Transcendental obstructions to rational points on general K3 surfaces

Anthony Várilly-Alvarado
Rice University
(joint work with Brendan Hassett and Patrick Varilly)

Ramification in Algebra and Geometry at Emory,
Emory University, May 2011
Fix a number field \( k \), and let \( \Omega_k \) be the set of places of \( k \). Let \( S \) be a class of nice (smooth, projective, geometrically integral) \( k \)-varieties. For \( X \in S \), we have the embedding

\[
\phi : X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v) = X(A)
\]

**Definition**

\( S \) satisfies the Hasse principle if for all \( X \in S \),

\[
X(A) \neq \emptyset \implies X(k) \neq \emptyset.
\]

**Definition**

\( S \) satisfies weak approximation if, for all \( X \in S \), the image of \( \phi \) is dense for the product of the \( v \)-adic topologies.
Manin (1970): the Brauer group $\text{Br} X$ of $X$ can be used to construct an intermediate “obstruction set”:

$$X(k) \subseteq X(A)^{\text{Br}} \subseteq X(A).$$

In fact, $X(A)^{\text{Br}}$ already contains the closure of $X(k)$ for the adelic topology:

$$\overline{X(k)} \subseteq X(A)^{\text{Br}} \subseteq X(A).$$

This set may be used to explain the failure of the Hasse principle and weak approximation on many kinds of varieties.
Three kinds of Brauer elements

- **Constant elements** $\text{Br}_0 X := \text{im}(\text{Br} k \to \text{Br} X)$
  No obstructions: $X(\mathcal{A})^A = X(\mathcal{A})$ for all $\mathcal{A} \in \text{Br}_0 X$

- **Algebraic elements** $\text{Br}_1 X := \ker(\text{Br} X \to \text{Br} X_{\bar{k}})$
  If $X(\mathcal{A}) \neq \emptyset$ then there is an isomorphism
  \[
  \frac{\text{Br}_1 X}{\text{Br}_0 X} \xrightarrow{\sim} H^1(\text{Gal}(\bar{k}/k), \text{Pic} X_{\bar{k}})
  \]
  (Hochschild-Serre spectral sequence)
  If $\text{Pic} X_{\bar{k}} = \mathbb{Z}$ then $\text{Br}_1 X$ gives no obstructions

- **Transcendental elements** $\text{Br} X \setminus \text{Br}_1 X$.
  “geometric: they survive base-change to an algebraic closure”
Where might we find transcendental classes?

For curves and surfaces of negative Kodaira dimension we have

$$\text{Br} \ X = \text{Br}_1 \ X,$$

i.e., these varieties have no transcendental Brauer classes.

Thus, if one is interested in transcendental classes, it is reasonable to start by looking at surfaces of Kodaira dimension 0. Within this class, we will consider K3 surfaces.

**Definition**

A K3 surface is a nice surface with

$$\omega_X \cong \mathcal{O}_X \quad \text{and} \quad h^1(X, \mathcal{O}_X) = 0.$$
Examples of K3 surfaces

- Double covers of $\mathbb{P}^2$ ramified along a smooth sextic plane curve:

  \[ \{w^2 - f(x, y, z) = 0\} \subseteq \mathbb{P}(1, 1, 1, 3) = \text{Proj } k[x, y, z, w], \]

  where $f(x, y, z) \in k[x, y, z]_6$.

- Smooth quartic surfaces in $\mathbb{P}^3$.

- Smooth complete intersections of 3 quadrics in $\mathbb{P}^5$. 
Transcendental elements: basic questions

Theorem (Harari, 1996)

There exist infinitely many explicit conic bundles $V$ over $\mathbb{P}^2$ with a transcendental Brauer-Manin obstruction to the Hasse principle.

Question

Are there nice algebraic surfaces that fail to satisfy the Hasse principle on account of a transcendental Brauer-Manin obstruction? Can we write down an example?

This body of work includes examples of transcendental Brauer-Manin obstructions to weak approximation on K3 surfaces. In all cases, the K3 surfaces considered are endowed with an elliptic fibration, which is used in an essential way to construct transcendental Brauer classes.
Question

Can we construct an explicit K3 surface $X$ with $\text{Pic} X_{\bar{k}} \cong \mathbb{Z}$ with a transcendental obstruction to weak approximation?

