k$-$p$-adic field, with $\text{index} = \text{period}^2$. 
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Cyclic algebras
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Define $y\lambda = \sigma(\lambda)y$ for all $\lambda \in L$ and $y^n = b$. 
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$A$ is denoted by $(L, \sigma, b)$. 
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The algebra $A$ is generated by two elements $x, y$ with relations

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In this case the algebra $A$ is also denoted by $(a, b)_n$. 
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**Question.** Let $A$ be a central simple algebra of prime degree. Is $A$ a cyclic algebra?

This question is open even for degree 5 algebras.
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**Theorem (Saltman(2007))**

Let $k$ be a $p$-adic field and $K$ be a function field of a curve over $k$. Let $\ell$ be a prime not equal to $p$. Every central simple algebra over $K$ of degree $\ell$ is cyclic.
Let $K$ be a function field of a $p$-adic curve and $q$ a prime not equal to $p$. 

Let $A$ be a central simple algebra of period $q$. To show that the index of $A$ divides $q^2$, we need to produce an extension $L/K$ of degree $q^2$ such that $A \otimes L$ is a split algebra.

Similarly to show that a degree $q$ algebra is cyclic, we need to find a cyclic extension of degree $q$ which splits the algebra.

In general it is difficult to verify when an extension splits the algebra. The condition whether the algebra $A \otimes L$ is "unramified" can be translated into a cohomological criterion.

Saltman's idea is to split the ramification in good extensions and appeal to known theorems on unramified classes (for instance the vanishing of unramified classes) to obtain his results.

We explain these ideas now.
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We explain these ideas now.
$R$ - a discrete valuation ring

$K$ - field of fractions of $R$

$k$ - residue field of $R$

$\nu : K^* \to \mathbb{Z}$ - the discrete valuation given by $R$.

$n$ - a natural number which is a unit in $R$.

We have a homomorphism $\partial : nBr(K) \to H^1(k, \mathbb{Z}/n\mathbb{Z})$ called the residue homomorphism.
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We have a homomorphism $\partial : \text{nBr}(K) \to H^1(k, \mathbb{Z}/n\mathbb{Z})$ called the **residue homomorphism**.

We say that a central simple algebra $A$ over $K$ is **unramified** at $R$ if $\partial(A)$ is trivial.
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By fixing a primitive $n^{th}$ root of unity, we fix an isomorphism $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$ and identify $H^1(k, \mathbb{Z}/n\mathbb{Z})$ with $k^*/k^{*n}$.
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For $A = (a, b)_n$, we have $\partial(A) = \bar{u} \in k^*/k^{*n}$, where $u = (-1)^{\nu(a)\nu(b)} a^{\nu(b)} b^{\nu(a)}$. 
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\]

In particular if \( A = (\pi, u) \), where \( u \) is unit in \( R \) and \( \pi \) is parameter in \( R \), then \( \partial(A) = \overline{u} \in k^*/k^{*n} \).
\(\mathcal{X}\) - a regular integral scheme of dimension 2,
\(\mathcal{X}^1\) - the set of codimension one points of \(\mathcal{X}\).
For \(x \in \mathcal{X}^1\), the local ring at \(x\) is a discrete valuation ring.
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We say that a central simple algebra \( A \) over \( K \) is **unramified on** \( \mathcal{X} \)
if \( A \) is unramified at every point of \( \mathcal{X}^1 \).
From now onwards $\mathcal{X}$ denotes a non-singular surface (i.e. two dimensional separated excellent integral Noetherian scheme quasi-projective over some affine scheme.

Theorem (Saltman) Let $K$ be as above. Assume that $K$ contains a primitive $n$-th root of unity. Let $A$ be a central simple algebra of period $n$. Then there exist $f, g \in K^*$ such that $A$ is unramified at every discrete valuation of $K$ $(n\sqrt[f]{f}, n\sqrt[g]{g})$. 

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Corollary (Saltman (1997))

Let \( k \) be a \( p \)-adic field and \( K \) be a function field of a curve over \( k \). Let \( A \) be a central simple algebra of period \( n \). Suppose that \( n \) is coprime to \( p \). Then \( \text{index}(A) \mid n^2 \).
Corollary (Saltman (1997))

Let $k$ be a $p$-adic field and $K$ be a function field of a curve over $k$. Let $A$ be a central simple algebra of period $n$. Suppose that $n$ is coprime to $p$. Then $\text{index}(A) | n^2$.

**Proof.** Proof if by induction on $n$ the period of $A$. 

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If $n = 1$, there is nothing to prove. Assume that $n \geq 2$. 

