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An Iterative Solver for the Navier-Stokes Equations in Velocity-Vorticity-Helicity Form

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Abstract

We study a variant of augmented Lagrangian (AL)-based block triangular preconditioners to accelerate the convergence of GMRES when solving linear algebraic systems arising from finite element discretizations of the 3D Navier-Stokes equations in Velocity-Vorticity-Helicity form. This recently proposed formulation couples a velocity-pressure system with a vorticity-helicity system, providing a numerical scheme with enhanced accuracy and superior conservation properties. We find that the resulting discrete systems can be solved efficiently by using AL preconditioning technique, together with the inner-outer FGMRES method for solving the sub-problems. Two numerical experiments are given which illustrate the effectiveness of the proposed method.

Key words Navier-Stokes equations; preconditioning; augmented Lagrangian method; vorticity; helical density

1 Introduction

We consider the incompressible Navier-Stokes (NS) system in velocity-vorticity-helicity (VVH) form, on a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary and for time interval $t \in (0, T]$,

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla P = \mathbf{f}, \\ \mathbf{w}_t - \nu\Delta\mathbf{w} + 2D(\mathbf{w})\mathbf{u} - \nabla\eta = \nabla \times \mathbf{f}, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{w} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ denotes velocity, $\mathbf{w}$ vorticity, η and P denote the helical density and Bernoulli pressure, $D(\mathbf{w}) := \frac{1}{2}(\nabla\mathbf{w} + [\nabla\mathbf{w}]^T)$ is the symmetric part of the vorticity gradient, and ν

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is the kinematic viscosity. The system is equipped with the initial conditions

$$\mathbf{u} = \mathbf{u}_0, \qquad \mathbf{w} = \nabla \times \mathbf{u}_0, \quad \text{for } t = 0, \tag{1.2}$$

and with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{w}|_{\partial\Omega} = \boldsymbol{\psi}, \tag{1.3}$$

where the natural choice of $\boldsymbol{\psi}$ is $\boldsymbol{\psi} = \nabla \times \mathbf{u}$ or $\boldsymbol{\psi} = 0$ for the far-field outflow boundaries. This formulation was first derived in [16], and has since been studied numerically in the case of equilibrium Navier-Stokes [13], and for the Boussinesq system [15]. All three of these studies have shown promising results.

The VVH system is particularly interesting from the physical point of view. It solves directly for velocity and vorticity, and it is believed that methods that do so are more physically accurate, particularly near boundaries [6]. VVH is also the first NS formulation to solve directly for helical density (which is related to the helicity through $H = \int_\Omega \eta \, d\mathbf{x}$), a quantity known to be of fundamental physical importance in fluid flow [14, 5, 8]. This formulation also explicitly enforces the vorticity to be incompressible, the third equation in (1.1), with helical density η in the second equation in (1.1) acting as a Lagrange multiplier corresponding to this constraint. Since $\nabla \cdot \nabla \times = 0$, that the vorticity is solenoidal is important both for physical relevance and mathematical consistency. Although it is possible to couple this constraint to the usual vorticity equation by adding an artificial Lagrange multiplier, VVH is the first velocity-vorticity method to *naturally* enforce this constraint.

A perceptible difficulty in computing with the fully coupled VVH system is solving the large sparse linear systems that arise in the discretizations. These systems couple two saddle point systems, each of which on its own is challenging to solve. The approach to this problem that we study herein is block preconditioned GMRES [18], where the block preconditioning is based on an augmented-Lagrangian (AL) approach developed earlier in [1, 3, 2] for velocity-pressure saddle point systems. Here we extend and study this approach for the dual-coupled saddle points systems resulting from a finite element discretization of (1.1)–(1.3).

This paper is arranged as follows. In Section 2, we present the finite element discretization for the VVH system (1.1)-(1.3) and the preconditioning technique employed herein. Two numerical experiments are presented in Section 3. The first is an analytic test problem, and the second is for a benchmark problem of 3D channel flow over a step. These numerical examples will illustrate the effectiveness of the proposed method. Finally, in Section 4, we draw conclusions and discuss future directions.

2 A solver

We present first in this section the finite element discretization for the equilibrium VVH system, along with a brief discussion. Then we will present the proposed block preconditioning technique for the linear systems arising from a Newton linearization of the discrete scheme.

2.1 Finite element discretization

We assume homogeneous boundary conditions for velocity to simplify the weak formulation and the following analysis. Extension to the case of other common boundary conditions for the velocity is straightforward, see e.g. [19], and both of our numerical experiments are for inhomogeneous boundary conditions for velocity. The Galerkin finite element method for the steady equations (1.1)-(1.3) is as follows.

