Algebra Colloquium

Counting points, counting fields, and heights on stacks

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Abstract: The basic objects of algebraic number theory are number fields, and the basic invariant of a number field is its discriminant, which in some sense measures its arithmetic complexity. A basic finiteness result is that there are only finitely many degree-d number fields of discriminant at most X; more generally, for any fixed global field K, there are only finitely many degree-d extensions L/K whose discriminant has norm at most X. (The classical case is where $K = \mathbb{Q}$.)

When a set is finite, we greedily ask if we can compute its cardinality. Write $N_d(K, X)$ for the number of degree-d extensions of K with discriminant at most d. A folklore conjecture holds that $N_d(K, X)$ is on order $c_d X$. In the case $K = \mathbb{Q}$, this is easy for $d = 2$, a theorem of Davenport and Heilbronn for $d = 3$, a much harder theorem of Bhargava for $d = 4$ and 5, and completely out of reach for $d > 5$. More generally, one can ask about extensions with a specified Galois group G; in this case, a conjecture of Malle holds that the asymptotic growth is on order $X^a(\log X)^b$ for specified constants a, b.

I'll talk about two recent results on this old problem:

1) (joint with TriThang Tran and Craig Westerland) We prove that $N_d(\mathbb{F}_q(t), X)) < c_\epsilon X^{1+\epsilon}$ for all d, and similarly prove Malles conjecture "up to epsilon" this is much more than is known in the number field case, and relies on a new upper bound for the cohomology of Hurwitz spaces coming from quantum shuffle algebras: https://arxiv.org/abs/1701.04541

2) (joint with Matt Satriano and David Zureick-Brown) The form of Malle's conjecture is very reminiscent of the Batyrev-Manin conjecture, which says that the number of rational points of height at most X on a Batyrev-Manin variety also grows like $X^a(\log X)^b$ for specified constants a, b. Whats more, an extension of $\mathbb{Q}$ with Galois group G is a rational point on a Deligne–Mumford stack called BG, the classifying stack of G. A natural reaction is to say the two conjectures is the same; to count number fields is just to count points on the stack BG with bounded height? The problem: there is no definition of the height of a rational point on a stack. I'll explain what we think the right definition is, and explain how it suggests a heuristic which has both the Malle conjecture and the Batyrev–Manin conjecture as special cases.

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