SHORT PATHS IN QUASI-RANDOM TRIPLE SYSTEMS WITH SPARSE UNDERLYING GRAPHS

JOANNA POLCYN, VOJTECH RÖDL, ANDRZEJ RUCIŃSKI, AND ENDRE SZEMERÉDI

A . The Frankl-Rödl regularity lemma for 3-uniform hypergraphs asserts that every large hypergraph can be decomposed into a bounded number of quasi-random structures consisting of a subhypergraph and a sparse underlying graph. In this paper we show that in such a quasi-random structure most pairs of the edges of the graph can be connected by hyperpaths of length at most twelve. Two applications are also given.

1. I

The Regularity Lemma from [9] is a powerful tool in contemporary graph theory and combinatorics. It allows one to partition every large graph into a bounded number of bipartite subgraphs, most of which are quasi-random, that is, they possess essentially all typical properties of corresponding random graphs. One of these properties, quite easy to prove, is that every two vertices with non-negligible neighborhoods can be connected by a path of length at most four (see, e.g., [6] and Corollary 8.5(a) in the Appendix below).

In this paper we study the much harder problem of the existence of short paths in 3-uniform, 3-partite hypergraphs with a certain regular structure related to the Frankl-Rödl regularity lemma in [1]. When this lemma is being applied, the initial hypergraph is broken into several quasi-random pieces and a desired structure is built from segments scattered among these highly regular substructures. It is then important to 'sew" them together by relatively short hyperpaths.

Two examples of this approach can be found in the forthcoming papers [8] and [2], where, respectively, the existence of Hamilton cycles in 3-uniform hypergraphs and the Ramsey numbers for hypercycles are treated. In both these applications, besides the Frankl-Rödl Lemma itself, a crucial role is played by the "Connection Lemma", analogous to, but much more complicated than the above mentioned result for graphs. The goal of this paper is to prove this " Connection Lemma" for quasi-random, 3-uniform hypergraphs.

In the next section, after some preliminary definitions, we state our main result, Theorem 2.9. Then, in section 3 we reformulate it in a more constructive way, specifying, in terms of their fourth neighborhoods, the edges that can be connected by short hyperpaths. Section 4 contains proofs of our main results, both relying on two lemmas, Lemma 4.1 and Lemma 4.2, which themselves will be proved in Sections 5 and 6. Section 7 presents briefly two applications of Theorem 2.9. One of them, a blow-up type result, guarantees a subhamiltonian path in a quasi-random 3-uniform hypergraph. The other approximates every large

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3-uniform hypergraph by finitely many pieces of small “diamater”. Finally, the Appendix collects elementary facts about ϵ-regular graphs which are used throughout the paper.

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2. P

Definition 2.1. *A 3-uniform hypergraph* is a pair $\mathcal{H} = (V, E)$, where V is a finite set of vertices and $E \subseteq \binom{V}{3}$ is a family of 3-element subsets of V called *hyperedges* or *triplets*. Throughout the paper we will often identify $\mathcal{H}$ with E.

We call $\mathcal{H}$ *3-partite* if there exists a partition $V = V_1 \cup V_2 \cup V_3$ such that for each $e \in E$ and for each $i = 1, 2, 3$ we have $e \cap V_i \neq \emptyset$. We refer to any 3-partite 3-uniform hypergraph $\mathcal{H}$ with a fixed 3-partition (V_1, V_2, V_3) as *a 3-graph*.

For an arbitrary hypergraph $\mathcal{H}$ and a graph G on the same vertex set, we denote by $\mathcal{H} - G$ the subhypergraph of $\mathcal{H}$ obtained by removing all hyperedges containing at least one edge of G.

The density and ϵ-regularity of bipartite graphs is measured by the ratio of edges to all potential edges (see the Appendix). For 3-graphs it is the ratio of hyperedges coinciding with the triangles of an underlying graph to all triangles in that graph.

Definition 2.2. For a 3-partite graph P with a fixed 3-partition $V_1 \cup V_2 \cup V_3$, we shall write $P = P^{12} \cup P^{23} \cup P^{13}$, where $P^{ij} = \{xy \in P : x \in V_i, y \in V_j\}$. Furthermore, let $Tr(P)$ be the set of all (vertex sets of) triangles formed by the edges of P. If $P = P^{12} \cup P^{23} \cup P^{13}$ is a 3-partite graph with the same vertex partition as $\mathcal{H}$, and moreover, $\mathcal{H} \subseteq Tr(P)$, then we say that P *underlies* $\mathcal{H}$.

The natural notion of density of $\mathcal{H}$ with respect to P counts the proportion of triangles of P which are triplets of $\mathcal{H}$, and then the δ-regularity of $\mathcal{H}$ means that for all $Q \subseteq P$ that contain a δ-fraction of $Tr(P)$, the densities of $\mathcal{H}$ with respect to such Q's are within δ from each other. However, it turns out that in some applications this is not strong enough. Therefore, the concept of so called (δ, r)-*regularity* was introduced in [1].

Definition 2.3. Let $r \geq 1$ be an integer and let $\mathcal{H}$ be a 3-graph with an underlying 3-partite graph $P = P^{12} \cup P^{23} \cup P^{13}$. Let $Q = (Q(1), ..., Q(r))$ be an r-tuple of 3-partite subgraphs $Q(s) = Q^{12}(s) \cup Q^{23}(s) \cup Q^{13}(s)$ satisfying that for all $s \in \{1, 2, ..., r\}$ and $1 \leq i < j \leq 3$, $Q^{ij}(s) \subseteq P^{ij}$. We define the *density* $d_{\mathcal{H}}(Q)$ of $\mathcal{H}$ with respect to Q as

$$d_{\mathcal{H}}(Q) = \frac{|\mathcal{H} \cap \bigcup_{s=1}^{r} Tr(Q(s))|}{|\bigcup_{s=1}^{r} Tr(Q(s))|}, \tag{1}$$

if $|\bigcup_{s=1}^{r} Tr(Q(s))| > 0$, and 0 otherwise.

Definition 2.4. Let an integer $r \geq 1$ and real numbers $0 < \alpha, \delta < 1$ be given. We say that a 3-graph $\mathcal{H}$ is (α, δ, r)-*regular* with respect to an underlying graph $P = P^{12} \cup P^{23} \cup P^{13}$ if for any r-tuple of subgraphs $Q = (Q(1), ..., Q(r))$ as above, if

$$|\bigcup_{s=1}^{r} Tr(Q(s))| > \delta |Tr(P)|,$$

then

$$|d_{\mathcal{H}}(Q) - \alpha| < \delta. \tag{2}$$

We say that $\mathcal{H}$ is (δ, r)*-regular* with respect to P if it is (α, δ, r)-regular for some α. Note that if $\mathcal{H}$ is (δ, r)-regular with respect to P, $\delta' \geq \delta$, and $r' \leq r$ is an integer, then $\mathcal{H}$ is also (δ', r')-regular with respect to P (with the same α). If $r = 1$, we just use the names δ*-regular* and (α, δ)*-regular*.