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Then the period of $B$ is $\frac{n}{q}$. 
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By the induction the index of $B$ divides $\frac{n^2}{q^2}$. Hence there exists an extension $L/K$ of degree $\frac{n^2}{q^2}$ which splits $B$. In particular the period of $A \otimes L$ is $q$. Thus it is enough to show that the index of $A \otimes L$ divides $q^2$. 
Let $M/L$ be the extension given by the $q^{th}$ roots of unity.

By the above theorem, there exists $f, g \in K^\ast$ such that $A$ is unramified over $K(q\sqrt{f}, q\sqrt{g})$. By a theorem of Grothendieck, the unramified Brauer group of $K(q\sqrt{f}, q\sqrt{g})$ is zero. Hence $A \otimes K(q\sqrt{f}, q\sqrt{g})$ is trivial. In particular the index of $A$ divides $q^2$. 

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Since $M$ is also a function field of a curve over a $p$-adic field, it is enough to prove the corollary when $K$ contains all the $q^{th}$ roots of unity and the period of $A$ is $q$.
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Hence $A \otimes K(\sqrt[q]{f}, \sqrt[q]{g})$ is trivial. In particular the index of $A$ divides $q^2$. 

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Let $M/L$ be the extension given by the $q^{th}$ roots of unity.

Since the degree of $M/L$ is coprime to $q$, the index of $A \otimes L$ is same as the index of $A \otimes M$.

Since $M$ is also a function field of a curve over a $p$-adic field, it is enough to prove the corollary when $K$ contains all the $q^{th}$ roots of unity and the period of $A$ is $q$.

By the above theorem, there exists $f, g \in K^*$ such that $A$ is unramified over $K(\sqrt[q]{f}, \sqrt[q]{g})$.

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Hence $A \otimes K(\sqrt[q]{f}, \sqrt[q]{g})$ is trivial. In particular the index of $A$ divides $q^2$. 
Theorem (Saltman (2008))

Let $K$ be as above and $q$ a prime which is a unit on $X$. Assume that $K$ contains a primitive $q^{th}$ root of unity. Let $A$ be a central simple algebra of degree $q$. Then there exists $f \in K^*$ such that $A$ is unramified at every discrete valuation of $K(\sqrt[n]{f})$.

Corollary (Saltman (2007))

Let $K$ be a function field of a p-adic curve and $q$ a prime not equal to $p$. Then every central simple algebra over $K$ of degree $q$ is cyclic.

Proof.
Let $A/K$ be a central simple algebra of degree $q$. Then by the above theorem, there exists $f \in K^*$ such that $A \otimes K(\sqrt[q]{f})$ is unramified. Once again by the theorem of Grothendieck, $A \otimes K(\sqrt[q]{f})$ is trivial. Hence $A$ is cyclic.
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$K$-contains a primitive $n^{th}$ root of unity.

For $x \in \mathcal{X}^1$, we have the residue homomorphism

$$\partial_x : n\text{Br}(K) \rightarrow \kappa(x)^*/\kappa(x)^{*n}$$
Two irreducible curves $C_1$ and $C_2$ on $\mathcal{X}$ said to have **normal crossings** if they intersect at a point $P$, then the maximal ideal $m_P$ at $P$ is generated by $\pi$ and $\delta$, where $\pi$ and $\delta$ define $C_1$ and $C_2$ at $P$ respectively.

A divisor $D$ on $\mathcal{X}$ is said to have **normal crossings** if any two distinct irreducible curves in the support of $D$ have normal crossings.

Given a divisor $D$ on $\mathcal{X}$ there exists a blow up $\mathcal{X}'$ of $\mathcal{X}$ such that the strict transform of $D$ and the exceptional curves on $\mathcal{X}'$ have normal crossings.
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The union of the curves \( C \) where \( \alpha \) is ramified is called the **ramification locus** of \( \alpha \).
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The union of the curves $C$ where $\alpha$ is ramified is called the \textbf{ramification locus} of $\alpha$.

After blowing up $\mathcal{X}$, we assume that the ramification locus of $\alpha$ is a union of regular curves $C_i$ with normal crossings.
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**Lemma**

Let $A$ be a two-dimensional regular local ring with maximal ideal $m$ and field of fractions $K$. Let $n$ be an integer which is a unit in $A$. Assume that $K$ contains a primitive $n^{th}$ root of unity. Let $\alpha \in n\text{Br}(K)$.

1. Suppose that $m = (\pi, \delta)$ and $\alpha$ is ramified only at $\pi$ on $A$. Then $\alpha = \alpha' + (u, \pi)$ for some unit $u \in A$ and $\alpha' \in \text{Br}(A)$.

