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## A Well-Connected Separator for Planar Graphs

by

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# A Well-Connected Separator for Planar Graphs

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## Abstract

We prove a separator theorem for planar graphs, which has been used to obtain polynomial time approximation schemes for the metric TSP [8] and the 2-edge-connected and biconnected spanning subgraph problems [6] in unweighted planar graphs. We also prove an extension of the theorem which was used in [3] to find quasi-polynomial time approximation schemes for the weighted cases of the latter two problems.

## 1 Introduction

Many graph optimization problems that are NP-hard to approximate may still admit a *polynomial time approximation scheme (PTAS)* for special classes of input graphs. A PTAS is an algorithm taking as input an instance of the optimization problem and some  $\varepsilon > 0$  and returning a solution which has weight at most  $(1 + \varepsilon)$  times the optimum (for minimization problems), and running in polynomial time for each fixed  $\varepsilon$ .

In the case of planar graph algorithms, the separator theorem of Lipton and Tarjan [12] inspired related separator theorems [1, 13] and many applications, e.g. in planar graph layouts for VLSI [15], nested dissection in numerical analysis [11] and planar flow algorithms [9]. In general, separator theorems for graphs can be used to design “divide-and-conquer” algorithms, in which the problem is divided into smaller subproblems and the (approximate) solutions to the subproblems are combined to obtain an (approximate) solution to the original problem.

In this paper we will give a complete proof of a separator theorem for planar graphs, which yields PTAS’s for the 2-edge-connected and 2-vertex-connected spanning subgraph problems and for the metric traveling salesperson problem in planar graphs.

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In the *2-edge-connected spanning subgraph* (2-ECSS) and *2-vertex-connected spanning subgraph* (2-VCSS) problems we want to find a 2-edge-connected or 2-vertex-connected spanning subgraph of a given graph which has minimum cost. Both problems are Max-SNP-hard even for unweighted graphs [5, 7]. The best currently known algorithms for arbitrary unweighted graphs achieve an approximation ratio of  $5/4$  for the 2-ECSS problem [10] and  $4/3$  for the 2-VCSS problem [16]. However, a PTAS for both problems in the case of unweighted planar graphs was found by Czumaj et al. [6]. Both algorithms heavily depend on Theorem 1. Theorem 2 is an extension of Theorem 1 and was used in [3] to obtain *quasi-polynomial time approximation schemes* (QPTAS's) for the weighted cases of both above mentioned problems. A QPTAS is the same as a PTAS, except that its running time is  $n^{O(\log(n))}$  for each fixed  $\varepsilon$ .

In the *metric traveling salesperson problem* (TSP) we are given a graph with edge costs forming a metric and ask for a simple closed tour through all vertices of the graph, which has minimum cost. This problem is also Max-SNP-hard [14] and the best known approximation algorithm of Christofides [4] achieves an approximation ratio of  $3/2$ . Despite the hardness result, Grigni et al. [8] found a PTAS for the metric TSP in which the metric comes from the shortest path distance in an unweighted planar graph. This result was improved by Arora et al. [2] by finding a PTAS for the metric TSP in weighted planar graphs.

However, neither [6] nor [3] present a proof of Theorem 1 or Theorem 2. The separator theorem used in [8] is a special case of Theorem 1 and its proof is only sketched. Therefore the aim of this paper is to give complete proofs of Theorems 1 and 2. The proofs use a similar approach as the proof of Miller's separator theorem [13]. The proof of the separator theorem in [2] instead uses arguments from Lipton's and Tarjan's separator theorem; however, it might also be proved using Theorem 1.

## 1.1 Statement of the theorem

In the following we will consider separators in an embedded weighted planar graph  $G$  that consist of cycles of  $G$  and of a closed Jordan curve  $J$ , which intersects edges of  $G$  (or of a contracted graph of  $G$ ) only at their endpoints. The *interior of a simple cycle*  $C$  ( $int(C)$ ) is the set of all vertices, edges and faces strictly inside  $C$  and  $w(int(C))$  is the total weight of all elements in  $int(C)$ . Likewise we define the *exterior of a simple cycle*  $C$  ( $ext(C)$ ) and  $w(ext(C))$ . Note that the vertices and edges on  $C$  are neither in  $int(C)$  nor in  $ext(C)$ . Similarly, we define the *interior* and *exterior of a closed Jordan curve*  $J$  and their respective weights. If the Jordan curve  $J$  intersects a vertex or a face of  $G$ , then the weight of this element counts towards neither the weight of the interior nor the weight of the exterior of  $J$ .