Setup 2.5. In what follows we always assume that $\mathcal{H}$ is a 3-graph and $P = P^{12} \cup P^{23} \cup P^{13}$ is a 3-partite graph, both with the same 3-partition $V = V(\mathcal{H}) = V(P) = V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = |V_3| = n$, and moreover, that P underlies $\mathcal{H}$, i.e., $\mathcal{H} \subseteq Tr(P)$.

Definition 2.6. Given $\mathcal{H}$ and P as in Setup 2.5, integers l and r and real numbers α, δ and ϵ, we call the pair $(\mathcal{H}, P)$ an $(\alpha, \delta, l, r, \epsilon)$*-triad* if

(i) each P^{ij}, $1 \leq i < j \leq 3$, is $(1/l, \epsilon)$-regular;
(ii) $\mathcal{H}$ is (α, δ, r)-regular with respect to P.

In particular, it follows that if $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r, \epsilon)$-triad then for all $1 \leq i < j \leq 3$ we have

$$(1/l - \epsilon)n^2 < |P^{ij}| < (1/l + \epsilon)n^2 \tag{3}$$

The hypergraph regularity lemma in [1] states that with the right choice of parameters, for every large 3-uniform hypergraph $\mathcal{H} = (V, E)$ the complete graph on V can be partitioned into finitely many graphs so that most triplets of $\mathcal{H}$ belong to $(\alpha, \delta, l, r, \epsilon)$-triad built upon these graphs. This paper studies the structure of $(\mathcal{H}, P)$ in such a typical situation.

There are several ways to define a path in a 3-uniform hypergraph, and we choose one in which the edges are glued along the path in the most tight way (see [5] and [3] for some study of paths and cycles defined in a "loose" way).

Definition 2.7. Let $\mathcal{H}$ be a 3-uniform hypergraph. *A hyperpath* of length $k \geq 0$ in $\mathcal{H}$ is a subhypergraph $\mathcal{P}$ of $\mathcal{H}$ consisting of $k + 2$ vertices and k hyperedges and whose vertices can be labelled $x_1, \ldots, x_{k+2}$ so that for each $i = 1, \ldots, k$, $x_i x_{i+1} x_{i+2} \in \mathcal{H}$. We then say that $\mathcal{P}$ *goes from the pair* $x_1 x_2$ *to the pair* $x_{k+2} x_{k+1}$ and these two pairs are called the *endpairs* of $\mathcal{P}$. The vertices $x_3, \ldots, x_k$ are called *internal*. Two paths are said to be *internally disjoint* if they do not share any internal vertex.

Remark 2.8. Note that the endpairs are ordered pairs of vertices. If $\mathcal{H}$ is a 3-partite hypergraph then the vertices of any hyperpath traverse the partition sets only in the cyclic order $V_1 \to V_2 \to V_3 \to V_1$, or in its reverse (see Figure 1). Hence, there are pairs of ordered pairs of vertices which, even in a complete 3-graph, are not connected by any hyperpath. Another consequence is that the lengths of paths connecting two given endpairs are equal modulo 3.

Throughout the paper we will be assuming that the cyclic ordering $V_1 \to V_2 \to V_3 \to V_1$ is canonical, and thus, specifying two unordered pairs of vertices, e and f, and saying that a hyperpath goes from e to f will not be ambiguous. (Note that under this convention a hyperpath from f to e is not a mere reverse of a path from e to f.)

Note also that unlike the graph case, the length of the shortest hyperpath between two given endpairs does not satisfy the triangle inequality, and thus cannot be called "distance".

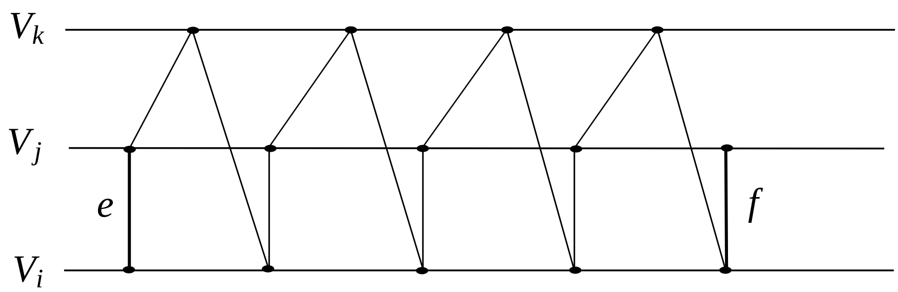
Our goal is to prove the following "Connection Lemma" which, in a way, extends a simple fact about graphs, Corollary 8.5(b) (see Appendix), to 3-uniform quasi-random hypergraphs. In addition, for the sake of future applications, we may force the hyperpaths to avoid a specified set of vertices S. A hyperpath $\mathcal{P}$ is called *S-avoiding* if $V(\mathcal{P}) \cap S = \emptyset$. Not to face the burden of computing yet another constant, we restrict S to have size only at most $n/\log n$. (The numerical constants are, clearly, not best possible.)

Theorem 2.9 (Connection Lemma)**.** *For all real $\alpha \in (0,1)$ and for all $\delta < \delta_0$, where*

$$\delta_0 = \frac{\alpha^{49}}{3^6 50^8 3000^{12}},$$

there exist two sequences $r(l)$ and $\epsilon(l)$ so that for all $\mathcal{H}$, P and for integer l if $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r(l), \epsilon(l))$-triad with $|V_1| = |V_2| = |V_3| = n$ sufficiently large, then there is a subgraph P_0 of at most $27\sqrt{\delta}n^2/l$ edges of P such that for every ordered pair of disjoint edges $(e, f) \in (P - P_0) \times (P - P_0)$, $e \cap f = \emptyset$, and for every set $S \subset V(\mathcal{H}) \setminus (e \cup f)$ of size $|S| \le n/\log n$, there is in $\mathcal{H} - P_0$ an S-avoiding hyperpath from e to f of length at most twelve.

Remark 2.10. In principle it might happen that an edge $e \in P - P_0$ is "isolated" in $\mathcal{H} - P_0$, that is, all triplets containing e also contain an edge of P_0. The conclusion of the above theorem ensures that this is not the case. In fact, all edges $e \in P - P_0$ are mutually connected by short hyperpaths within $\mathcal{H} - P_0$.



F 1. A hyperpath of length 12 from e to f. Every 3 consecutive vertices on the path form a hyperedge.

3. C

As mentioned earlier, in the case of (d, ϵ)-regular graphs, it is easy to see that for every pair of vertices with at least ϵn neighbors each, there is a short path (of length at most four) between them (see, e.g., [6] and the Appendix below). In fact, see [7], every two vertices of degree at least $16(\epsilon^2/d)n$ can be connected by a path of length at most five.

The quantification of Theorem (note that "there exist sequences $r(l)$ and $\epsilon(l)$" translates to "for all l there exist r and ϵ") indicates the possibility of the following hierarchy of constants:

$$\alpha \gg \delta \gg 1/l \gg 1/r, \epsilon,$$

where $\beta \gg \gamma$ means that γ is sufficiently smaller than β, or that γ is chosen only after β is being fixed.

