MATH CIRCLE: CATALAN NUMBERS

Counting:

1. **Balanced parentheses.** How many ways can you write \( n \) balanced pairs of parentheses ()? Try for \( n = 1, 2, 3, 4, 5 \). When you get to \( n = 4 \), write your answers on the board.

2. **ASCII-art mountain ranges.** How many mountain ranges can you draw using \( n \) upstrokes / and \( n \) downstrokes \( \backslash \)? Try for \( n = 1, 2, 3, 4, 5 \). When you get to \( n = 4 \), draw your mountains on the board.

3. **Diagonal-avoiding paths.** Imagine you’re in a part of Manhattan where the streets are on an \( n \times n \) grid, and they’re all one-way to the east and the south. Also, everything below the diagonal is in a really bad neighborhood. How many ways are there to get from the northwest corner to the southeast? Try for \( n = 1, 2, 3, 4, 5 \). When you get to \( n = 4 \), draw your paths on the board.

4. **Polygon triangulations.** How many ways are there of dividing a regular \( (n + 2) \)-gon into triangles by drawing lines between vertices? Try for \( n = 1, 2, 3, 4, 5 \). When you get to \( n = 4 \), draw your polygons on the board.

5. **Handshakes across a table.** King Arthur has \( 2n \) knights sitting at his round table. How many different ways are there for \( n \) pairs of knights to shake hands across the table if different pairs of knights don’t want to cross their arms? Try for \( n = 1, 2, 3, 4, 5 \). When you get to \( n = 4 \), draw your diagrams on the board.

Find a way to convince your group members and the rest of the class that the number you get from your favorite counting problem above is always the same as the number you get from another one of the counting problems. Is there some correspondence between the things you’re counting?

Let \( D_n \) be the number of things in your counting problem of size \( n \). Try using a “divide and conquer” argument to get a recursive formula for the number \( D_n \) in terms of \( D_0, D_1, \ldots, D_{n-1} \). Try figuring out how the argument works for \( n = 4, 5, 6 \) first.

Use this to calculate \( D_6, D_7, D_8 \).

Do you get the same answer as for binary trees?

**Catalan’s Triangle.** Let’s focus on problem (3), counting diagonal avoiding paths. On an \( n \times n \) grid, count how many diagonal-avoiding paths there are that reach each intersection above the diagonal. Try for \( n = 2, 3, 4, 5 \). Do you notice any patterns? Is there an easier way to come up with the answer?

**An aside on binomial coefficients.** The binomial coefficient \( \binom{m}{n} \) is defined to be the number of ways of choosing \( n \) things among a collection of \( m \). For example, if you have 10 different shirts and you want to pack 5 of them in your suitcase, there are \( \binom{10}{5} \) different ways of doing so.

It turns out that

\[
\binom{m}{n} = \frac{m(m-1)(m-2) \cdots (m-n+1)}{n!} = \frac{m!}{n!(m-n)!}
\]

where \( n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \).

1. Calculate \( \binom{6}{2}, \binom{5}{3}, \binom{7}{3} \).
(2) In the new state lottery, you choose three numbers from 1 to 8. The lottery consists of randomly drawing three balls numbered 1 through 8. You win if the numbers you chose match the balls that were drawn. What are your chances of winning?

\[
\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3.
\]

(3) \[\sum_{i=0}^{n} \binom{n}{i} = ?\]

(4) Evaluate \[\binom{n}{1} + 3 \binom{n}{3} + 5 \binom{n}{5} + \cdots\] in closed form.

Paths through a \(k \times m\) grid. Now we’re in a different section of Manhattan. The streets are laid out on a \(k \times m\) grid, and they’re still all one-way streets to the east and the south, but it’s all a good neighborhood. How many paths are there from the northwest corner to the southeast? Try for 2 \(\times\) 3, 3 \(\times\) 3, 3 \(\times\) 4, 4 \(\times\) 4 grids first, at least until you see the pattern.

Aside on generating functions. Given a sequence of numbers \(a_0, a_1, a_2, \ldots\), form the generating function

\[f(z) = a_0 + a_1 z + a_2 z^2 + \cdots.\]

We won’t worry so much about infinite sums, and whether or not this is a bona fide function. For example, if \(a_i = \binom{n}{i}\) then

\[f(z) = \binom{n}{0} + \binom{n}{1} z + \binom{n}{2} z^2 + \cdots + \binom{n}{n-1} z^{n-1} + \binom{n}{n} z^n = (1 + z)^n.\]

Note that the binomial formula

\[\binom{m}{n} = \frac{m(m-1)(m-2) \cdots (m-n+1)}{n!}\]

works when \(m\) is a fraction as well. For example,

\[\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3} \left( \frac{1}{3} - 1 \right)}{2!} = \frac{\frac{1}{3} (-\frac{2}{3})}{2} = -\frac{1}{9}.\]

(1) Calculate \(\binom{\frac{1}{2}}{2}, \binom{\frac{1}{2}}{3}, \binom{\frac{1}{2}}{4}\).

(2) Find a formula for \(\binom{n}{1/n}\).

For more on Catalan numbers, see Tom Davis’ explanation at

http://www.geometer.org/mathcircles/catalan.pdf