Let $(\mathbf{X}_h, Q_h) \subset (\mathbf{H}^1(\Omega), L^2(\Omega))$ be conforming finite element spaces on a regular mesh τ_h of a polyhedral domain Ω, satisfying the LBB condition, inverse inequality, and the standard approximation properties, see e.g. [9]:

$$\begin{aligned} \inf_{\mathbf{v}_h \in \mathbf{X}_h} (\|\boldsymbol{\phi} - \mathbf{v}_h\|_0 + h\|\boldsymbol{\phi} - \mathbf{v}_h\|_1) &\leq Ch^{\ell+1}|\boldsymbol{\phi}|_{\ell+1} \\ \inf_{q_h \in Q_h} \|r - q_h\|_0 &\leq Ch^{\ell}|r|_{\ell}, \quad \text{for } \ell = 0, \ldots, k, \end{aligned}$$

with some integer $k \geq 1$. Define the subspaces $\mathbf{X}_{h0} := \mathbf{X}_h \cap \mathbf{H}_0^1(\Omega)$ and $\mathbf{V}_{h0} := \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$.

The finite element formulation reads: Given forcing $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and kinematic viscosity $\nu > 0$, find $(\mathbf{u}_h, \mathbf{w}_h, P_h, \eta_h) \in (\mathbf{X}_{h0}, \mathbf{X}_h, Q_h, Q_h)$ satisfying $\forall(\mathbf{v}_h, \boldsymbol{\chi}_h, q_h, r_h) \in (\mathbf{X}_{h0}, \mathbf{X}_{h0}, Q_h, Q_h)$,

$$\begin{cases} (\mathbf{w}_h \times \mathbf{u}_h, \mathbf{v}_h) - (P_h, \nabla \cdot \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \gamma_1(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \\ (\nabla \cdot \mathbf{u}_h, q_h) = 0 \\ 2(D(\mathbf{w}_h)\mathbf{u}_h, \boldsymbol{\chi}_h) + (\eta_h, \nabla \cdot \boldsymbol{\chi}_h) + \nu(\nabla \mathbf{w}_h, \nabla \boldsymbol{\chi}_h) + \gamma_2(\nabla \cdot \mathbf{w}_h, \nabla \cdot \boldsymbol{\chi}_h) = (\nabla \times \mathbf{f}, \boldsymbol{\chi}_h) \\ (\nabla \cdot \mathbf{w}_h, r_h) = 0, \\ \mathbf{w}_h - I_h(\nabla \times \mathbf{u}_h) = 0 \text{ on } \partial\Omega \end{cases} \tag{2.4}$$

Here I_h denotes a generic interpolant such that $\int_{\partial\Omega} I_h(\nabla \times \mathbf{u}_h) \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is an outward normal vector to $\partial\Omega$, e.g. a Clement type interpolant I_h^C based on local averaging.

This method was analyzed in [13], and was found to be stable and optimally convergent. More precisely, the following result is valid: Let $(\mathbf{u}, p)$ be the solution to the incompressible Navier-Stokes equations in a bounded domain $\Omega \in \mathbb{R}^3$ with a sufficiently regular boundary and homogeneous Dirichlet boundary conditions for $\mathbf{u}$. Assume $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{k+1}(\Omega)$. If $(\mathbf{u}_h, P_h)$, $(\mathbf{w}_h, \eta_h)$ are the solutions to (2.4), with $\mathbf{w}_h = I_h^C(\nabla \times \mathbf{u})$ on $\partial\Omega$ and a small data assumption on $\|\mathbf{f}\|$, then the *a priori* error estimate

$$\begin{aligned} &\|\nabla(\mathbf{u} - \mathbf{u}_h)\|^2 + \|\mathbf{w} - \mathbf{w}_h\|^2 \\ &\qquad \leq C\,(h^{2k} + \|(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u})) \otimes \mathbf{n}\|^2_{-\frac{1}{2},\partial\Omega} + \|(\nabla \times \mathbf{u} - I_h^C(\nabla \times \mathbf{u})) \cdot \mathbf{n}\|^2_{-\frac{1}{2},\partial\Omega}) \end{aligned}$$

holds with $\mathbf{w} = \nabla \times \mathbf{u}$. Moreover, if the extra regularity of the Navier-Stokes velocity is assumed, $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{k+2}(\Omega)$, then it holds

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|^2 + \|\nabla(\mathbf{w} - \mathbf{w}_h)\|^2 + \|P - P_h\| + \|\eta - \eta_h\| \leq C\,h^{2k}.$$

The above convergence result assumes $\mathbf{w}_h = I_h^C(\nabla \times \mathbf{u})$ for the vorticity boundary condition instead of the more practical $\mathbf{w}_h = I_h^C(\nabla \times \mathbf{u}_h)$. Still, computations suggest these are shortcomings in the analysis, and not the method itself.

Note the use of grad-div stabilization with parameters γ_1 and γ_2 both in the velocity and vorticity equations. Although the convergence result has been proved with $\gamma_1 = \gamma_2 = 0$, it can be easily extended to the case of $\gamma_1, \gamma_2 > 0$, with the constants C possibly dependent on γ-s. In practice, when using an element pair that does not provide pointwise enforcement of the solenoidal constraints (e.g., Taylor-Hood), the addition of this term improves the divergence error, and also reduces the effect of the possibly large Bernoulli pressure and helical density errors on the velocity and vorticity errors, respectively [12, 17].