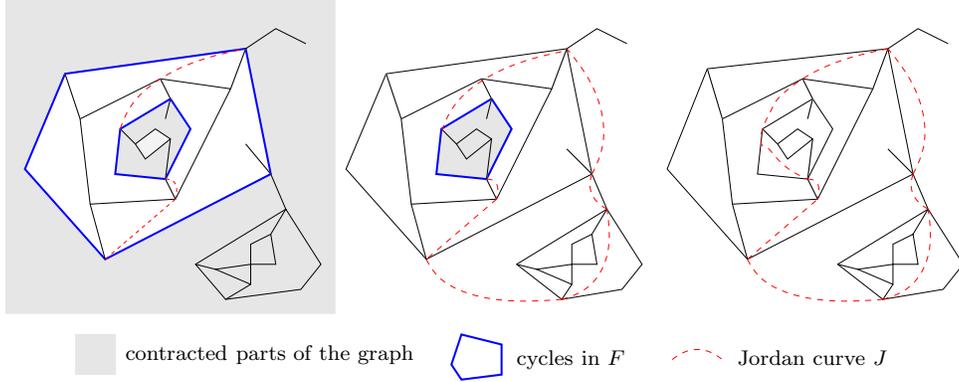


Figure 1: The three different types of the separator. In the first two cases the Jordan curve is closed after the contraction.

We now state our main theorem.

**Theorem 1** *Let  $G$  be a connected planar graph on  $n \geq 3$  vertices embedded in the plane. Suppose  $G$  has non-negative weights on its vertices, edges and faces, and non-negative costs on its edges. Let  $W$  be the total weight of the graph and let  $M$  be its total cost and assume that no edge has weight more than  $(3/4)W$ .*

*Then, for any positive integer  $k$ , we can find a subgraph  $F$  of  $G$  and a closed Jordan curve  $J$  in  $O(n)$  time such that:*

1.  *$F$  is the union of at most two vertex-disjoint simple cycles (maybe none). The total cost of the edges on each cycle is at most  $M/k$ . If  $F$  contains two cycles  $A$  and  $C$ , then  $\text{int}(C) \subset \text{int}(A)$ .*
2. *The interior of  $C$  and the exterior of  $A$  (if they exist) both have weight at most  $W/2$ .*
3. *Denote by  $G'$  the embedded graph that results after deleting the interior of  $C$  and the exterior of  $A$  (if they exist) and contracting each cycle in  $F$  to a vertex of weight 0. Then  $J$  is a Jordan curve through the new contracted vertices, which intersects edges of  $G'$  only at their endpoints. The interior and exterior of  $J$  both have weight at most  $(3/4)W$ .*
4. *The set of vertices of  $G'$  on  $J$  has size  $O(k)$ .*

The three possible types of the separator (according to the number of cycles in  $F$ ) are illustrated in Figure 1. Note that it is necessary to assume that each edge

weighs at most  $(3/4)W$ . If we had a graph in which an edge  $e$  has weight larger than  $(3/4)W$  and cost exceeding  $M/k$ , no such separator would exist. Because of its high cost  $e$  cannot be on any of the cycles in  $F$  and due to its high weight it cannot be in the contracted interior of  $C$  or in the contracted exterior of  $A$ . Hence  $e$  is also an edge in  $G'$  and any Jordan curve in the contracted graph  $G'$  has  $e$  either in its interior or exterior and so would not satisfy the third property.

## 2 Cycle separator lemma

The following lemma is needed in the proof of Theorem 1 and is a modification of Lipton's and Tarjan's cycle separator theorem [12].

Let  $G$  be a weighted planar graph embedded in the plane and assume for simplicity that we have scaled the weights such that the total weight of  $G$  is  $W = 1$ . A simple cycle  $C$  in  $G$  is called an  $\alpha$ -*separating cycle* if its interior and exterior both have weight at most  $\alpha$ . We call  $G$  *triangulated*, if the boundary of every face is a triangle. The planar dual  $G^*$  of  $G$  is the planar graph whose vertex set is the set of faces of  $G$  in which two vertices are joined by an edge if and only if the corresponding faces share an edge in  $G$ .