Polcyn [6], working under a comfortable assumption that $\delta \ll 1/l$, proved that most edges of P can be mutually connected by hyperpaths of length at most seven. Typical edges were defined in [6] in terms of the first and second neighborhood in $\mathcal{H}$. Here, with δ and $1/l$ swapped in the hierarchy, to formulate a constructive version of Theorem 2.9, we need to look into the fourth neighborhood of an edge.

Let us begin by defining the first neighborhood.

Definition 3.1. Let $\mathcal{H}$ be a 3-uniform hypergraph and let $e = \{x, y\}$ be a pair of vertices in $V = V(\mathcal{H})$. We define the *hypergraph neighborhood* of e to be $\Gamma_{\mathcal{H}}(e) = \{z \in V : \{z, x, y\} \in \mathcal{H}\}$. The vertices in $\Gamma_{\mathcal{H}}(e)$ will be called *neighbors of* e.

Note that in a 3-graph $\mathcal{H}$ with an underlying graph $P = P^{12} \cup P^{23} \cup P^{13}$, if $e \in P^{ij}$ then $\Gamma_{\mathcal{H}}(e) \subseteq V_k$, where $\{i, j, k\} = \{1, 2, 3\}$.

Imagine that both, $\mathcal{H}$ and P are chosen at random as a result of the following 2-round experiment. First, create P by tossing a coin over each pair in $(V_1 \times V_2) \cup (V_2 \times V_3) \cup (V_1 \times V_3)$ independently with the success probability $1/l$, then create $\mathcal{H}$ by selecting each triangle of P with probability α. In such a random hypergraph the expected number of triplets is $\alpha n^3/l^3$ and, for a given edge of P (here we condition that e has been selected), the expected value of $|\Gamma_{\mathcal{H}}(e)|$ equals $\alpha n/l^2$. It is proved in [6] that if $(\mathcal{H}, P)$ is an (deterministic) $(\alpha, \delta, l, 1, \epsilon(l))$-triad, then for almost all edges of P, $|\Gamma_{\mathcal{H}}(e)|$ is close to the above expectation.

Fact 3.2 ([6]). *For all real $\alpha > 0$ and $\delta > 0$, there exists a sequence $\epsilon(l) > 0$ such that for all integer $l \geq 1$, whenever $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, 1, \epsilon(l))$-triad then all but at most $7\sqrt{\delta}n^2/l$ edges of P^{ij}, $1 \leq i < j \leq 3$, satisfy the inequalities*

$$n\left(\frac{1}{l} - \epsilon\right)^2 (\alpha - \delta) < |\Gamma_{\mathcal{H}}(e)| < (\alpha + \delta)\left(\frac{1}{l} + \epsilon\right)^2 n.$$

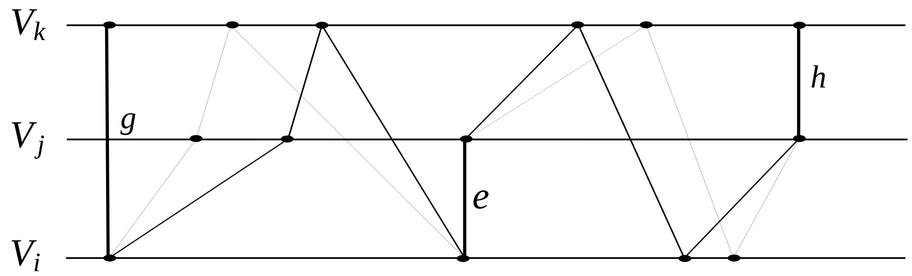
Remark 3.3. In [6], the above inequalities contain only the term $\sqrt[4]{\delta}$ in place of δ, but the same proof yields also Fact 3.2 in the present form. (The constant 7 instead of 6 in [6] comes from considering here all edges of P^{ij}, and not just the proper ones.)

However, for reasons which will be explained later, to guarantee short connections (via hyperpaths) of an edge $e \in P$ with most of the other edges of $\mathcal{H}$, we will need to look four steps ahead.

Definition 3.4. Let e_1, e_2 be edges of P. We say that e_1 *reaches* e_2 *within* $\mathcal{H}$ *in* k *steps and in* t *ways* if there exist at least t internally disjoint hyperpaths in $\mathcal{H}$ of length k from e_1 to e_2. For $t = 1$ we will skip the phrase "in t ways". For an edge $e \in P$, we denote by

$Four^{+}(e, \mathcal{H})$ the set of those edges of P, which are reached from e within $\mathcal{H}$ in four steps and in $\gamma_0 n$ ways, and by $Four^{-}(e, \mathcal{H})$ the set of all edges of P which reach e within $\mathcal{H}$ in four steps and in $\gamma_0 n$ ways (see Figure 2), where

$$\gamma_0 = \frac{\alpha^4}{5000 l^7}.$$



F 2. The fourth neigborhoods of e ($g \in Four^{-}(e, \mathcal{H})$, $h \in Four^{+}(e, \mathcal{H})$).

Let us now provide some intuition for why it is necessary to consider the fourth hypergraph neighborhood of a graph edge. Suppressing α, δ, ϵ, most edges of P belong to about n/l^2 triplets of $\mathcal{H}$ (see Fact 3.2), but any such edge e can be completely cut off from the rest of $\mathcal{H}$ if no stronger assumption is made. Indeed, the total number of triplets extending triplets containing e is of the order n^2, and clearly the removal of such a tiny fraction of triplets cannot affect the δ-regularity which “controls” only sets of hyperedges of size, roughly, n^3/l^3.

In two steps, only about n^2/l^4 edges are reached from a typical edge. Most of them extend to about n/l^2 triplets, a total of n^3/l^6 – still much less than n^3/l^3 if l is large. To estimate the number of edges reached from a typical edge in three steps, the quantity n^3/l^6 has to be divided, due to repetitions, by, roughly, n/l^4 (the number of vertices forming triangles with two given, disjoint edges), yielding only n^2/l^2 edges. Again, they belong to about $n^3/l^4 \ll \delta n^3/l^3$ triplets – a quantity not under control. Hence, the shortest distance at which a typical edge can reach a substantial number of other edges is four.

Theorem 3.5 below states that, indeed, most edges have large fourth neighborhood, and, more importantly, edges with large fourth neighborhood are mutually connected by short hyperpaths.