2.2 A block AL-based preconditioner

We consider the Newton method to converge to the solution of the nonlinear problem (2.4). For higher Reynolds numbers the Newton method should be combined with a continuation technique with respect to ν. Suppressing the spatial discretization notation, the Newton linearization of the system (2.4) reads: Given the velocity and vorticity approximations U and W solve for the updates $\mathbf{u}, \mathbf{w}, P, \eta$ the system

$$\begin{cases} -\nu\Delta\mathbf{u} - \gamma_1\nabla\nabla\cdot\mathbf{u} + W\times\mathbf{u} + \nabla P + \mathbf{w}\times U = \mathbf{f}_u, \\ \nabla\cdot\mathbf{u} = g_u, \\ 2D(W)\mathbf{u} - \nu\Delta\mathbf{w} - \gamma_2\nabla\nabla\cdot\mathbf{w} + 2D(\mathbf{w})U - \nabla\eta = \mathbf{f}_v, \\ \nabla\cdot\mathbf{w} = g_v, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{w}|_{\partial\Omega} - \nabla\times\mathbf{u}|_{\partial\Omega} = g_{bc}. \end{cases} \tag{2.5}$$

with $\{\mathbf{f}_u, g_u, \mathbf{f}_v, g_v, g_{bc}\}$ standing for a (nonlinear) residual. We remark that the last equation in (2.5), representing the boundary coupling of the vorticity and velocity, requires a special treatment while solving the discrete linear system iteratively. In particular, we enforce in the iteration that $\mathbf{w}|_{\partial\Omega}$ be equal to the nodal average of $\nabla\times\mathbf{u}$, on the boundary, from the previous iteration. We also tried using different projections of $\nabla\times\mathbf{u}$ onto $\partial\Omega$ to define $\mathbf{w}|_{\partial\Omega}$, but nodal averaging gave the best results.

For the sake of clarity, we assume for a moment that the vorticity boundary conditions are decoupled from the velocity, say $\mathbf{w}|_{\partial\Omega} = 0$, and do not contribute to the vorticity d.o.f. Given the structure of the system in (2.5), the algebraic form of the finite element linearized equations in our case is the following coupled system:

$$\begin{pmatrix} A_u & -B^T & M & 0 \\ -B & 0 & 0 & 0 \\ N & 0 & A_v & B^T \\ 0 & 0 & B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ P \\ \mathbf{w} \\ \eta \end{pmatrix} = \begin{pmatrix} f_u \\ g_u \\ f_v \\ g_v \end{pmatrix}. \tag{2.6}$$

More specifically, the four blocks in the upper left corner

$$\begin{pmatrix} A_u & -B^T \\ -B & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & -B_1^T \\ A_{21} & A_{22} & A_{23} & -B_2^T \\ A_{31} & A_{32} & A_{33} & -B_3^T \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix} \tag{2.7}$$

correspond to the rotation form of the linearized Navier–Stokes equations. The diffusive term multiplied by the viscosity ν is contained in the diagonal blocks of A_u, and the cross-product terms are included in off-diagonal blocks $A_{ij}, i \neq j$. The grad-div stabilization terms with parameter γ_1 are in all 9 blocks of A_u. The 4 blocks in the lower right corner of (2.6) $\begin{pmatrix} A_v & B^T \\ B & 0 \end{pmatrix}$, which arise from the vorticity-helicity saddle point system, are similar in form to the convection form of the linearized Navier–Stokes equation, but the convection term is distributed in all 9 blocks of A_v due to the definition of $D(\mathbf{w})$.

Observe that the coupled VVH system (2.6) is singular for the Ethier-Steinman and step problem we consider in this paper. In both problems, the Bernoulli pressure P and helical density η are unique up to an additive constant, making the linear system in (2.6) rank deficient by 2 (because B is rank deficient by 1.). One may either remove these singularities by setting a single Dirichlet degree of freedom for both P and η, but as is the case for velocity-pressure systems as well, when using Krylov solvers these singularities need not be removed provided the iterations take place in an appropriate subspace.

To build a preconditioner for (2.6), assume we are given a generalized saddle point system of the form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad \text{or} \qquad \mathcal{A}\,\mathbf{x} = \mathbf{b}\,. \tag{2.8}$$

The augmented Lagrangian (AL) approach from [1] consists first of replacing the original system (2.8) with the equivalent one

$$\begin{pmatrix} A + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad \text{or} \qquad \mathcal{A}_\gamma\,\mathbf{x} = \mathbf{b}\,, \tag{2.9}$$

followed by preconditioning (2.9) with a block triangular preconditioner of the form

$$\mathcal{P}_\gamma = \begin{pmatrix} \widehat{A}_\gamma & B^T \\ 0 & -\widehat{S}_\gamma \end{pmatrix}. \tag{2.10}$$