2. Suppose that $m = (\pi, \delta)$ and $\alpha$ is ramified only at $\pi$ and $\delta$ on $A$. Then either $\alpha = \alpha' + (u, \pi) + (v, \delta)$ or $\alpha = \alpha' + (u\delta^s, \pi)$ for some units $u, v \in A$, $s$ coprime to $n$ and $\alpha' \in \text{Br}(A)$.
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Theorem (Saltman)
Let $K$ be as above. Assume that $K$ contains a primitive $n$th root of unity. Let $\alpha \in n\text{Br}(K)$. Then there exist $f, g \in K^*$ such that $A$ is unramified at every discrete valuation of $K(\sqrt[n]{f}, \sqrt[n]{g})$.
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Proof. After blowing up, we assume that the ramification locus of $\alpha$ is union of two regular curves $C$ and $E$ with normal crossings. Choose $f \in K^*$ as follows:

$$\text{div}_\alpha(f) = C + E + F$$

for some divisor $F$ such that $F$ does not contain any component of either $C$ or $E$ and $F$ does not pass through the points of $C \cap E$. 
Let $P$ be closed point in $C \cap F$. 
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By the above lemma, we have $\alpha = \alpha' + (\pi, u)$ for some $\alpha'$ unramified at $P$, $u$ a unit at $P$ and $\pi$ defines $C$ at $P$. Let $u_P = u$. 

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Similarly if $P \in E \cap F$, we have $\alpha = \alpha' + (\delta, v)$. Let $u_P = v$. 
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Suppose that $P \in C \cap E$.

We have $f = \pi \delta w$ for some unit $w$ at $P$ and $\pi$ and $\delta$ define $C$ and $E$ at $P$ respectively.
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By the above lemma, we have $\alpha = \alpha' + (\pi, u)$ for some $\alpha'$ unramified at $P$, $u$ a unit at $P$ and $\pi$ defines $C$ at $P$. Let $u_P = u$.

Similarly if $P \in E \cap F$, we have $\alpha = \alpha' + (\delta, v)$. Let $u_P = v$.

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We have $\alpha = \alpha' + (u \delta^s, \pi)$ or $\alpha = \alpha' + (\pi, u) + (\delta, v)$.

In the first case, let $u_P = uw$ and in the second case let $u_P = uv$. 

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By the Chinese remainder theorem, we get a function $g \in K^*$ which is a unit at all the above closed points and components of $C, E, F$ such that $g(P)^{-1}u_P(P)$ is an $n^{th}$ power for all $P \in C \cap E, C \cap F, E \cap F$. Then $\alpha$ is unramified at every discrete valuation of $K(n\sqrt{f}, n\sqrt{g})$.

Corollary Let $X, K$ and $n$ as above. Suppose that for every closed point $P$ of $X$ the residue field $\kappa(P)$ is algebraically closed. Let $A$ be a central simple algebra over $K$ of period $n$. Then there exists $f \in K^*$ such that $A$ is unramified over $K(n\sqrt{f})$. 

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**Corollary**

Let $\mathcal{X}, K$ and $n$ as above. Suppose that for every closed point $P$ of $\mathcal{X}$ the residue field $\kappa(P)$ is algebraically closed. Let $A$ be a central simple algebra over $K$ of period $n$. Then there exists $f \in K^*$ such that $A$ is unramified over $K(\sqrt[n]{f})$. 
$\mathcal{X}$ - non-singular surface and $K$ its field of fractions and $\alpha \in qBr(K)$, $q$ a prime.
\( \mathcal{X} \) - non-singular surface and \( K \) its field of fractions and \( \alpha \in \mathbb{qBr}(K), \mathbb{q} \text{ a prime.} \)

Assume that the ramification locus of \( \alpha \) on \( \mathcal{X} \) is a union of regular curves with normal crossings.
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Let \( P \) be a nodal point and \( C_1 \) and \( C_2 \) are the two curves in the ramification locus of \( \alpha \).
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Let \( A = \mathcal{O}_{\mathcal{X}, P} \). Then \( A \) is a two dimensional regular local ring with maximal ideal \( m_P = (\pi, \delta) \) for some primes \( \pi, \delta \in A \) defining \( C_1 \) and \( C_2 \) at \( P \) respectively. Let \( \kappa(P) \) be the residue field at \( P \).
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By the above lemma, we have either $\alpha = \alpha' + (u, \pi) + (v, \delta)$ or $\alpha = \alpha' + (u\delta^s, \pi)$ for some units $u, v$ in $A$, $s$ coprime to $q$ and $\alpha' \in Br(A)$. 
$P$ is a **cold** point if $\alpha = \alpha' + (u\delta^m, \pi)$ for some unit $u$ at $P$ and $m$ coprime to $q$. 