**Lemma 1** *Let  $G = (V, E)$  ( $|V| \geq 3$ ) be an embedded triangulated planar graph with non-negative weights  $w$  on its vertices, edges and faces, with total weight 1. Furthermore, assume that every face has weight at most  $1/4$ .*

*Let  $T$  be a spanning tree of  $G$  and for any  $u, v \in V$  denote by  $P_{uv}$  the path between  $u$  and  $v$  in  $T$ . Then there exists a  $3/4$ -separating cycle  $P_{uv} \cup \{uv\}$  for some non-tree edge  $uv \in E$ .*

*Proof:* Consider a non-tree edge  $e = uv$  of  $G$ , which induces a cycle  $C_e = P_{uv} \cup uv$  in the tree  $T$ . Denote by  $H_e$  the “heavy side” of  $C_e$  and by  $L_e$  the “light side” of  $C_e$  (breaking ties arbitrarily). Hereby we mean that, say  $L_e$ , contains all vertices, edges and faces which contribute to the weight of either the interior or exterior of  $C_e$ , whichever has a smaller total weight.

Also note that the dual edges of all non-tree edges form a tree  $T^*$  in  $G^*$ , because a dual cycle would imply that the original spanning tree  $T$  is not connected, and any two faces of  $G$  can be connected by only non-tree edges (otherwise  $T$  would have a cycle). We now orient the dual edge of each non-tree edge  $e$  to point from  $L_e$  to  $H_e$ . By starting at some arbitrary dual vertex and following a directed path in  $T^*$  as far as possible, we eventually reach a dual vertex with outdegree 0 in  $T^*$ . Let  $\Delta$  be the face of  $G$  corresponding to this dual vertex and denote the three vertices (in  $G$ ) which lie on the boundary of  $\Delta$  by  $t$ ,  $u$  and  $v$ .

We distinguish three cases corresponding to the number of non-tree edges on the boundary of  $\Delta$ . If  $\Delta$  is bounded by two tree edges and one non-tree edge, then the cycle bounding  $\Delta$  may be chosen to be the separator, since its interior and exterior both have weight at most  $1/4$ .

In the second case, one of the edges bounding  $\Delta$ , say  $ut$ , is an edge in  $T$  and in the third case none of those edges is in  $T$ . Both situations are depicted in Figure 2.