Here and in the following $\widehat{A}_\gamma$ denotes a preconditioner for the velocity block A_γ and $\widehat{S}_\gamma$ is a preconditioner to the Schur complement of the augmented system $S_\gamma = B(A + \gamma B^T W^{-1} B)^{-1} B^T$. Based on the identity

$$S_\gamma^{-1} = S_0^{-1} + \gamma\, W^{-1},$$

a reasonable choice of $\widehat{S}_\gamma$ is the scaled W matrix, e.g. $\widehat{S}_\gamma = \gamma^{-1}\, W$, where W is typically a diagonal matrix, for example an approximation of the pressure mass matrix in the case of a linearized Navier-Stokes problem. Eigenvalue bounds for $\mathcal{P}_\gamma^{-1}\mathcal{A}_\gamma$ have been established in [1] and [3], and field of values type bounds for $\mathcal{P}_\gamma^{-1}\mathcal{A}_\gamma$, which lead to rigorous convergence estimates for GMRES, have been proved in [2].

In this paper we study the augmented Lagrangian preconditioning, when the augmentation is introduced on the differential level, the so called “first augment, then discretize” method. This approach allows us both to improve accuracy of the finite-element solution and to build an efficient preconditioner. Indeed, the matrix A_u can be decomposed as $A_u = A + \gamma_1 G$, where A corresponds to the discretization of $-\nu\Delta + \mathbf{w}\times$ operator, while G discretizes $-\nabla\nabla\cdot$. Thus adding $\gamma_1 G$ is similar from an algebraic point of view to the addition of $\gamma_1 B^T W^{-1} B$ with W given by the pressure mass matrix. The same observation is valid for the matrix A_v. Since (2.7) can be regarded as the augmented Lagrangian linear system, we consider the variant of the AL preconditioner:

$$\begin{pmatrix} \widehat{A}_u & B^T \\ 0 & -\widehat{S}_u \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & B_1^T \\ 0 & A_{22} & A_{23} & B_2^T \\ 0 & 0 & A_{33} & B_3^T \\ 0 & 0 & 0 & -\widehat{S}_u \end{pmatrix} \tag{2.11}$$

where $\widehat{S}_u^{-1} = \gamma_1 \widehat{M}_p^{-1}$ and $\widehat{M}_p$ is the main diagonal of the pressure mass matrix M_p. The presence of the grad-div stabilization terms in A_{11}, A_{22} and A_{33} makes the preconditioner (2.11) of augmented Lagrangian type. For the vorticity-helicity system, a similar block upper-triangular preconditioner is used except that the grad-div stabilization parameter is γ_2 instead of γ_1. To solve subproblems with A_{11}, A_{22} and A_{33} in the velocity and vorticity blocks, we consider both exact and inexact solvers which we will discuss in detail later.

For the coupled system (2.6), we define the following block lower triangular approximation

$$\begin{pmatrix} \widehat{A}_u & -B^T & 0 & 0 \\ 0 & -\widehat{S}_u & 0 & 0 \\ N & 0 & \widehat{A}_v & B^T \\ 0 & 0 & 0 & -\widehat{S}_v \end{pmatrix}, \tag{2.12}$$

as the *global preconditioner*, where $\widehat{A}_s$ and $\widehat{A}_v$ are corresponding block upper triangular approximations of A_u and A_v. The reason for using a block lower triangular matrix is that keeping N, a discrete analogue of the $2D(W)\mathbf{u}$ operator, appeared to be superior to including M, a discrete analogue of the $\mathbf{w} \times U$ operator.

In this paper we investigate the solution of (2.6) using a sparse direct solver ("backslash"in MATLAB) and preconditioned GMRES with the global preconditioner (2.12). The iterative solver does not require the coefficient matrix to be full rank, but the singularity of (2.6) may make the solve more difficult, so we choose to solve the nonsingular system in numerical experiments. In (2.12), since S_s and S_v are both diagonal, the major computation lies in solving linear systems with the diagonal blocks of A_s and A_v. Here we compare incomplete LU factorization `ilu` in MATLAB and algebraic multigrid method (AMG) MI20 [4] written in Fortran with a MATLAB interface. Note that the incomplete LU factorization `ilu` has been optimized and built into Matlab and MI20 is compiled by `mex` of MATLAB, so they are both quite efficient.

We also investigate an inner-outer Flexible GMRES (FGMRES) scheme. For the latter we use the implementation based on the simpler GMRES algorithm described in [10]. Here, to solve the linear systems with the velocity-pressure equation and the vorticity-helicity equation, instead of applying one action of the AL-type preconditioners, two inner GMRES iterations with corresponding preconditioners are adopted. This inevitably increases the cost, but we find it significantly reduces the outer FGMRES iterations and thus total iteration time. This method is found to be, by far, the most efficient of those tested.

3 Numerical experiments

We now describe two numerical examples, to illustrate the effectiveness of the proposed method. All experiments have been computed on a Sun Microsystems SunFire V40z, with 4 Dual Core AMD Opteron Processors and 32 GB of memory running Linux.