Let us denote by $R_0(\mathcal{H}) = R_0$ the set of all edges of P, for which

$$\min\left(|Four^{+}(e, \mathcal{H})|, |Four^{-}(e, \mathcal{H})|\right) < \left(\frac{\alpha^4}{2000}\right)\frac{n^2}{l}.$$

Theorem 3.5. *For all real $\alpha \in (0, 1)$ and $\delta < \delta_0$, where δ_0 is as in Theorem 2.9 there exist two sequences $r(l)$ and $\epsilon(l)$ such that for all $\mathcal{H}$, P and for all integer l, if $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r(l), \epsilon(l))$-triad with $|V_1| = |V_2| = |V_3| = n$ sufficiently large, then*

(i) $|R_0| \leq 27\sqrt{\delta}n^2/l$, *and*

(ii) *for every ordered pair of disjoint edges* $(e, f) \in (P - R_0) \times (P - R_0)$ *and for every set* $S \subset V(\mathcal{H}) \setminus (e \cup f)$ *of size* $|S| \le n/\log n$*, there is in* $\mathcal{H}$ *an* S*-avoiding hyperpath from* e *to* f *of length at most twelve.*

4. T

Theorems 2.9 and 3.5 are straightforward consequences of two technical lemmas. A subgraph A of $P = P^{12} \cup P^{23} \cup P^{13}$ is called *framed* if for some $1 \le i < j \le 3$, $A \subseteq P^{ij}$. Our first lemma needs only the assumption that $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r(l), \epsilon(l))$-triad, where $r \equiv 1$.

Lemma 4.1. *For all* $c \in (0, 1)$ *and* $\alpha \in (0, 1)$ *and for all* $\delta < \delta_1$*, where*

$$\delta_1 = \frac{\alpha c^{12}}{3^6 50^8},$$

there exists a sequence $\epsilon(l)$ *so that for all* $\mathcal{H}$*,* P *and integer* l *if* $(\mathcal{H}, P)$ *is an* $(\alpha, \delta, l, 1, \epsilon(l))$*-triad with* $|V_1| = |V_2| = |V_3| = n$ *sufficiently large, then the following is true: For every subgraph* $P_1 \subset P$*, where* $|P_1| \le 29\sqrt{\delta}n^2/l$*, and for every pair of framed subgraphs* A *and* B *of* $P - P_1$*, each of size at least* cn^2/l*, there exist edges* $a \in A$ *and* $b \in B$ *and a hyperpath in* $\mathcal{H} - P_1$ *from* a *to* b *of length at most four.*

Our second lemma asserts that for a typical $(\mathcal{H}, P)$, apart from a small set of edges P_0, all other edges of P have their fourth neighborhood substantial, even if the edges of P_0 are to be avoided. This lemma needs the whole strength of the (δ, r)-regularity.

Lemma 4.2. *For all real* $\alpha \in (0, 1)$ *and for all* $\delta < \delta_2$*, where*

$$\delta_2 = \frac{\alpha^2}{180^2},$$

there exist two sequences $r(l)$ *and* $\epsilon(l)$ *such that for all* $\mathcal{H}$*,* P *and integers* l *if* $(\mathcal{H}, P)$ *is an* $(\alpha, \delta, l, r(l), \epsilon(l))$*-triad with* $|V_1| = |V_2| = |V_3| = n$ *sufficiently large, then there exists* $P_0 \subset P$*,* $|P_0| \le 27\sqrt{\delta}n^2/l$*, such that*

$$\min\left(\left|Four^+(e, \mathcal{H} - P_0)\right|, \left|Four^-(e, \mathcal{H} - P_0)\right|\right) \ge \left(\frac{\alpha^4}{2000}\right)\frac{n^2}{l} \tag{4}$$

for all $e \in P - P_0$*.*

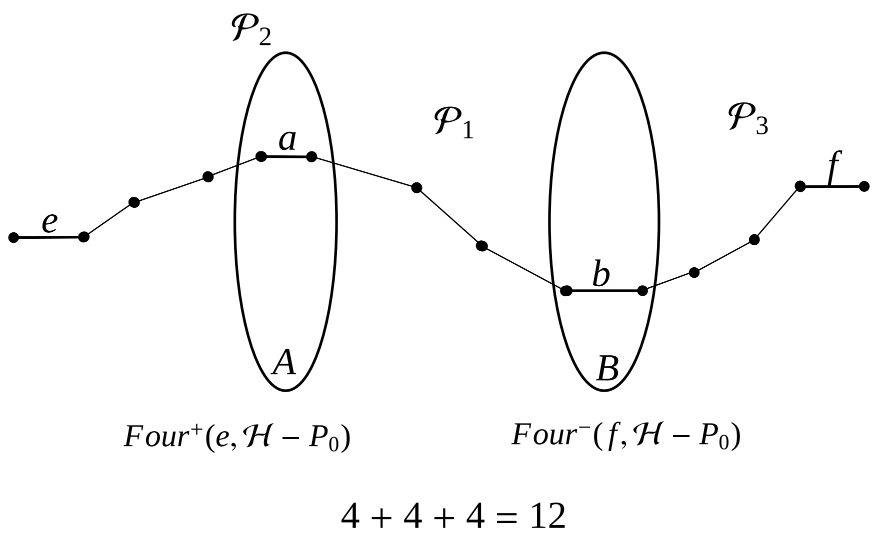
¿From Lemmas 4.1 and 4.2 we immediately derive our main result.

Proof of Theorem 2.9. Note that for $c = \alpha^4/3000$, $\delta_0 = \delta_1 < \delta_2$. Given α and $\delta < \delta_0$, let $\epsilon_1(l)$ satisfy Lemma 4.1 with $c = \frac{\alpha^4}{3000}$, and let sequences $r(l)$, and $\epsilon_2(l)$ satisfy Lemma 4.2. We claim that Theorem 2.9 is true with the above choice of $r(l)$ and with $\epsilon(l) = \min(\epsilon_1(l), \epsilon_2(l))$.

Indeed, consider any $\mathcal{H}$, P and l such that $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r(l), \epsilon(l))$-triad and apply Lemma 4.2. It follows that there exists $P_0 \subset P$, $|P_0| \le 27\sqrt{\delta}n^2/l$, such that (4) holds for all $e \in P - P_0$. Fix disjoint $e, f \in P - P_0$, and a set $S \subset V(\mathcal{H}) \setminus (e \cup f)$ of size $|S| \le n/\log n$.

Define $P_S = \{e \in P : S \cap e \neq \emptyset\}$ and observe that $|P_S| = o(n^2)$, and thus, for large n, $|P_0 \cup P_S| \leq 29\sqrt{\delta}n^2/l$, and

$$\min\left(|Four^+(e, \mathcal{H} - P_0)|, |Four^-(e, \mathcal{H} - P_0)|\right) - |P_S| \geq \left(\frac{\alpha^4}{3000}\right)\frac{n^2}{l}.$$



F 3. A hyperpath from e to f. (An illustraction of the proof of Theorem 2.9)

Since $(\mathcal{H}, P)$ is also an $(\alpha, \delta, l, 1, \epsilon_1(l))$-triad, we may apply Lemma 4.1 with $c = \frac{\alpha^4}{3000}$ to

$$A = Four^+(e, \mathcal{H} - P_0) \setminus P_S, \quad B = Four^-(f, \mathcal{H} - P_0) \setminus P_S \quad \text{and} \quad P_1 = P_0 \cup P_S,$$

obtaining edges $a \in A$ and $b \in B$, and a hyperpath $\mathcal{P}_1$ in $\mathcal{H} - (P_0 \cup P_S)$ from a to b of length at most four.