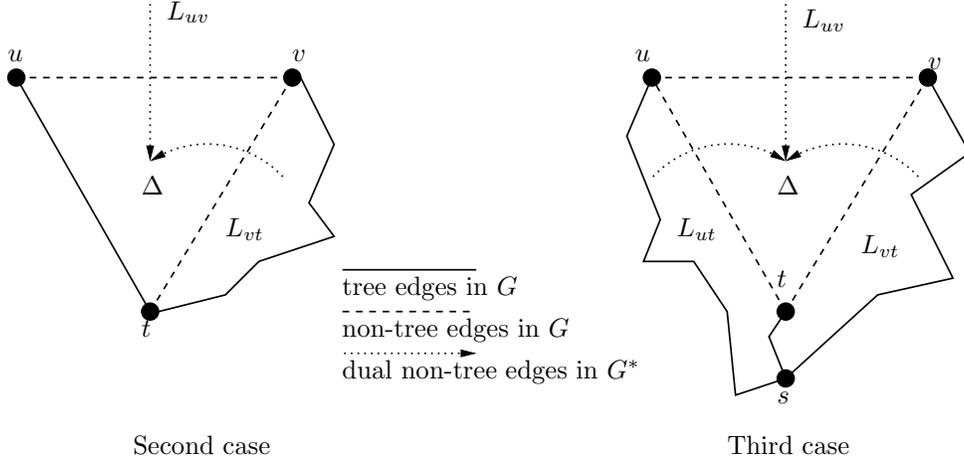


Figure 2: The latter two cases in the proof of Lemma 1

For the second case assume w.l.o.g. that  $t$  lies on  $P_{uv}$  (the other case being that  $u$  lies on  $P_{vt}$ ). Now  $H_{uv} = L_{vt} \cup \{\Delta, vt\}$  and  $H_{vt} = L_{uv} \cup \{\Delta, uv, ut, u\}$  (cf. Figure 2). Since  $\Delta$  is the only element in  $H_{uv} \cap H_{vt}$ , we have that  $w(H_{uv}) + w(H_{vt}) \leq 5/4$  and therefore at least one of  $H_{uv}$  and  $H_{vt}$  has weight at most  $3/4$ . Hence  $P_{uv} \cup \{uv\}$  or  $P_{vt} \cup \{vt\}$  is the desired  $3/4$ -separating cycle.

We now show how to find a  $3/4$ -separating cycle in the third case. Assume w.l.o.g. that  $t$  is the vertex which lies either on  $P_{uv}$  or in  $\text{int}(C_{uv})$ . In either case, let  $s$  be the vertex on  $P_{uv}$  which is closest to  $t$  in  $T$ . Note that in case  $t$  lies on  $P_{uv}$  we have  $s = t$ . We have the following equalities:

$$H_{uv} = L_{ut} \cup L_{vt} \cup \{\Delta, ut, vt\} \cup (P_{st} \setminus \{s\}) \quad (1)$$

$$H_{ut} = L_{uv} \cup L_{vt} \cup \{\Delta, uv, vt\} \cup (P_{sv} \setminus \{s\}) \quad (2)$$

$$H_{vt} = L_{uv} \cup L_{ut} \cup \{\Delta, uv, ut\} \cup (P_{su} \setminus \{s\}). \quad (3)$$

Note that each weighted element of  $G$  appears at most twice in the right-hand sides of (1)-(3), except for  $\Delta$ , which appears three times. Remembering that the total weight of  $G$  equals 1 and that  $w(\Delta) \leq 1/4$ , we have that  $w(H_{uv}) + w(H_{ut}) +$

$w(H_{vt}) \leq 2 + 1/4$ . Therefore at least one of the cycles  $C_{uv}$ ,  $C_{ut}$  and  $C_{vt}$  must be a  $3/4$ -separating cycle.  $\square$

If the planar graph  $G$  is allowed to have multiple edges and loops and we require every face to have at most three corners on its boundary, we get a similar result. The additional possible shapes of a face that have to be considered in the proof of Lemma 2 are depicted in Figure 3 and the corresponding arguments are similar to those in the proof of Lemma 1.

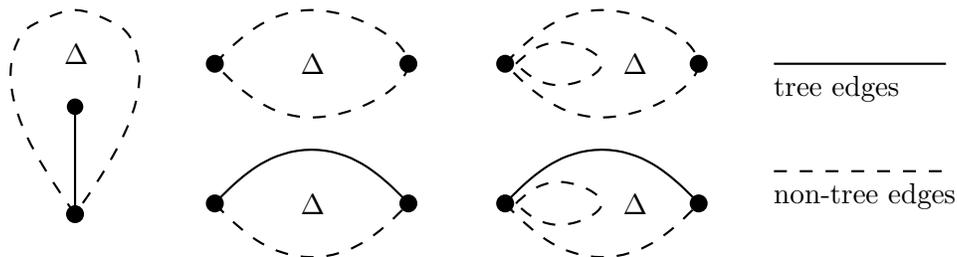


Figure 3: Possible faces if  $G$  has multiple edges or loops.

**Lemma 2** *Let  $G = (V, E)$  be an embedded planar graph with non-negative weights  $w$  on its vertices, edges and faces, whose total weight is 1 and no element weighs more than  $1/4$ .  $G$  is allowed to have multiple edges and loops. Moreover, assume that every face of  $G$  has at most three corners on its boundary (if the face is non-simple, multiple occurrences of a vertex are counted as multiple corners).*

*Let  $T$  be a spanning tree of  $G$  and for any  $u, v \in V$  denote by  $P_{uv}$  the path between  $u$  and  $v$  in  $T$ . Then there exists a  $3/4$ -separating cycle  $P_{uv} \cup \{uv\}$  for some non-tree edge  $uv \in E$ .*

### 3 Proof of Theorem 1

For the remainder of this paper we assume that  $G$  is a weighted planar graph on  $n$  vertices with total weight on its vertices, faces and edges equal to 1, and total cost on its edges equal to  $M$ .