3.1 Experiment 1: The steady Ethier-Steinman problem

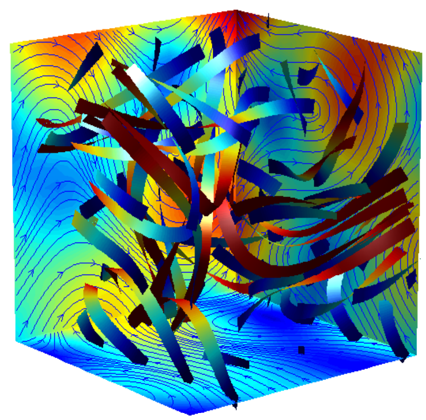

Figure 1: The velocity solution used in numerical experiment 1, on the $(-1,1)^3$ domain. The complex flow structure is seen in the streamribbons in the box and the velocity streamlines and speed contours on the sides.

For our first numerical experiment, we compute on $\Omega = (-1,1)^3$ approximations to the chosen analytical solution

$$\begin{aligned}
u_1 &= -\left(e^x \sin(y+z) + e^z \cos(x+y)\right) && (3.13)\\
u_2 &= -\left(e^y \sin(z+x) + e^x \cos(y+z)\right) && (3.14)\\
u_3 &= -\left(e^z \sin(x+y) + e^y \cos(z+x)\right) && (3.15)\\
p &= -\frac{1}{2}(e^{2x} + e^{2y} + e^{2z} + 2\sin(x+y)\cos(z+x)e^{(y+z)} \\
&\quad +2\sin(y+dz)\cos(x+y)e^{(z+x)} \\
&\quad +2\sin(z+dx)\cos(y+z)e^{(x+y)}). && (3.16)
\end{aligned}$$

This velocity-pressure combination (3.13)-(3.16) is an Ethier-Steinman solution [7] without the time dependence. Although unlikely to be physically realized, it is a good test problem because it has non-trivial helicity which implies the existence of complex structure [14] in the velocity field, which can be seen in a visualization of the flow in Figure 1.

Solutions are computed using the steady VVH scheme (2.4) with (P_2, P_1) Taylor-Hood elements on quasi-uniform meshes, functions $\mathbf{f}$ and $\nabla \times \mathbf{f}$ computed from $\mathbf{u}$, p and ν, and a

ν	h	Vel	Vort	Newton	Time
1	1/2	1.25e-3	1.33e-3	3	0,63, 1.26
1	1/4	1.50e-4	1.53e-4	3	11.40, 24.15
1	1/8	1.84e-5	1.85e-5	3	364.29, 1059.71
0.1	1/2	1.33e-3	2.57e-3	3	0.59, 1.21
0.1	1/4	1.55e-4	2.44e-4	3	10.87, 23.53
0.1	1/8	1.86e-5	2.29e-5	3	379.21, 1062.70
0.01	1/2	2.76e-3	6.52e-3	3	0.59, 1.49
0.01	1/4	4.41e-4	1.09e-3	3	12.50, 27.43
0.01	1/8	4.77e-5	1.21e-4	3	382.67, 1058.29

Table 1: Error and timings of backslash for $\nu = 1,\ 0.1,\ 0.01$ for Ethier-Steinman problem

Newton method is used to converge the nonlinear iteration. Dirichlet boundary conditions for velocity and vorticity are enforced as the interpolant of $\mathbf{u}$ and $\nabla \times \mathbf{u}$ on the boundary. Newton's method stops if two consecutive solutions differ by a relative error less than 10^{-4}. For linear solves, restarted GMRES(50) is used. It stops if the relative residual norm has been reduced by a factor at least 10^{-11} or a total of 500 iterations (10 restarts) has been reached. We found that using looser GMRES tolerances cannot make the L_2 velocity and vorticity errors reach the order of 10^{-5} (a desired level obtained with backslash) when $\nu = 1$ and $h = 1/8$. We currently do not have a satisfactory explanation for this phenomenon. Here, we take $\gamma_1 = \gamma_2 = 1$.

We perform experiments for $\nu = 1,\ 0.1$, and 0.01 using Matlab's backslash, GMRES with an incomplete LU factorization (ILU) to solve the subproblems (drop tolerance of 10^{-2}), GMRES with one iteration of MI20 to solve the subproblems, and inner-outer FGMRES schemes with ILU, and with MI20, as inexact subproblem solvers.

Tables 1-5 show the timings (in seconds) and errors from using each of these solvers. The errors are given as the L_2 velocity and vorticity errors (in "Vel" and "Vort" columns). The time includes two parts. The first represents the time of the first solve, while the second the average time of the rest. This is because the first solve requires significantly less time than the rest due to the symmetry of A_s and A_v resulting from an all-zero initial guess.

Table 1 shows the results with Matlab's backslash. For each viscosity, Newton's methods needs 3 iterations for all values of h considered, and the errors decrease by a factor around 8 each time the mesh is refined (which is expected with (P_2, P_1) Taylor-Hood elements if solved accurately). However, as expected, the time required to solve the system grows quickly.