For such a surface

- $\text{Br}_1 X / \text{Br}_0 X \xrightarrow{\sim} H^1 (\text{Gal}(\bar{k}/k), \text{Pic} X_{\bar{k}}) = 0$, i.e., there are no algebraic Brauer-Manin obstructions to weak approximation.
- there are no elliptic fibrations: we have to bring some fresh geometric insight to the table.
Weak Approximation

**Theorem (Hassett, Varilly, V-A; 2010)**

Let $X$ be the K3 surface of degree 2 given by

$$w^2 = \det \begin{pmatrix}
2(2x + 3y + z) & 3x + 3y & 3x + 4y & 3y^2 + 2z^2 \\
3x + 3y & 2(2x + 3y + z) & 3z & 4y^2 \\
3x + 4y & 3z & 2(x + 3z) & 4x^2 + 5xy + 5y^2 \\
3y^2 + 2z^2 & 4y^2 & 4x^2 + 5xy + 5y^2 & 2(2x^3 + 3x^2z + 3xz^2 + 3z^3)
\end{pmatrix}$$

in $\mathbb{P}_\mathbb{Q}(1, 1, 1, 3)$. Then $\text{Pic} \, X_{\overline{\mathbb{Q}}} \cong \mathbb{Z}$, and there is a transcendental Brauer-Manin obstruction to weak approximation on $X$. The obstruction arises from a quaternion Azumaya algebra $A \in \ker (\text{Br} \, X \hookrightarrow \text{Br} \, \kappa(X))$. Explicitly, if $M_i$ denotes the $i$-th leading principal minor of the above matrix, then

$$A = \left( -\frac{M_2}{M_1^2}, -\frac{M_3}{M_1 M_2} \right).$$

Note: $X$ has rational points! e.g., $[15 : 15 : 16 : 13752]$
Consider the following polynomials in $\mathbb{Q}[x, y, z]$:

- $A := 62x^2 - 16xz + 16y^2 + 16z^2$
- $B := -5x^2 - 2y^2 - 2z^2$
- $C := -4x^2 - 6xz - 4y^2 - 2yz - 3z^2$
- $D := 48x^2 - 16xz + 30y^2 + 48yz + 32z^2$
- $E := -2x^2 - y^2 - 4yz - 6z^2$
- $F := 16x^2 + 48xz + 32y^2 + 62z^2$
Hasse principle

Theorem (Hassett, V-A; 2011)

Let $X$ be the K3 surface of degree 2 given by

$$w^2 = \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}$$

in $\mathbb{P}_Q(1, 1, 1, 3)$. Then $\text{Pic} X_{\overline{Q}} \cong \mathbb{Z}$, and there is a transcendental Brauer-Manin obstruction to the Hasse principle on $X$. The obstruction arises from a quaternion Azumaya algebra $\mathcal{A} \in \text{im}(\text{Br} X \hookrightarrow \text{Br} \kappa(X))$. Explicitly, if $M_i$ denotes the $i$-th leading principal minor of the above matrix, then

$$\mathcal{A} = \left(-\frac{M_2}{M_1^2}, -\frac{M_3}{M_1 M_2}\right).$$
Hodge theoretic motivation

How did we know where to look for these examples?
Let $X$ be a complex projective K3 surface. Let

$$T_X := \text{NS}(X) \perp \subseteq \text{H}^2(X, \mathbb{Z})$$

be the \textit{transcendental lattice} of $X$. Write

$$\Lambda_{K3} = U^3 \oplus E_8(-1)^2$$

for the abstract K3 lattice.
The exponential sequence shows there is a one-to-one correspondence

$$\left\{ \alpha \in \text{Br} \ X \text{ of exact order } n \right\} \leftrightarrow \left\{ \text{surjections } T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \right\}$$

Hence, to $\alpha$ as above, we may associate $T_\alpha \subseteq T_X$:

$$T_\alpha = \ker(\alpha: T_X \to \mathbb{Z}/n\mathbb{Z}).$$
Theorem (van Geemen; 2005)

Let $X$ be a complex projective $K3$ surface of degree 2 with $\text{Pic } X \cong \mathbb{Z}$, and let $\alpha \in (\text{Br } X)[2]$. Then one of the following three things must happen:

1. There is a unique primitive embedding $T_\alpha \hookrightarrow \Lambda_{\Lambda 3}$. This gives a degree 8 $K3$ surface $Y$ associated to the pair $(X, \alpha)$.

2. $T_\alpha(-1) \cong \langle h^2, P \rangle^\perp \subseteq H^4(Z, \mathbb{Z})$, where $Z$ is a cubic fourfold with a plane $P$ ($h$ is the hyperplane class).

3. $T_\alpha(-1) \cong \langle h_1^2, h_1 h_2, h_2^2 \rangle^\perp \subseteq H^4(W, \mathbb{Z})$, where $W$ is a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a type $(2, 2)$ divisor ($h_1, h_2$ are the pullbacks to $W$ of the hyperplane classes of $\mathbb{P}^2$ under the two projections $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$).

