GENERATING FUNCTIONS

1. A Factorization Problem: An Introduction to Infinite Sums

In mathematics, we often encounter several very interesting patterns. Often, these patterns can be encoded into a function that is easy to study. These functions allow us to express these patterns very compactly. We call these functions generating functions. We start with a factorization problem. Let \( N \) be a positive integer. We want to factor the expression \( x^{N+1} - 1 \).

This polynomial has a root at \( x = 1 \), since

\[
1^{N+1} - 1 = 1 - 1 = 0.
\]

So \( x^{N+1} - 1 \) can be written as \( (x - 1)f(x) \), where \( f(x) \) is some other polynomial.

What should \( f(x) \) be? We start with the identity

\[
x^2 - 1 = (x - 1)(x + 1).
\]

A similar identity exists for cubes:

\[
x^3 - 1 = (x - 1)(x^2 + x + 1).
\]

Exercise 1.1. Show that

\[
(x - 1)(x^3 + x^2 + x + 1) = x^4 - 1.
\]

Exercise 1.2. Show that if \( R \geq 1 \) is an integer, then

\[
(x - 1)(1 + x + x^2 + \cdots + x^{R-1} + x^R) = x^{R+1} - 1.
\]

Dividing both sides by \( x - 1 \) (when \( x \neq 1 \)), we obtain the identity

\[
\frac{x^{R+1} - 1}{x - 1} = 1 + x + x^2 + x^3 + \cdots + x^{R-1} + x^R.
\]

What happens when \( R \) gets very big? If \( |x| > 1 \) is big, then \( x^{R+1} \) become incredibly big. But if \( |x| < 1 \) and \( x \neq 1 \), then as \( R \) grows, it turns out that \( x^{R+1} \) gets smaller, and in fact goes to zero. We conclude the following result.

Theorem 1.3. Suppose that \( x \) is a real number such that \( |x| < 1 \). We have

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots
\]

Exercise 1.4. We have not stated what happens to the identity

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots
\]

when \( x = -1 \). What happens when \( x = -1 \)? What happens when \( x = 1 \)?
The identity
\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \]
is an example of an \textbf{infinite sum} since the right-hand side has infinitely many terms. Infinite sums occur everywhere in mathematics, and this particular infinite sum is of very high importance. We now explore some interesting consequences of the identity.

2. Generating Functions

We will use the identity
\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \]
to give some examples of \textbf{generating functions}. As an example, we start with
\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \]

We now square both sides which involve a LOT of FOILing:
\[
\frac{1}{(1-x)^2} = (1 + x + x^2 + \cdots + x^n + \cdots)^2 \\
= (1 + x + x^2 + \cdots + x^n + \cdots)(1 + x + x^2 + \cdots + x^n + \cdots) \\
= 1(1 + x + x^2 + \cdots + x^n + \cdots) + x(1 + x + x^2 + \cdots + x^n + \cdots) \\
+ x^2(1 + x + x^2 + \cdots + x^n + \cdots) + \cdots + x^n(1 + x + x^2 + \cdots + x^n + \cdots) + \cdots
\]
but in the end, when we collect all the constant terms, then all the terms with \(x\), and then all the terms with \(x^2\), etc., we see that
\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots + (n+1)x^n + \cdots
\]

Multiplying both sides by \(x\), we see that
\[
\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots + nx^n + \cdots
\]

If you notice, the \(n\)-th term in this sum is \(x^n\) times the \(n\)-th nonnegative integer. Therefore,
\[
\frac{x}{(1-x)^2}
\]
is the generating function for the integers.

We can use this construct a generating function for all of the odd integers. Starting with the identity
\[
\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots + nx^n + \cdots,
\]
we multiply both sides by \(x\) to obtain
\[
\frac{x^2}{(1-x)^2} = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + \cdots + nx^{n+1} + \cdots,
\]
Now, we add the two generating functions together to obtain

\[
\frac{x^2 + x}{(1 - x)^2} = \frac{x}{(1 - x)^2} + \frac{x^2}{(1 - x)^2} \\
= x + (2 + 1)x^2 + (3 + 2)x^3 + (4 + 3)x^4 + \cdots \\
= x + 3x^2 + 5x^3 + 7x^4 + \cdots
\]

Therefore,

\[
\frac{x^2 + x}{(1 - x)^2}
\]

is the generating function for the odd numbers.

**Exercise 2.1.** Use the identity

\[
\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots + (n + 1)x^n + \cdots
\]

to come up with a generating function for the even numbers. That is, what function equals the infinite sum

\[
2 + 4x + 6x^2 + 8x^3 + 10x^4 + \cdots + 2nx^n + \cdots?
\]