Next we turn to GMRES preconditioned by the global preconditioner for the nonsingular coupled system. In Table 2 we present results when the subproblems are solved by ILU with dropping tolerance 10^{-2}. Compared to the direct solve, we see some deterioration in the velocity and vorticity errors when $\nu = 0.01$. However, the method is significantly more efficient than the direct solver. Interestingly, we see an improvement of the method's efficiency as ν decreases from 1 to 0.1, but then it deteriorates somewhat as ν is decreased to 0.01. In the last column we report the number of GMRES iterations for the first Newton step, followed by the average number of iterations for the remaining steps. The notation 500* means that the total number of 500 GMRES iterations (10 cycles of GMRES(50)) was performed without reaching the desired stopping tolerance. Note that, nevertheless, the accuracy of the final solution obtained by Newton's method is acceptable.

ν	h	Vel	Vort	Newton	Setup time	Iter time	Iterations
1	1/2	1.25e-3	1.33e-3	3	0.28, 0.08	2.30, 3.22	106, 142
1	1/4	1.50e-4	1.53e-4	3	1.63, 1.59	15.90, 55.20	208, 500*
1	1/8	1.83e-5	1.83e-5	3	46.64, 41.80	124.15, 520.16	185, 497
0.1	1/2	1.33e-3	2.57e-3	3	0.14, 0.07	1.87, 2.97	85, 130
0.1	1/4	1.55e-4	2.44e-4	3	1.47, 1.42	6.23, 14.55	81, 113
0.1	1/8	1.85e-5	2.27e-5	3	32.13, 32.13	80.64, 314.22	120, 350
0.01	1/2	2.75e-3	6.51e-3	4	0.17, 0.07	2.37, 11.37	116, 500*
0.01	1/4	5.96e-4	2.45e-3	4	1.26, 1.35	14.17, 54.98	157, 500*
0.01	1/8	6.64e-5	3.76e-4	3	28.57, 28.01	114.54, 444.59	178, 500*

Table 2: Error, timings and iteration counts of GMRES with global preconditioner (ILU) for $\nu = 1,\ 0.1,\ 0.01$ for Ethier-Steinman problem.

The results for GMRES preconditioned by the global preconditioner when the subproblems are solved with MI20 are given in Table 3. The symbol "-" means that MI20 fails; more specifically, the preconditioner generated by MI20 is not effective, causing the norm of the preconditioned residual vector resulting from applying one step of MI20 to blow up. This is not surprising because algebraic multigrid methods have difficulties when dealing with problems with small viscosity. For $\nu = 0.1$, we see some deterioration of the efficiency compared to the $\nu = 1$ case, but the method fails for $\nu = 0.01$ on all but the coarsest mesh. When the method works, it provides the same level of accuracy for velocity and vorticity as that obtained by using ILU as the approximate subproblem solver.

Lastly, we tried the inner-outer FGMRES scheme, both with ILU and MI20 as subproblem solvers. The inner GMRES stops if the relative residual norm has been reduced by at least six orders of magnitude or when GMRES reaches 50 iterations (and it reaches 50 iteration in almost all cases.). The results are presented in Tables 4 and 5. Comparing with the global preconditioner, in terms of iteration counts, the inner-outer scheme gives nearly grid-independent convergence rates. When the subsystems are solved with ILU, this is also the most efficient method of those tested.

3.2 Experiment 2: 3D channel flow over a step

For our next experiment, we investigate the effectiveness of the method on a 3D steady channel flow over a step with $\nu = 1/10$. The domain for the problem is shown in Figure 2, and we note the step has height of one unit. We use no-slip boundary conditions on the top, bottom, and sides of the channel, as well as the step. A zero-traction condition is used at the outflow, and for the inflow we use the velocity profile of steady 3D channel

ν	h	Vel	Vort	Newton	Setup time	Iter time	Iterations
1	1/2	1.25e-3	1.33e-3	3	0.25, 0.04	2.86, 4.16	100, 137
1	1/4	1.50e-4	1.53e-4	3	0.36, 0.37	15.03, 18.30	90, 99
1	1/8	1.83e-5	1.83e-5	3	3.89, 4.17	148.02, 471.58	99, 268
0.1	1/2	1.33e-3	2.57e-3	3	0.34, 0.04	2.66, 3.52	84, 113
0.1	1/4	1.55e-4	2.44e-4	3	0.44, 0.44	14.89, 24.06	81, 108
0.1	1/8	1.85e-5	2.28e-5	3	5.27, 4.72	143.88, 864.19	77, 408
0.01	1/2	2.76e-3	6.52e-3	3	0.24, 0.04	3.06, 14.12	108, 448
0.01	1/4	-	-	-	-	-	-
0.01	1/8	-	-	-	-	-	-

Table 3: Error, timings and iteration counts of GMRES with global preconditioner (MI20) for $\nu = 1,\ 0.1,\ 0.01$ for Ethier-Steinman problem.