First, we may assume that no vertex and no face weighs more than  $1/4$ , since otherwise we could let  $J$  be a nearly trivial loop through that heavy element, which then counts towards neither the interior nor the exterior of  $J$ . Moreover, if for some edge  $e$  we have  $1/4 \leq w(e) \leq 3/4$ , we may choose  $J$  to go through the endpoints of  $e$  and having only  $e$  in its interior to find a separator satisfying Theorem 1. Therefore we can also assume that each edge weighs at most  $1/4$ .

The main idea of the proof is to find cycles of low edge cost, whose interior or exterior, respectively, have small weight. After deleting the interiors or exteriors of these cycles and appropriately triangulating, the remaining graph will have diameter  $O(k)$ . The desired cycle(s) and the desired Jordan curve can then be constructed from the 3/4-separating cycle obtained from applying Lemma 2.

### Constructing the cycle tree

The idea of constructing a cycle tree  $\mathcal{T}$  of  $G$  is due to Miller [13], which essentially uses a breadth-first search through the vertices and faces of  $G$ . We choose an arbitrary vertex  $v_0$  on the infinite face and label it 0. Each face which has  $v_0$  on its boundary gets label 1 (including the infinite face). Now each unlabeled vertex which lies on the boundary of a face with label 1 gets label 2, etc. In this way all vertices will get even labels and all faces will get odd labels (cf. Figure 4). Moreover, vertices around a face of label  $2m + 1$  can only have label  $2m$  or  $2m + 2$ . Consequently, the endpoints of an edge either have the same label or their labels differ by two. The same holds for the two faces adjacent to an edge.

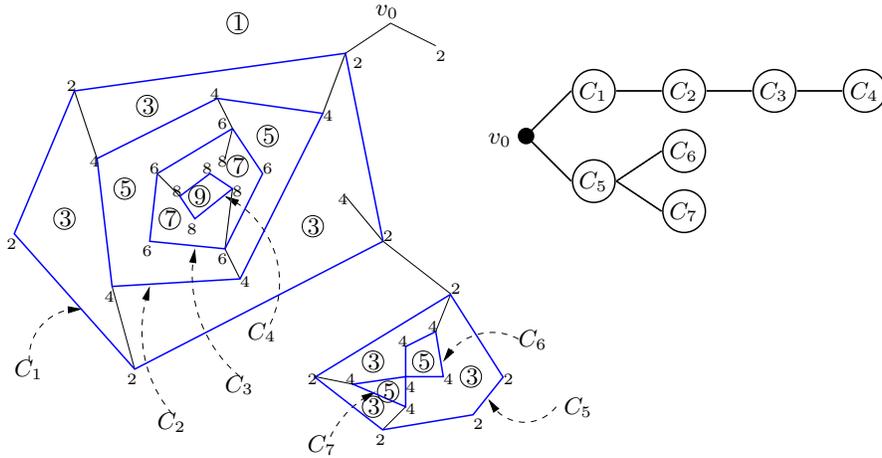


Figure 4: A labelling of  $G$  and the corresponding cycle tree  $\mathcal{T}$ .

Local arguments [13] show that the set of edges whose adjacent faces have different labels forms an edge-disjoint collection of simple cycles. All vertices lying on such a cycle  $C$  have the same (even) label, and so we define that to be the label of  $C$ . Further, we observe that cycles with the same label are non-nesting, and that a vertex  $v$  on a label  $2m + 2$  cycle shares a face  $f$  with some vertex  $w$  on a  $2m$  cycle. We will refer to this fact as the *visibility property* of the decomposition.

We define the *cycle tree*  $\mathcal{T}$  of  $G$  as follows:  $V(\mathcal{T})$  is the collection of cycles in the above decomposition plus the root vertex  $v_0$ . All cycles with label 2 are adjacent to  $v_0$  and two cycles are adjacent if they share a face and differ in label (necessarily by two). The properties of the decomposition insure that  $\mathcal{T}$  is a tree. Furthermore, if  $C'$  is a descendant of  $C$  in  $\mathcal{T}$ , then  $C'$  lies in the interior of  $C$ . Cycles which are not descendants of each other in  $\mathcal{T}$  have disjoint interiors.