Let $I = V(\mathcal{P}_1) \cup f \setminus a$. Among at least $\gamma_0 n > |I \cup S|$ (for large n) internally disjoint hyperpaths from e to a in $\mathcal{H} - P_0$ choose one which is disjoint from $I \cup S$, obtaining an S-avoiding hyperpath $\mathcal{P}_2$ in $\mathcal{H} - P_0$ from e to b of length at most eight. Finally, set $J = V(\mathcal{P}_2) \setminus b$ and choose a hyperpath $\mathcal{P}_3$ in $\mathcal{H} - P_0$ from b to f which avoids the vertices of $J \cup S$. This way we obtain an S-avoiding hyperpath in $\mathcal{H} - P_0$ from e to f of length at most twelve (see Figure 3).

□

Proof of Theorem 3.5. Since $R_0 \subseteq P_0$, where P_0 is as in Lemma 4.2, part (i) follows from the estimate on $|P_0|$. The proof of part (ii), is very similar to that of Theorem 2.9. We define P_S as before and apply Lemma 4.1 with $c = \frac{\alpha^4}{3000}$ to

$$A = Four^+(e, \mathcal{H}) \setminus P_S, \quad B = Four^-(f, \mathcal{H}) \setminus P_S \quad \text{and} \quad P_1 = P_S,$$

obtaining edges $a \in A$ and $b \in B$, and a hyperpath $\mathcal{P}_1$ in $\mathcal{H} - P_S$ from a to b of length at most four. Finally, we extend $\mathcal{P}_1$ to an S-avoiding hyperpath in $\mathcal{H}$. □

Remark 4.3. It will follow from the proof of Lemma 4.1 that, in fact, depending on the position of the sets A and B, the promised hyperpath is precisely of length two, three or four. Consequently, depending on the position of e and f, the length of a hyperpath from e to f, guaranteed by Theorems 2.9 and 3.5, is precisely ten, eleven or twelve.

5. S

In this section we prove Lemma 4.1. We begin with formulating a claim from which the lemma will follows quite easily. Let E be any framed subgraph of P. Then $First^+(E,\mathcal{H})$ and $Second^+(E,\mathcal{H})$ denote the sets of all edges $h \in P$ reached in $\mathcal{H}$ by an edge $g \in E$ in one and, respectively, in two steps. Sets $First^-(E,\mathcal{H})$ and $Second^-(E,\mathcal{H})$ are defined similarly, by replacing the phrase "reached in $\mathcal{H}$ by an edge $g \in E$" by "reaching in $\mathcal{H}$ an edge $g \in E$". Throughout, ijk always stands for any one of the sequences: 123 or 231 or 312, that is, sequences which follow the cyclic ordering 1231.

Claim 5.1. *For all $c \in (0,1)$ and $\alpha \in (0,1)$, all $0 < \delta < \min(\alpha, c^6/50^8)$ and sequences $0 < \epsilon(l) \le \frac{\sqrt{\delta}}{10l^3}$, and all integers $l \ge 1$, if $(\mathcal{H}, P)$ is an $(\alpha,\delta,l,1,\epsilon(l))$-triad with $|V_1| = |V_2| = |V_3| = n$ sufficiently large, then for all $P_1 \subset P$ of size $|P_1| \le 29\sqrt{\delta}n^2/l$ and for all sets $E \subseteq P^{ij} - P_1$ of size $|E| \ge cn^2/l$,*

$$\min\left(|First^+(E,\mathcal{H}-P_1)|, |First^-(E,\mathcal{H}-P_1)|\right) \ge \frac{c}{6}\frac{n^2}{l}, \tag{5}$$

$$\min\left(|Second^+(E,\mathcal{H}-P_1)|, |Second^-(E,\mathcal{H}-P_1)|\right) \ge \left(1 - \frac{4\delta^{1/8}}{\sqrt{c}}\right)\frac{n^2}{l}, \tag{6}$$

In order to derive Lemma 4.1 from Claim 5.1 we need one more simple fact about vertex-disjoint subgraphs of bipartite graphs.

Fact 5.2. *Let A and B be two bipartite graphs with the same bipartition $V_1 \cup V_2$, $|V_1| = |V_2| = n$. Then there exist $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \ge \frac{1}{2}|A| - \frac{1}{2}\Delta_2(A)$, $|B'| \ge \frac{1}{2}|B|$ and $V(A') \cap V(B') \cap V_2 = \emptyset$, where $\Delta_2(A)$ is the maximum degree in A among the vertices of V_2.*

Proof. Let us put vertices of the set V_2 in two lines: one ordered by their degrees in A in the descending manner (line A), the second – the same with respect to B (line B). Now include the first vertex on line B to B' and remove it from both lines. We repeat this step for line A and then again for B and so on until all vertices are placed in one of the sets A' or B'. (Note that $|V(A')| = \lfloor n/2 \rfloor$ and $|V(B')| = \lceil n/2 \rceil$.)

Along the way, let us match each vertex of B included to B' with the one included to A' in the very next step (if n is odd, the vertex included to B' last remains unmatched). Because we have started with the vertex of the largest degree in B, its match has a smaller or equal degree in B, and this is true for each matched pair. Therefore, we have $|B'| \ge \frac{1}{2}|B|$. To prove that $|A'| \ge \frac{1}{2}|A| - \frac{1}{2}\Delta_2(A)$ we apply the same reasoning to the set A minus all edges incident to the first vertex included to B'. □

Proof of Lemma 4.1. Given c and α, let

$$\delta < \delta_1 = \frac{\alpha c^{12}}{3^6 50^8} < \frac{\alpha (c/3)^6}{50^8} \tag{7}$$

and

$$\epsilon(l) = \frac{\sqrt{\delta}}{10l^3}$$

Note that $\delta < \alpha$, and

$$1 - \frac{4\delta^{\frac{1}{8}}}{\sqrt{c/3}} + \frac{c}{12} > 1 + \epsilon(l)l, \tag{8}$$

the latter by inequalities $\delta^{\frac{1}{8}} < c\sqrt{c}/(50\sqrt{3})$ and $\epsilon(l)l < c/50^2$.

Let $\mathcal{H}$, P, l, A, B, and P_1 be as in Lemma 4.1. Without loss of generality we assume that $A \subseteq P^{12}$ and will consider all three cases for B.

If $B \subseteq P^{13}$, apply Claim 5.1 with $E = A$ to obtain a set $A^{13} = Second^+(A, \mathcal{H} - P_1) \subseteq P^{13} - P_1$ of at least

$$\left(1 - \frac{4\delta^{\frac{1}{8}}}{\sqrt{c}}\right)\frac{n^2}{l}$$

edges. By (3) and (8), we conclude that $B \cap A^{13} \neq \emptyset$, implying the existence of a hyperpath within $\mathcal{H} - P_1$ from an edge $a \in A$ to an edge $b \in B$ of length two.