ν	h	Vel	Vort	Newton	Setup time	Iter time	Iterations
1	1/2	1.25e-3	1.33e-3	3	0.13, 0.08	3.01, 3.00	4, 5
1	1/4	1.50e-4	1.53e-4	3	1.52, 1.54	13.54, 25.64	5, 8
1	1/8	1.83e-5	1.83e-5	3	44.22, 41.90	153.70, 212.24	5, 7
0.1	1/2	1.33e-3	2.57e-3	3	0.12, 0.07	1.61, 2.57	3, 5
0.1	1/4	1.55e-4	2.44e-4	3	1.38, 1.33	6.02, 17.22	3, 6
0.1	1/8	1.83e-5	2.29e-5	3	31.33, 31.73	93.71, 219.42	4, 7
0.01	1/2	2.76e-3	6.52e-3	3	0.12, 0.07	2.49, 7.72	4, 11
0.01	1/4	4.41e-4	1.09e-3	3	1.21, 1.26	14.08, 60.59	5, 13
0.01	1/8	4.82e-5	1.21e-4	3	25.52, 26.63	126.72, 474.08	5, 14

Table 4: Error, timings and iteration counts of inner-outer GMRES (ILU) for $\nu = 1,\ 0.1,\ 0.01$ for Ethier-Steinman problem.

flow (without a step), $Re = 10$ based on the height of the step. For the vorticity boundary condition, we use a Dirichlet condition to enforce it equal the nodal average of the curl of the velocity at the boundary.

A similar problem was considered by V. John in [11], but with constant inflow velocity $\mathbf{u}_{in} =< 0, 1, 0 >^T$. Such a boundary condition is not physical, although once the flow is into the channel several units, a more realistic velocity profile takes shape. However, since as meshwidth tends to zero, the components of vorticity tend to infinity (due to no slip on channel walls), and thus such an inflow condition is inappropriate for a velocity-vorticity method such as the one studied herein. Thus we alter the problem by first solving a channel flow problem without a step in primitive variables that has inflow as in [11], and use the outflow velocity as the inflow for our problem - this provides the same overall flow rate, but now with a more physical inflow condition that does not have very large inflow vorticity.

Solutions are computed for this problem using (P_2, P_1) Taylor-Hood elements, $\mathbf{f} = \nabla \times$

ν	h	Vel	Vort	Newton	Setup time	Iter time	Iterations
1	1/2	1.25e-3	1.33e-3	3	0.27, 0.04	2.53, 4.68	3, 5
1	1/4	1.50e-4	1.53e-4	3	0.44, 0.44	26.30, 39.43	4, 6
1	1/8	1.85e-5	1.84e-5	3	4.66, 4.60	224.47, 384.98	4, 7
0.1	1/2	1.33e-3	2.57e-3	3	0.07, 0.04	2.59, 4.34	3, 5
0.1	1/4	1.55e-4	2.44e-4	3	0.49, 0.44	15.38, 47.54	3, 6
0.1	1/8	1.86e-5	2.29e-5	3	6.05, 5.55	220.91, 607.12	3, 8
0.01	1/2	2.76e-3	6.52e-3	3	0.14, 0.04	4.64, 12.72	4, 12
0.01	1/4	4.41e-4	1.09e-3	3	0.44, 0.42	42.85, 523.39	6, 61
0.01	1/8	-	-	-	-	-	-

Table 5: Error, timings and iteration counts of inner-outer GMRES (MI20) for $\nu = 1,\ 0.1,\ 0.01$ for Ethier-Steinman problem.

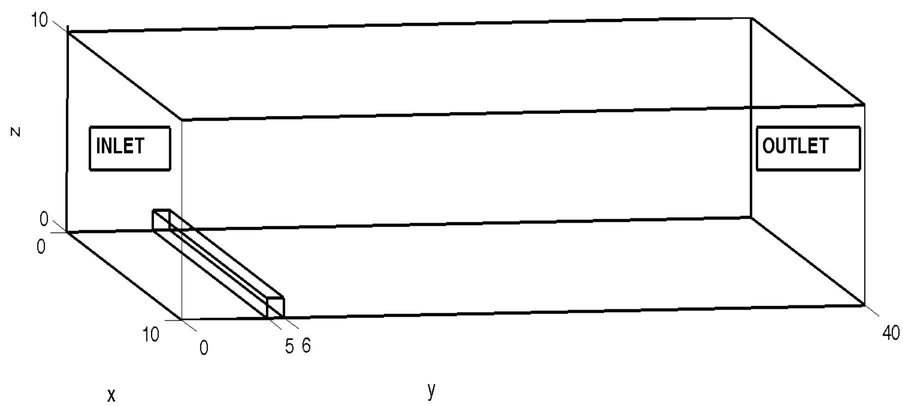

Figure 2: Shown above is the domain for the 3D channel flow over a step problem.