### Pruning the graph

If  $\mathcal{T}$  (as a rooted tree with root  $v_0$ ) has height at most  $k$ , then we do not change  $G$  and continue with the triangulation described below with  $A = v_0$  and  $\mathcal{C} = \emptyset$ . Otherwise we try to simplify the graph  $G$  and find cycles of low cost whose interiors have small weight.

We will call the perimeter of a cycle in the cycle tree *heavy* if the total cost of its edges exceeds  $M/k$ , otherwise we call the perimeter of the cycle *light*. We will call the interior of a cycle *heavy* if the total weight of the vertices, edges and faces inside the cycle exceeds  $1/2$ , otherwise we call it *light*. Similarly, we define the exterior of a cycle to be light or heavy.

Starting at  $v_0$ , we now choose a path to a leaf cycle in  $\mathcal{T}$  such that from each cycle on the path we go to its child whose interior has the largest weight, breaking ties arbitrarily.

Let  $B$  denote the first cycle on this path with a light interior. If no such  $B$  exists then  $B$  denotes the leaf cycle on this path. From  $B$  we walk back towards the root and denote by  $A$  the first cycle found with light perimeter. If no such cycle is found, then  $A = v_0$ . In either case we require  $A \neq B$ . Also, denote by  $d_{\mathcal{T}}$  the distance metric in  $\mathcal{T}$ .

We now define  $\mathcal{C}_i = \{C \in V(\mathcal{T}) : C \text{ is a descendant of } A \text{ and } d_{\mathcal{T}}(A, C) = i\}$  for  $i \geq 1$ . Then we search for the smallest  $j \geq 1$ , such that the total cost of all edges of all cycles in  $\mathcal{C}_j$  is at most  $M/k$  and so that the interior of every  $C \in \mathcal{C}_j$  has weight at most  $1/2$ . If we let  $b = d_{\mathcal{T}}(A, B)$ , then the latter condition certainly holds for all  $j > b$  and also for  $j = b$  if  $B$  has light interior. Moreover, the first condition can only hold for  $j \geq b$ , since every cycle on the path between  $A$  and  $B$  in  $\mathcal{T}$  has heavy perimeter. However, since the total cost of all edges in  $G$  is  $M$ , we must have  $j < k + 1$ . Note, that  $j = k + 1$  is only possible if  $B$  has heavy interior. We abbreviate  $\mathcal{C}_j = \mathcal{C}$  and summarize the properties of  $A$  and  $\mathcal{C}$ .

1. Either  $A = v_0$  or  $A$  has light exterior and light perimeter.
2.  $\mathcal{C}$  is a (possibly empty) collection of cycles whose total edge cost is at most  $M/k$ .
3. For every  $C \in \mathcal{C}$  we have:

- $w(\text{int}(C)) \leq 1/2$ , and
- $d_{\mathcal{T}}(A, C) \leq k + 1$ .

If  $A \neq v_0$ , then we replace the exterior of  $A$  by an empty face with weight  $w(\text{ext}(A))$ , and we replace  $\mathcal{T}$  by its subtree rooted at  $A$ . Similarly, for every  $C \in \mathcal{C}$  we remove its interior and replace it by an empty face with weight  $w(\text{int}(C))$ , and we remove the descendants of  $C$  in  $\mathcal{T}$ .

By the choice of  $A$  and  $\mathcal{C}$  we ensure that the new faces all have weight at most  $1/2$ . If any of the new faces has weight exceeding  $1/4$ , we may safely choose the cycle bounding that face to be the separator, since it must have light perimeter and the weight of its exterior (interior for  $A$ ) cannot exceed  $3/4$ . Therefore we may still assume that the modified graph satisfies the assumption that every face weighs at most  $1/4$ .

### Triangulating the graph and finding the separator

We will now triangulate the modified graph and build a spanning tree  $T$  to be able to apply the cycle separator lemma.

Because of the visibility property, each face  $f$  of label  $2m + 1$  is adjacent to at least one vertex  $w$  of label  $2m$ . Therefore, we can triangulate each face by adding edges from  $w$  to all vertices around that face. If  $f$  is non-simple, then this may create multiple edges or loops (cf. Figure 5). We distribute the weight of  $f$  arbitrarily among the new faces.