If $B \subseteq P^{12}$, we use Fact 5.2 to obtain two subgraphs $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq (c/3)n^2/l$ (for n sufficiently large), $|B'| \geq (c/2)n^2/l$ and $V(A') \cap V(B') \cap V_2 = \emptyset$. Then by Claim 5.1 applied with $c := c/3$, the set $A^{13} = Second^+(A', \mathcal{H} - P_1) \subseteq P^{13} - P_1$ has cardinality at least

$$\left(1 - \frac{4\delta^{\frac{1}{8}}}{\sqrt{c/3}}\right)\frac{n^2}{l},$$

and taking $B^{13} = First^-(B', \mathcal{H} - P_1) \subseteq P^{13} - P_1$, by Claim 5.1 applied with $c := c/2$, we have

$$|B^{13}| \geq \frac{c}{12}\frac{n^2}{l}.$$

Again, by (3) and (8), we conclude that $B^{13} \cap A^{13} \neq \emptyset$. Let $zu \in B^{13} \cap A^{13}$ and let $xyzu$ and zuv be hyperpaths, respectively, from $a = xy$ to uz and from zu to $b = vu$. Note that by the disjoint choice of A' and B' we have $y \neq v$, and so $xyzuv$ is a hyperpath within $\mathcal{H} - P_1$ from $a \in A$ to $b \in B$ of length three.

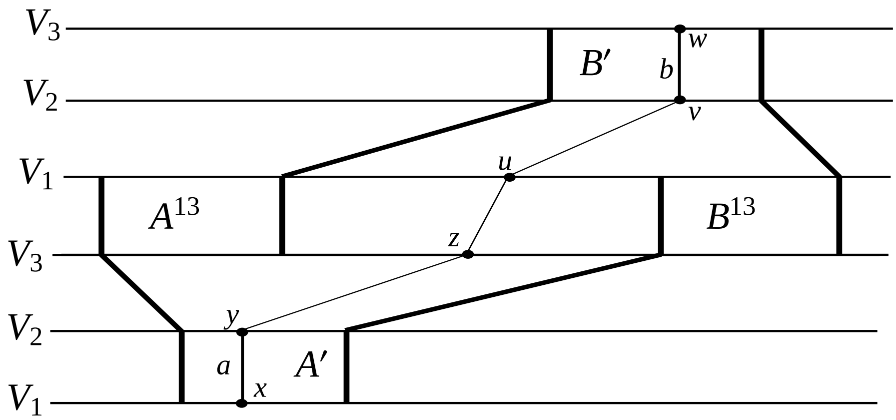
The last case is when $B \subseteq P^{23}$. Here also we apply Fact 5.2 to obtain two subgraphs $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq (c/3)n^2/l$, $|B'| \geq (c/2)n^2/l$ and $V(A') \cap V(B') \cap V_2 = \emptyset$. (Technically, we identify for a moment sets V_1 and V_3 to treat A and B as two bipartite graphs on the same vertex set.) By Claim 5.1 applied with $c := c/3$, the set $A^{13} = Second^+(A', \mathcal{H} - P_1) \subseteq P^{13} - P_1$ consists of at least

$$\left(1 - \frac{4\delta^{\frac{1}{8}}}{\sqrt{c/3}}\right)\frac{n^2}{l}$$

edges, and taking $B^{13} = Second^-(B', \mathcal{H} - P_1) \subseteq P^{13} - P_1$, by Claim 5.1 applied with $c := c/2$, we get

$$|B^{13}| \geq \left(1 - \frac{4\delta^{\frac{1}{8}}}{\sqrt{c/2}}\right)\frac{n^2}{l}.$$

Again, by (3), (8) and (7), we conclude that $B^{13} \cap A^{13} \neq \emptyset$. Let $zu \in B^{13} \cap A^{13}$ and let $xyzu$ and $zuvw$ be hyperpaths, respectively, from $a = xy$ to uz and from zu to $b = wv$. Note that by the disjoint choice of A' and B' we have $y \neq v$, and so $xyzuvw$ is a hyperpath within $\mathcal{H} - P_1$ from $a \in A$ to $b \in B$ of length four (see Figure 4).



F 4. An illustraction of the last case of the proof of Lemma 4.1.

□

It remains to prove Claim 5.1. We first show a simple but crucial fact which will be applied twice in the proof of Claim 5.1.

Fact 5.3. *For any real $\alpha, \delta \in (0, 1)$, integer l and $\epsilon \leq \sqrt{\delta}/(10l^3)$, let $(\mathcal{H}, P)$ be an $(\alpha, \delta, l, 1, \epsilon)$-triad. Let, further, $A \subseteq V_j$, $B \subseteq V_i$ and $Q \subseteq P$ be such that $|P^{ki} - Q| \leq 29\sqrt{\delta}n^2/l$, $|A| \geq an$, $|B| \geq bn$, and every vertex of A has in Q at least $\beta|B|/l$ neighbors in B and at least $\gamma n/l$ neighbors in V_k. If $\min(\gamma, b\beta) > \epsilon l$ and $ab\beta\gamma \geq 43\sqrt{\delta}$ then $|\mathcal{H} \cap Tr(Q)| > 2(\alpha - \delta)\delta n^3/l^3$.*

Proof. By the $(1/l, \epsilon)$-regularity of P^{ki}, we have

$$|Tr(Q \cup P^{ik})| \geq \sum_{y \in A} deg_Q(y, B) deg_Q(y, V_k) \left(\frac{1}{l} - \epsilon\right) \geq \frac{9}{10} ab\beta\gamma \frac{n^3}{l^3}.$$

On the other hand, setting $Q' = P^{ij} \cup P^{jk} \cup (P^{ki} - Q)$, by Corollary 8.4, we have

$$|Tr(Q')| < 29\sqrt{\delta}(1.21)\frac{n^3}{l^3} + 4\epsilon n^3 < 36\sqrt{\delta}\frac{n^3}{l^3},$$

and so, by our assumptions and Fact 8.6, we may estimate

$$|Tr(Q)| \geq |Tr(Q \cup P^{ki})| - |Tr(Q')| \geq \left(\frac{9}{10} 43\sqrt{\delta} - 36\sqrt{\delta}\right)\frac{n^3}{l^3} > 2\delta \frac{n^3}{l^3} > \delta |Tr(P)|.$$

Therefore, by the $(\alpha, \delta, 1)$-regularity of $\mathcal{H}$,

$$d_{\mathcal{H}}(Q) = \frac{|\mathcal{H} \cap Tr(Q)|}{|Tr(Q)|} > \alpha - \delta. \quad \square$$

Proof of Claim 5.1. By symmetry, it is enough to prove only that $|First^+(E, \mathcal{H} - P_1)| \geq (c/6)n^2/l$ and similarly, that $|Second^+(E, \mathcal{H} - P_1)| \geq \left(1 - 4\delta^{1/8}/\sqrt{c}\right)n^2/l$. Let us fix α and c, $0 < \alpha, c < 1$, and let

$$\delta < \min\left(\alpha, \frac{c^6}{50^8}\right). \tag{9}$$

Further, with δ given above, let for all l

$$\epsilon(l) \leq \frac{\sqrt{\delta}}{10l^3} < \frac{\delta^{\frac{1}{4}}}{l\sqrt{c}} < \frac{c}{120l}. \tag{10}$$