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$P$ is a **cool** point if $u$ and $v$ are $q^{th}$ powers modulo the maximal ideal $m_P$.

$P$ is a **hot** point if images of $u$ and $v$ do not generate the same subgroups of $\kappa(P)^*/\kappa(P)^*q$.

$P$ is a **chilly** point if the images of $u$ and $v$ generate the same non-trivial subgroups $\kappa(P)^*/\kappa(P)^*q$. In the last case write $v = u^s_P$ up to an $n$th power. The integer $s_P$ is called the coefficient of $P$ with respect to $C_1$. 
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In the last case write $\bar{v} = \bar{u}^s$ up to an $n^{th}$ power. The integer $s_P$ is called the **coefficient** of $P$ with respect to $C_1$. 
Let $P$ be a cool point. Let $\mathcal{X}'$ be the blow up of $\mathcal{X}$ at $P$. Then it is easy to see that $\alpha$ is unramified at the exceptional curve. Thus replacing $\mathcal{X}$ by $\mathcal{X}'$, we can assume that there are no cool points.
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Consider the following graph:
vertices are the curves in the ramification locus of $\alpha$. The edges are the chilly points. Two vertices have an edge if both the curves intersect at a chilly point.
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After blowing up several times, we ensure that there are no chilly loops and no cool points.

Suppose $C_i$ are all the curves in the ramification locus. Then we can choose, for each $C_i$, a non-zero $s_i \in \mathbb{Z}/n\mathbb{Z}$ such that: Suppose $P$ is a chilly point on $C_i$ and $C_j$ with coefficient $s$ with respect to $C_i$. Then $s = s_j/s_i \in \mathbb{Z}/n\mathbb{Z}$. 
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Let $R_i$ be the discrete valuation ring at $C_i$ and $S_i$ be the integral closure of $R$ in $K(\sqrt[q]{f})$. 
Let $\kappa(C_i)$ be the residue field of $R_i$. Since $\nu_{C_i}(f) = s_i$ coprime to $q$, $S_i$ is a discrete valuation ring unramified over $R_i$, with residue field $\kappa(C_i)$. 

$\alpha$ is unramified at $S_i$. Hence the specialization of $\alpha$ gives an element $\beta_{C_i} \in \text{Br}(\kappa(C_i))$. $\beta_{C_i}$ is called the residual Brauer class of $\alpha$ with respect to $f$. Let $L_i/\kappa(C_i)$ be the residue of $\alpha$ at $C_i$. If $\alpha$ has index $q$, then the residual Brauer class of $\alpha$ is split by the extension $L_i$. Suppose that we find $f \in K^*$ such that $\alpha$ is unramified over $K(q^{1/2}f)$. Then it is easy to see that all the residual Brauer class $\beta_{C_i}$ is unramified on the curve $C_i$. 

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Suppose that we find $f \in K^*$ such that $\alpha$ is unramified over $K(\sqrt[q]{f})$. Then it is easy to see that all the residual Brauer class $\beta_{C_i}$ is unramified on the curve $C_i$. 
Using a local patching argument, one chooses $f \in K^*$ such that $\text{div}_\mathcal{X}(f) = \sum s_i C_i + F$ for some divisor $F$ on $\mathcal{X}$ which does not contain any of the curves $C_i$ and does not pass through the nodal points. Further all the residual Brauer classes with respect to $f$ are trivial (in particular unramified). This choice of $f$ kills most of the ramification of $\alpha$, except those valuations centered at the closed points which are in the intersection of $F$ and $C_i$. Finally one needs a further modification of $f$ such that $\alpha$ is unramified over $K^q \sqrt{f}$. We skip the details of this and go to the consequences of the results and the methods.
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Galois cohomology

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\[
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An element of the form $(a_1) \cdot (a_2) \cdot \ldots \cdot (a_n)$ is called a symbol.
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$a_1, \cdots, a_n \in k^*$. The cup product gives an element

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An element of the form $(a_1) \cdot (a_2) \cdots \cdot (a_n)$ is called a symbol.

The image of $(a) \cdot (b) \in H^2(k, \mu_\ell)$ in $\ell \text{Br}(k)$ is the cyclic algebra $(a, b)_\ell$. 
Let $K$ be a function field of a $p$-adic curve and $D$ a central division algebra over $K$. 

