Folding Perfect Thirds

It is easy to fold the side of a square into halves, or fourths, or eighths, etc. But folding odd divisions, like thirds, exactly is more difficult. The below procedure is one was to fold thirds.

(1)  

(2)  

(3) \(\frac{1}{3}\)

**Question 1:** Prove that this method actually works.

**Question 2:** How could you generalize this method, say, to make perfect 5ths or nths (for n odd)?
Exploring a Basic Origami Move

Origami books display many different folding moves that can be made with paper. One common move, especially in geometric folding, is the following:

Given two points \( p_1 \) and \( p_2 \) and a line \( L \), fold \( p_1 \) onto \( L \) so that the resulting crease line passes through \( p_2 \).

Let’s explore this basic origami operation by seeing exactly what is happening when we fold a point to a line.

**Activity:** Take a sheet of regular writing paper, and let one side of it be the line \( L \). Choose a point \( p \) somewhere on the paper, perhaps like below. Your task is to fold \( p \) onto \( L \) over and over again.

It is easier, actually, to fold \( L \) to \( p \), by bending the paper until \( L \) touches \( p \) and then flattening the crease. Do this many times—as many as you can stand!—choosing different points \( p' \) where \( p \) lands on \( L \).

**Question 1:** Describe, as clearly as you can, exactly what you see happening. What are the crease lines forming? How does your choice of the point \( p \) and the line \( L \) fit into this? Prove it.
What’s This Doing?

Take a square piece of paper and fold a line from the lower-left corner going up at some angle, $\theta$. Then fold the paper in half from top to bottom and unfold. Then fold the bottom 1/4 crease line. That should give you something like the left figure below.

Then do the operation in the middle figure: Make a fold that places point $p_1$ onto line $L_1$ and at the same time places point $p_2$ onto line $L_2$. You will have to curl the paper over, line up the points, and then flatten.

Lastly, with the flap folded, extend the $L_1$ crease line shown in the right-most figure. Call this crease line $L_3$.

**Question 1:** Unfold everything. Prove that we if we extend $L_3$ then it will hit the lower-left corner, $p_1$.

**Question 2:** What is crease line $L_3$ in relation to the other lines in the paper? Can you prove it, or is this just a coincidence?
A More Complicated Fold

The origami angle trisection method is able to do what it does by using a rather complex origami move:

Given two points $p_1$ and $p_2$ and two lines $L_1$ and $L_2$, we can make a crease that simultaneously places $p_1$ onto $L_1$ and $p_2$ onto $L_2$.

**Question 1:** Will this operation always be possible to do, no matter what the choice of the points and lines are?

**Question 2:** Remember that when we fold a point $p$ to a line $L$ over and over again, we can interpret the creases as being tangent to a parabola with focus $p$ and directrix $L$. What does this tell us about this more complex folding operation? How can we interpret it geometrically? Draw a picture of this.
**Activity:** Let’s explore what this operation is doing in a different way. Take a sheet of paper and mark a point $p_1$ (somewhere near the center is usually best) and let the bottom edge be the line $L_1$.

Pick a second point $p_2$ to be anywhere else on the paper. Our objective is to see where $p_2$ goes as we fold $p_1$ onto $L_1$ over and over again.

So pick a spot on $L_1$ (call it $p'_1$) and fold it up to $p_1$. Using a marker or pen, draw a point where the folded part of the paper touches $p_2$. (If no other parts of the paper touch $p_2$, try a different choice of $p'_1$.) Then unfold. You should see a dot (which we could call $p'_2$) that represents where $p_2$ went as we make the fold.

Now choose a different $p'_1$ and do this over and over again. Make enough $p'_2$ points so that you can connect the dots and see what kind of curve you get.

**Question 3:** What does this curve look like? Look at other people’s work in the class. Do their curves look like yours? Do you know what kind of equation would generate such a curve?