CONTINUED FRACTIONS

1. Approximations

Suppose we want to approximate a fraction like \( \frac{6995}{1203} \) by something a little simpler. How might we do this? Let’s start with something a bit simpler like \( \frac{51}{19} \). One way is by drawing a rectangle like shown and literally divide it into squares

\[
\begin{array}{ccc}
19 & 19 & 13 \\
& & 6 \\
\end{array}
\]

The first approximation is \( \frac{51}{19} \approx 2 \). The remainder is \( \frac{13}{19} \). The \( 13 \times 19 \) rectangle is also a \( 19 \times 13 \) rectangle, and so we can approximate it in the same way, so our second approximation is \( 2 + \frac{1}{1+\frac{1}{19}} \approx 2 + \frac{1}{1} = 3 \). The third approximation is \( 2 + \frac{1}{1+\frac{1}{2+\frac{1}{19}}} = \frac{8}{3} \), and the third is \( 2 + \frac{1}{1+\frac{1}{2+\frac{1}{19}}} = \frac{51}{19} \). These representations are called continued fractions. We usually represent a continued fraction by the list of remainders, so in this example we write \( \frac{51}{19} \) as \([2; 1, 2, 6]\). Notice the integer part is set off by a semicolon. The intermediate approximations are called the \( n \)th convergents, are usually good approximations, with simpler denominators.

- is the continued fraction representation of a number unique?
- What is the continued fraction expansion of \( \frac{6995}{1203} \)? What is the best fraction approximation with denominator having no more than 2 digits?
- What is the connection between the continued fraction representation of a fraction \( \frac{a}{b} \) and \( \frac{b}{a} \)?
- What is the continued fraction representation of \( \frac{25}{16} \), \( \frac{49}{36} \), and \( \frac{81}{64} \)? what about \( \frac{5}{3} \)?
- What happens if you compute the continued fraction of an un-reduced fraction like \( \frac{52}{18} \) or \( \frac{102}{42} \)?
2. **Infinite continued fraction.**

We can also approximate irrational numbers by continued fractions, but these fractions cannot terminate. For instance, we can write $\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$, which we write as $[1; \overline{2}]$. Here the overline means that this part repeats. The convergents are defined the same as before: 1, [1; 2], [1; 2, 2],[1; 2, 2, 2], etc.

- What are the first few convergents of $\sqrt{2}$? how closely do these approximate it?
- What number is represented by $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}$?

- What is $[1; \overline{2}, 1]$?
- What is the Continued Fraction representation of $e$?
- What kinds of numbers have a repeating representation?

3. **Pell’s Equation**

Pell’s equation is an equation of the form $x^2 - Dy^2 = \pm 1$ for some $D$. For instance we might take $D = 2$. This is a hyperbola, and has infinitely many solutions, but the problem we care about is to find integer solutions to the equation. It turns out that what we need is for $\frac{x}{y}$ to be very close to $\sqrt{D}$, so a good guess might be to look at the convergents of the continued fraction expansion of $\sqrt{D}$. In fact, all the integer solution are convergents of $\sqrt{D}$, but not all the convergents necessarily give solutions.

- Find some integer solutions to the equation $x^2 - 2y^2 = \pm 1$.
- Find some integer solutions to the equation $x^2 - 14y^2 = \pm 1$.
- What convergents give solutions?