$\mathbf{f} = \mathbf{0}$, and $\gamma_1 = \gamma_2 = 1$. To test the method, we compute solutions on four different mesh levels; plots of the level 4 solution, which is resolved, are given in Figure 3, and we see the expected behavior: a smooth flow away from the step and recirculation behind the step (which can be seen in the zoomed in picture).

We now present the number of Newton's iterations, GMRES timings and iteration counts in Table 6 for exact solve, Table 7 for GMRES with global preconditioner (ILU) and Table 8 for inner-outer FGMRES.

The inner-outer FGMRES with ILU as the preconditioner for the inner iterations is the best of all the methods we tried. While the timings are comparable to the global preconditioner that does not converge, it converges to the desired tolerance and yields grid-independent convergence. Although GMRES with global preconditioner failed, a plot of its last iteration (omitted) shows a solution that appears to be correct. As expected, the direct solver is ineffective on larger systems, due to memory and speed.

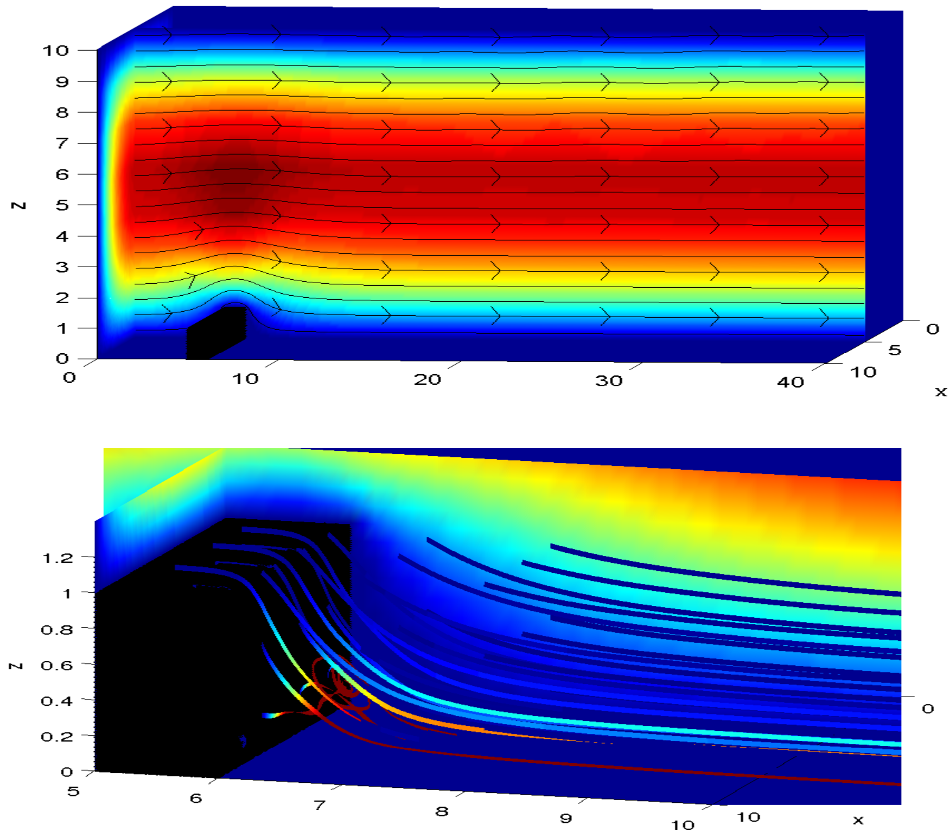

Figure 3: Streamlines over speed contours for the velocity solution, from the finest mesh, of the $\nu = 0.1$ step problem (top), and streamribbons of the velocity solution near the step (bottom).

4 Conclusions and future directions

We have found that the linear systems arising from a Newton linearization of the Galerkin finite element for the steady coupled VVH system can be solved effectively with an inner-outer FGMRES with ILU being used for the subproblems. This method is robust on our test problems, and is significantly faster than a direct solve.

Although we have found a solver much better than Matlab's sparse direct solver for this problem, the fact that we are solving the coupled VVH system makes our solves much less efficient than could be done for the velocity-pressure system. Hence only when the desired physical accuracy provided by the VVH system should one consider its use for equilibrium incompressible flow problems.

DOF	Newton	Time
Direct solve (backslash in MATLAB)		
25812	4	4.21
54324	5	63.15
115486	5	506.41
244060	-	-

Table 6: Timings of backslash

DOF	Newton	Setup time	Iter time	Iterations
GMRES with global preconditioner (ILU)				
25812	4	0.34	23.86	500*
54324	5	3.17	94.50	500*
115486	5	14.99	308.52	500*
244060	5	40.11	659.95	500*

Table 7: Timings of GMRES with global preconditioner (ILU)

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DOF	Newton	Setup time	Iter time	Iterations
Inner-outer GMRES (ILU)				
25812	4	0.37	20.98	13.5
54324	5	3.12	99.35	15.4
115486	5	14.41	345.11	17
244060	5	40.50	754.98	16.8

Table 8: Error, timings and iteration counts of inner-outer GMRES (ILU).

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