Let $\mathcal{H}$, P, and l be as in Claim 5.1. Set $\epsilon = \epsilon(l)$ for convenience and fix $1 \leq i < j \leq 3$. Let E be a set of at least cn^2/l edges of $P^{ij} - P_1$. Define

$$E_1 = \{yz \in P^{jk} : \quad xyz \in \mathcal{H} \text{ and } xy \in E \text{ and } xz \notin P_1 \text{ for some } x \in V_i\}$$

and assign to each edge yz of E_1 one (arbitrary) vertex $x = x_{yz} \in V_i$ which together with yz satisfies the conditions in the definition of E_1. Finally, let

$$E_2 = \bigcup_{yz \in E_1 - P_1} \{zw \in P^{ki} : \quad w \neq x_{yz} \text{ and } yzw \in \mathcal{H} \text{ and } yw \notin P_1\}.$$

Note that $E_1 - P_1 = First^+(E, \mathcal{H} - P_1)$, and that by avoiding $w = x_{zy}$, $E_2 - P_1 \subseteq Second^+(E, \mathcal{H} - P_1)$.

Observation 5.4. Trivially, if $xy \in E$, $yz \in P^{jk} - E_1$ and $xz \notin P_1$ then $xyz \notin \mathcal{H}$. Similarly, but more subtly, if $yz \in E_1 - P_1$, $zw \in P^{ki} - E_2$, and $yw \notin P_1$, then $yzw \notin \mathcal{H}$ unless $w = x_{yz}$, which implies that the edges of $E_1 - P_1$, $P^{ki} - E_2$ and $P^{ij} - P_1$ span at most $|E_1 - P_1|$ hyperedges in $\mathcal{H}$.

Using these observations and Fact 5.3 we will first show that a significant fraction of vertices $y \in V_j$ have large (close to n/l) neighborhood in E_1, and so subgraph $E_1 - P_1$ is large. Then we will argue that most vertices of V_k have large degree in E_2, meaning that the set E_2 must be very large (close to n^2/l), and so must be $E_2 - P_1$.

Let

$$L_0 = \left\{y \in V_j : \ deg_{P^{jk}}(y) < \left(\frac{1}{l} - \epsilon\right)n\right\}.$$

By (19) with $A = V_k$, we have $|L_0| \leq \epsilon n$. Next, let us consider the set

$$L = \left\{y \in V_j - L_0 : \ deg_E(y) \geq \frac{cn}{2l}\right\}.$$

Observe that $|L| \geq cn/3$. Indeed, otherwise, using (18) and (10), we obtain a contradiction

$$|E| < |L|n\left(\frac{1}{l} + \epsilon\right) + |L_0|n\left(\frac{1}{l} + \epsilon\right) + \epsilon n^2 + n\frac{cn}{2l} < \frac{cn^2}{l}.$$

We proceed with the following fact. Set $\overline{E}_1 = P^{jk} - E_1$ and

$$L' = \left\{y \in L : deg_{\overline{E}_1}(y) > 7\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\frac{n}{l}\right\}.$$

Fact 5.5. $|L'| \leq 13\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}n$

Proof. Assume $|L'| > 13(\delta^{\frac{1}{4}}/\sqrt{c})n$ and apply Fact 5.3 with $A = L'$, $B = V_i$ (and so $b = 1$), $\beta = c/2$, $a = 13\delta^{\frac{1}{4}}/\sqrt{c}$, and $\gamma = 7\delta^{\frac{1}{4}}/\sqrt{c}$ to the 3-partite subgraph $Q = Q^{ij} \cup Q^{jk} \cup Q^{ki}$, where

$$\begin{aligned} Q^{ij} &= E[V_i, L'], \\ Q^{jk} &= \overline{E}_1[L', V_k] \\ Q^{ki} &= P^{ki} - P_1. \end{aligned}$$

As $\min(\gamma, b\beta) > \epsilon l$ and $ab\beta\gamma > 43\sqrt{\delta}$, it follows that $\mathcal{H} \cap Tr(Q) \neq \emptyset$. However, by the construction of Q (see Observation 5.4) we have $\mathcal{H} \cap Tr(Q) = \emptyset$. This contradiction ends the proof of Fact 5.5. □

We set $L'' = L - L'$. By (9),

$$|L''| = |L| - |L'| \geq \frac{1}{3}cn - 13\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}n > \frac{1}{4}cn > \epsilon n. \tag{11}$$

Note that every vertex $y \in L''$ has

$$deg_{E_1}(y) > \frac{n}{l} - \epsilon n - 7\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\frac{n}{l},$$

and thus, by (9), (10) and (11),

$$|E_1| \geq |L''|\left(\frac{n}{l} - \epsilon n - 7\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\frac{n}{l}\right) > \frac{c}{5}\frac{n^2}{l}.$$

To complete the proof of the inequality (5) we count the number of edges in $E_1 - P_1$

$$|E_1 - P_1| > \frac{c}{5}\frac{n^2}{l} - 29\sqrt{\delta}\frac{n^2}{l} > \frac{c}{6}\frac{n^2}{l},$$

the latter by (9).

We continue with the proof of the inequality (6). Let $\overline{E}_2 = P^{ki} - E_2$. Note that by the $(1/l, \epsilon)$-regularity of P^{jk} and P^{ki}, Fact 8.3 and (11), the set

$$Ł_0 = \left\{z \in V_k : deg_{P^{jk}}(z, L'') < \left(\frac{1}{l} - \epsilon\right)|L''| \text{ or } deg_{P^{ki}}(z) < \left(\frac{1}{l} - \epsilon\right)n\right\}$$

has size $|Ł_0| \leq 2\epsilon n$. Next, let us consider the set

$$Ł_1 = \left\{z \in V_k : deg_{\overline{E}_1}(z, L'') > 7\frac{\delta^{\frac{1}{8}}}{l}|L''|\right\}.$$

Since each vertex of L'' has in $\overline{E}_1$ degree at most $7(\delta^{\frac{1}{4}}/\sqrt{c})\frac{n}{l}$, a simple, double counting argument shows that $|Ł_1| \leq (\delta^{\frac{1}{8}}/\sqrt{c})n$. Further, let

$$Ł_2 = \left\{z \in V_k : deg_{P_1}(z, L'') > 116\frac{\delta^{\frac{1}{4}}}{\sqrt{c}l}|L''|\right\}.$$

Clearly, $|\text{Ł}_2| < (\delta^{\frac{1}{4}}/\sqrt{c})n$, since otherwise $|P_1| > 29\sqrt{\delta}n^2/l$ – a contradiction. Set $\text{Ł} = V_k \setminus (\text{Ł}_0 \cup \text{Ł}_1 \cup \text{Ł}_2)$ and define

$$\text{Ł}' = \left\{ z \in \text{Ł} : deg_{\overline{E}_2}(z) > 9\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\frac{n}{l} \right\}.$$

Observe, by (9) and (10), that for all $z \in \text{Ł}$ (and thus for all $z \in \text{Ł}'$) we have

$$deg_{E_1 - P_1}(z, L'') > \left(\frac{1}{l} - \epsilon - 7\frac{\delta^{\frac{1}{8}}}{l} - 116\frac{\delta^{\frac{1}{4}}}{\sqrt{c}l} \right)|L''| > \frac{4}{5l}|L''|.$$

Fact 5.6. $|\text{Ł}'| \le 24\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}n$.