To apply the cycle separator lemma we now construct a breadth-first search tree  $T$ , starting from  $v_0$  if  $A = v_0$  or starting from the vertex on  $A \neq v_0$ , from which the new empty face in the exterior of  $A$  was triangulated.

Since  $d_{\mathcal{T}}(A, C) \leq k + 1$  for every  $C \in \mathcal{C}$ , the diameter of  $T$  is at most  $2k + 2$  and every vertex on a cycle  $C \in \mathcal{C}$  has the same distance from  $v_0$  in  $T$ . In particular, none of the edges in the interior of a cycle  $C \in \mathcal{C}$  is an edge in  $T$ . Furthermore, to be able to construct the Jordan curve  $J$  later, we double each edge from the original graph, set the weight of both edges to zero and set the weight of the face between both edges to the original weight of that edge. However, we do not double edges that arose from the triangulation.

Now we can apply Lemma 2 to the modified graph and  $T$  and obtain a  $3/4$ -separating cycle  $C_{sep}$ . By the arguments above, there can be at most one  $C \in \mathcal{C}$  for which  $C_{sep}$  passes through its interior. If this is the case, then  $C$  is a cycle in  $F$ . Moreover, if an edge of  $C_{sep}$  passes through the exterior of  $A \neq v_0$ , then also  $A \in F$ .

To construct the Jordan curve  $J$  we first delete the exterior of  $A$  and the interior of  $C$  and contract  $A$  and  $C$ , depending on whether  $A \in F$  or  $C \in F$  or both. For each edge  $e$  in  $C_{sep}$  that arose from the triangulation,  $J$  will contain an arc

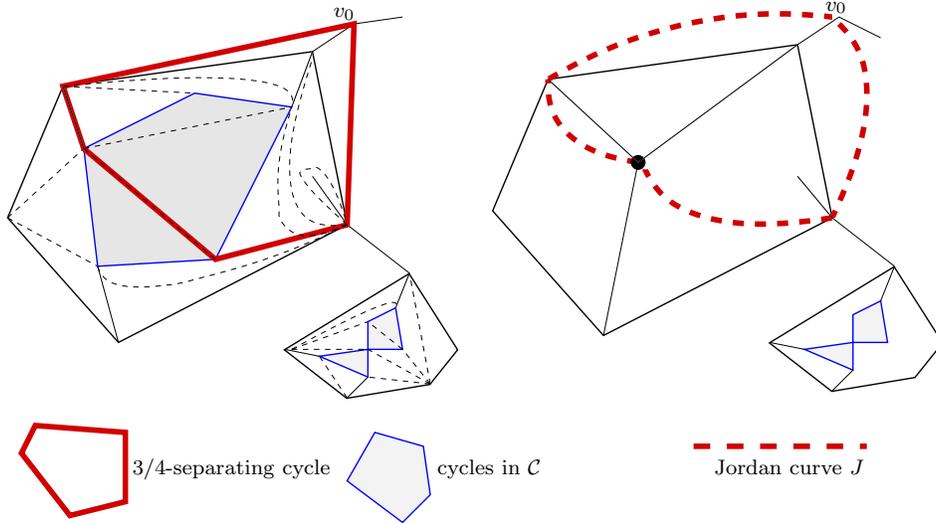


Figure 5: A 3/4-separating cycle in the triangulated graph and the corresponding Jordan curve in the contracted graph. (For readability the infinite face is not triangulated.)

connecting the endpoints of  $e$  and passing through the face that contained  $e$ . For each edge  $e$  in  $C_{sep}$  which arose from doubling the original edges in  $G$ ,  $J$  will contain an arc passing through the endpoints of  $e$  and being parallel to the original edge. We choose the arc to be in that face containing the original edge, which also has the edge  $e \in C_{sep}$  on its boundary. This ensures that  $J$  has the edge  $e$  with weight  $w(e)$  on the same “side” as  $C_{sep}$  has the face between the copies of  $e$ , which also has weight  $w(e)$ . Therefore, in the contracted graph  $G'$ , we have  $int(J) \subseteq int(C_{sep})$  and  $ext(J) \subseteq ext(C_{sep})$  and thus  $w(int(J))$  and  $w(ext(J))$  are bounded by  $3/4$ .