Theorem (Suresh (2007))

Let $K$ be a function field of a $p$-adic curve. Let $D$ be a central division algebra over $K$ of degree $q$. Suppose that $q$ is a prime not equal to $p$ and $K$ contains a primitive $q$th root of unity. Then $D$ is either a cyclic algebra or a tensor product of two cyclic algebras.
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Suppose the period of $D$ is 2 and $p \neq 2$. By the Saltman’s result, the degree of $A$ is at most 4. In particular, either a quaternion algebra or $A$ is isomorphic to a tensor product of two quaternion algebras (a theorem of Albert).
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Hence $D \otimes (f, g)^{-1} = (a, b)$ and $D \simeq (a, b) \otimes (f, g)$. 
Using the methods of *Saltman* on the study of ramification of algebras over surfaces, we have proved the following local-global principle.
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- $\mathcal{X}$ - a regular, integral surface
- $K$ - the function field of $\mathcal{X}$
- $\ell$ - a prime not equal to the characteristic of $K$
- Assume that $\ell$ is a unit in $\mathcal{O}_\mathcal{X}$
- Also assume that $K$ contains a primitive $\ell^{th}$ root of unity.
- $\mathcal{X}^1$ - the set of all codimension one points of $\mathcal{X}$
- For $x \in \mathcal{X}^1$, $K_x$ - completion of $K$ with respect to the discrete valuation $\nu_x$ given by $x$.
- $\kappa(x)$ - the residue field at $x$
Unramified cohomology

\[ H^\text{nr}_n(K/X, \mathbb{Z}/\ell\mathbb{Z}) = \{ \zeta \in H^n(K, \mathbb{Z}/\ell\mathbb{Z}) | \zeta \in \text{Image}(H^n(\mathcal{O}_X, x, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H^n(K, \mathbb{Z}/\ell\mathbb{Z})) \text{ for all } x \in \mathcal{X}^1 \} \]
Unramified cohomology

\[ H^n_{nr}(K/\mathcal{X}, \mathbb{Z}/\ell \mathbb{Z}) = \{ \zeta \in H^n(K, \mathbb{Z}/\ell \mathbb{Z}) \mid \zeta \in \text{Image}(H^n(\mathcal{O}_{\mathcal{X}, x}, \mathbb{Z}/\ell \mathbb{Z}) \to H^n(K, \mathbb{Z}/\ell \mathbb{Z})) \text{ for all } x \in \mathcal{X}^1 \} \]

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\[ H_{nr}^n(K/\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z}) = \{ \zeta \in H^n(K, \mathbb{Z}/\ell\mathbb{Z}) \mid \zeta \in \text{Image}(H^n(\mathcal{O}_{\mathcal{X}, x}, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H^n(K, \mathbb{Z}/\ell\mathbb{Z})) \text{ for all } x \in \mathcal{X}^1 \} \]

\[ H_{nr}^n(K/\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z}) \] is called the **unramified cohomology**.

For a field \( F \), let

\[ H_{nr}^n(F, \mathbb{Z}/\ell\mathbb{Z}) = \{ \zeta \in H^n(F, \mathbb{Z}/\ell\mathbb{Z}) \mid \zeta \in \text{Image}(H^n(\mathcal{O}_v, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H^n(F, \mathbb{Z}/\ell\mathbb{Z})) \text{ for all discrete valuations } v \text{ of } F \} \]
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**Theorem (Parimala-Suresh (2010))**

Let $\mathcal{X}$ and $K$ be as above. Suppose that for every irreducible closed curve $C$ on $\mathcal{X}$, $\kappa(C)$ is a global field or a local field. Let $\zeta \in H^3(K, \mathbb{Z}/\ell\mathbb{Z})$ and $\alpha \in H^2(K, \mathbb{Z}/\ell\mathbb{Z})$ is a symbol. If for every $x \in \mathcal{X}^1$, there exists $f_x \in K_x^*$ such that $\zeta - \alpha \cdot (f_x) \in H^3_{nr}(K_x, \mathbb{Z}/\ell\mathbb{Z})$, then there exists $f \in K^*$ such that $\zeta - \alpha \cdot (f) \in H^3_{nr}(K/\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z})$. 
Using the above local-global principle and the results of Saltman mentioned above, we have proved the following:
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Let $K$ be the function field of a $p$-adic curve and $q$ a prime not equal to $p$. Suppose that $K$ contains a primitive $n^{th}$ root of unity. Then every element in $H^3(K, \mathbb{Z}/q\mathbb{Z})$ is a symbol.
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Theorem (Parimala-Suresh(2007))
Let $K$ be the function field of a $p$-adic curve. If $p \neq 2$, then every quadratic form over $K$ in at least 9 variables is isotropic.