Idea: “go backwards” and work over any field $k$. 
The pair \((X, \mathcal{A})\) in the \(K3\) counter-example to weak approximation is naturally associated to a cubic fourfold \(Y \subseteq \mathbb{P}_\mathbb{Q}^5 := \text{Proj} \mathbb{Q}[X_1, X_2, X_3, Y_1, Y_2, Y_3]\) given by

\[
2X_1^2 Y_1 + 3X_1^2 Y_2 + X_1^2 Y_3 + 3X_1 X_2 Y_1 + 3X_1 X_2 Y_2 + 3X_1 X_3 Y_1 \\
\quad + 4X_1 X_3 Y_2 + 3X_1 Y_2^2 + 2X_1 Y_3^2 + X_2^2 Y_3 + 3X_2 X_3 Y_3 \\
\quad + 4X_2 Y_2^2 + X_3^2 Y_1 + 3X_3^2 Y_3 + 4X_3 Y_1^2 + 5X_3 Y_1 Y_2 \\
\quad + 5X_3 Y_2^2 + 2Y_1^3 + 3Y_1^2 Y_3 + 3Y_1 Y_3^2 + 3Y_3^3 = 0.
\]

This fourfold contains the plane \(\{Y_1 = Y_2 = Y_3 = 0\}\).
Theorem

Let $Y$ be a cubic fourfold smooth over a field $k$. Suppose that $Y$ contains a plane $P$, and let $\tilde{Y}$ denote the blow up of $Y$ along $P$, $q : \tilde{Y} \rightarrow \mathbb{P}^2$ the corresponding quadric surface bundle, and $r : \mathcal{W} \rightarrow \mathbb{P}^2$ its relative variety of lines. Assume that there exists no plane $P' \subset Y_{\overline{k}}$ such that $P'$ meets $P$ along a line. Then the Stein factorization

$$r : \mathcal{W} \xrightarrow{\pi_1} X \xrightarrow{\phi} \mathbb{P}^2$$

consists of a smooth $\mathbb{P}^1$-bundle followed by a degree-two cover of $\mathbb{P}^2$, which is a K3 surface.

Note: The case $k = \mathbb{C}$ goes back at least to Voisin (1985).
The pair \((X, \mathcal{A})\) in the \(K3\) counter-example to the Hasse principle is naturally associated to a double cover of \(\mathbb{P}^2 \times \mathbb{P}^2 = \text{Proj} \mathbb{Q}[x_0, x_1, x_2] \times \text{Proj} \mathbb{Q}[y_0, y_1, y_2]\) ramified along the divisor of type \((2, 2)\) given by

\[
62x_0^2y_0^2 + 16x_1^2y_0^2 - 16x_0x_2y_0^2 + 16x_2^2y_0^2 - 5x_0^2y_0y_1 - 2x_1^2y_0y_1 - 2x_2^2y_0y_1 + 48x_0^2y_1^2 + 30x_1^2y_1^2 - 16x_0x_2y_1^2 + 48x_1x_2y_1^2 + 32x_2^2y_1^2 - 4x_0^2y_0y_2 - 4x_1^2y_0y_2 - 6x_0x_2y_0y_2 - 2x_1x_2y_0y_2 - 3x_2^2y_0y_2 - 2x_0^2y_1y_2 - x_1^2y_1y_2 - 4x_1x_2y_1y_2 - 6x_2^2y_1y_2 + 16x_0^2y_2^2 + 32x_1^2y_2^2 + 48x_0x_2y_2^2 + 62x_2^2y_2^2 = 0.
\]
Let \( W \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \) be a smooth type \((2, 2)\) divisor. The projections \( \pi_i: W \to \mathbb{P}^2 \) define conic bundle structures ramified along smooth sextic curves \( C_1, C_2 \) (respectively).

Let \( \phi_i: X_i \to \mathbb{P}^2 \) be a double cover ramified along \( C_i \). Then \( W \times_{\mathbb{P}^2} X_i \to X_i \) is a smooth \( \mathbb{P}^1 \)-bundle for the étale topology. This is our element \( A \in (\text{Br} X)[2] \).
The constructions above don’t necessarily yield K3 surfaces for which \( \text{Pic } X_{\overline{k}} \cong \mathbb{Z} \).

We use ideas of van Luijk, Elsenhans and Jahnel to construct surfaces for which we can prove this is the case. This requires some intensive point counts over finite fields.

In the Hasse principle counter-example, we need the primes of bad reduction of \( X \). A Groebner basis computation over \( \mathbb{Z} \) shows these primes divide

\[
15759613890094825604372865835346781138746844525753305159054222247848 \\
6447126409097867530208903148583982831806707465087640413725725692102 \\
066052438935787890195630829862880978409927736101780460331885931216 \\
3834734316641126951561545616484413034372280131483926551238024753488 \\
7685949966462337614350040948449859415529952750002506898422776271778 \\
4930382467087815253432944143520
\]

(366 digits!)