This establishes the third property claimed in Theorem 1. The first two properties are satisfied by the choice of  $A$  and  $C$  and the fourth property follows from the fact that the diameter of the breadth-first search tree  $T$  is at most  $2k + 2$ .

Moreover, all steps in the algorithm can be implemented in linear time. The most complicated step is to find the edge which according to Lemma 2 induces the 3/4-separating cycle. However, the weights of the interior and the exterior of each cycle induced by a non-tree edge can be computed starting at the leaves of the dual tree and using dynamic programming. For more details we refer the reader to [13].

## 4 Extension

In order to find a QPTAS for the problem of finding a minimum-weight 2-edge-connected or 2-vertex-connected spanning subgraph of a weighted planar graph, the following extension of Theorem 1 was used in [3]. Here a *separation* of  $G$  is one in the sense of Theorem 1, i.e. a subgraph consisting of at most two vertex-disjoint cycles and a closed Jordan curve.

**Theorem 2** *Let  $G$  be a planar graph on  $n$  vertices with non-negative edge costs, embedded in the plane, and let  $k \geq 1$ . Then we can find a list of  $O(n^2)$  separations of  $G$ , such that for any valid weight scheme of the vertices, edges and faces of  $G$ , some separation in this list satisfies the conclusions of Theorem 1.*

*Proof:* We will construct the labelling of the vertices and faces as in the proof of Theorem 1. Note, that this labelling does not depend on the weights. Hence the tree of cycles  $\mathcal{T}$  is also independent of the vertex, edge and face weights.

During the process of finding the separation of  $G$  (with known weights) as in Theorem 1, we find the cycles  $A$  and  $B$ , the collection  $\mathcal{C}$  of cycles, and a separating cycle in the triangulated modified graph.

We claim that there are  $O(n^2)$  possible ways of choosing all these cycles. Since  $G$  is planar, there are at most  $n$  choices for  $B$ . Then  $A$  has to be the first cycle on the path from  $B$  to  $v_0$  in  $\mathcal{T}$ , which has light perimeter. We can make the choice of  $\mathcal{C}$  also independent of the weights by not allowing  $j = b$  in the proof of Theorem 1. Therefore, after having determined  $A$  and  $B$ , the choice of  $\mathcal{C} = \mathcal{C}_j$  is unique with this slight modification.

Finally, after possible modifications of the graph and after the triangulation, there are  $O(n)$  possible edges which can induce the 3/4-separating cycle. Therefore, we can first choose  $B$  in at most  $n$  different ways and after the same steps as in the proof of Theorem 1 we can choose the edge that induces the 3/4-separating cycle in  $O(n)$  ways to find a list of  $O(n^2)$  possible separations.

If now  $G$  is given with a valid weighting of the vertices, edges and faces, at least one of the separations must be the one constructed in the proof of Theorem 1. Hence it must also satisfy the conclusions of the theorem as claimed.  $\square$

## 5 Conclusions

In this paper we show a separator algorithm for planar graphs, which is useful in obtaining approximation schemes for combinatorial optimization problems on planar graphs.

For the 2-connected spanning subgraph problems mentioned in the introduction, it yields a PTAS for the unweighted case and a QPTAS for the weighted case of these problems. In order to find a PTAS for the weighted 2-connected spanning subgraph problems, it would be sufficient to find a separator as in Theorem 1, where the cost of each contracted cycle in the separator is bounded by  $OPT/k$  instead of  $M/k$ , where  $OPT$  is the cost of a minimum 2-connected spanning subgraph of the input graph.

Arora et al. [2] present a result resembling Theorem 1, with an argument based on the method of Lipton and Tarjan [12] rather than of Miller [13]. However that method has the drawback that the components of the separator are paths rather than cycles, and this is insufficient for our 2-ECSS and 2-VCSS applications.

One can investigate if there are similar separators which would be well suited to find approximation algorithms for higher connected spanning subgraph problems. In particular, for the 3-ECSS problem, one may try a separator built from two consecutive cycles in  $\mathcal{T}$ ; this is a topic for future work.

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