Proof. The proof of this fact is very similar to the proof of Fact 5.5. We will argue that the inequality $|\text{Ł}'| > 24(\delta^{\frac{1}{4}}/\sqrt{c})n$ contradicts a conclusion of Fact 5.3. Define a 3-partite subgraph $Q = Q^{ij} \cup Q^{jk} \cup Q^{ki}$ as follows:

$$\begin{aligned} Q^{ij} &= P^{ij} - P_1, \\ Q^{jk} &= E_1[\text{Ł}', L''] - P_1, \\ Q^{ki} &= \overline{E}_2[\text{Ł}', V_i]. \end{aligned}$$

By the construction of Q and Observation 5.4 we have $|\mathcal{H} \cap Tr(Q)| \le |E_1| = O(n^2)$. Apply Fact 5.3 with $A = \text{Ł}'$, $B = L''$ (and so $b = c/4$), $\beta = 4/5$, $a = 24\delta^{\frac{1}{4}}/\sqrt{c}$, and $\gamma = 9\delta^{\frac{1}{4}}/\sqrt{c}$ to yield $|\mathcal{H} \cap Tr(Q)| = \Omega(n^3)$. For large enough n, this is a contradiction which ends the proof of Fact 5.6. □

To complete the proof of the inequality (6), set

$$\text{Ł}'' = \text{Ł} \setminus \text{Ł}' = V_k \setminus (\text{Ł}_0 \cup \text{Ł}_1 \cup \text{Ł}_2 \cup \text{Ł}')$$

and note that for every vertex $z \in \text{Ł}''$, by (10), we have

$$deg_{E_2}(z) \ge \frac{n}{l} - \epsilon n - 9\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\frac{n}{l} > (1 - 10\frac{\delta^{\frac{1}{4}}}{\sqrt{c}})\frac{n}{l}.$$

Note also that all the exceptional sets $\text{Ł}'$, Ł_0, Ł_1 and Ł_2 contain together less than $2(\delta^{\frac{1}{8}}/\sqrt{c})n$ vertices and therefore $|\text{Ł}''| > (1 - 2\delta^{\frac{1}{8}}/\sqrt{c})n$. Thus, by (9),

$$|E_2| \ge |\text{Ł}''|\left(1 - 10\frac{\delta^{\frac{1}{4}}}{\sqrt{c}}\right)\frac{n}{l} > \left(1 - 3\frac{\delta^{\frac{1}{8}}}{\sqrt{c}}\right)\frac{n^2}{l},$$

hence

$$|E_2 - P_1| > \left(1 - 3\frac{\delta^{\frac{1}{8}}}{\sqrt{c}}\right)\frac{n^2}{l} - 29\sqrt{\delta}\frac{n^2}{l} > \left(1 - 4\frac{\delta^{\frac{1}{8}}}{\sqrt{c}}\right)\frac{n^2}{l}.$$

□

6. T

In this section we prove Lemma 4.2. Let us begin with some heuristic. We call an edge $\mathcal{H}$-*good*, or just *good* if, say, $|\Gamma_{\mathcal{H}}(e)| \geq \frac{2}{9}\alpha n/l^2$. As proved in [6] (see Fact 3.2 above), for most edges of P we have $|\Gamma_{\mathcal{H}}(e)| \sim \alpha n/l^2$, so most edges are good, but unfortunately, some of these good edges may have small fourth, and even second neighborhood. Indeed, it might happen that for a good edge $e = xy$, whenever $xyz \in \mathcal{H}$ then yz has a very small neighborhood.

To find a large subset of good edges e with large fourth neighborhoods $Four^+(e, \mathcal{H})$ and $Four^-(e, \mathcal{H})$, one could argue as follows. Suppose that the set of bad (=not good) edges has size ρn^2. Then, for each $i = 1, 2, 3$, at most $\sqrt{\rho}n$ vertices of V_i are incident to at least $\sqrt{\rho}n$ bad edges (let us call these vertices bad), and, provided $\sqrt{\rho} \ll 1/l^2$, one could start at a good edge with good endpoints and move four steps, avoiding both, bad edges and bad vertices. The problem is that Fact 3.2 yields only ρ of order $1/l$ – too large for our needs.

To get around this problem we will find a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with much less bad edges. This sounds paradoxical, since removing hyperedges can only decrease $|\Gamma_{\mathcal{H}}(e)|$. Note, however, that edges e with $\Gamma_{\mathcal{H}}(e) = \emptyset$ are not so bad – there is no way to get to them! Let us call them $\mathcal{H}$-*dead*. To distinguish between $\mathcal{H}$-dead and other bad edges, we will alter our previous definition and call an edge $e \in P$ $\mathcal{H}$-bad if

$$0 < |\Gamma_{\mathcal{H}}(e)| < \frac{2}{9}\alpha\frac{n}{l^2}.$$

So, for any $\mathcal{H}' \subseteq \mathcal{H}$, every edge of P is either $\mathcal{H}'$-good or $\mathcal{H}'$-bad or $\mathcal{H}'$-dead. Let us denote these three subgraphs by $G_{\mathcal{H}'}$, $B_{\mathcal{H}'}$ and $D_{\mathcal{H}'}$. For technical reasons we distinguish also a class F_0 of atypical edges of P. For all $1 \leq i < j \leq 3$, an edge $e \in P^{ij}$ belongs to the subgraph F_0, if either it is not typical or at least one of its ends is not typical in P^{ij}. Note, that by Fact 8.3 and Corollary 8.4, $|F_0| < 24\epsilon n^2$.

Claim 6.1. *For all $\alpha \in (0, 1)$ and for all $\delta < \alpha/9^2$ there exist two sequences $r(l)$ and $\epsilon(l)$ so that for all $\mathcal{H}$, P and integer l if $(\mathcal{H}, P)$ is an $(\alpha, \delta, l, r(l), \epsilon(l))$-triad then there exists a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ such that $|B_{\mathcal{H}'}| \leq \delta n^2/l^4$, $|D_{\mathcal{H}'}| < 22\sqrt{\delta}n^2/l$ and $F_0 \subseteq D_{\mathcal{H}'}$.*

Proof of Lemma 4.2. With given α and

$$\delta < \delta_2 = \frac{\alpha^2}{180^2} < \frac{\alpha}{9^2},$$

let $r(l)$ and $\epsilon_1(l)$ be such that Claim 6.1 holds. Set

$$\epsilon(l) = \min\left(\epsilon_1(l), \frac{\alpha^4}{20{,}000 l^8}\right).$$

We will prove Lemma 4.2 with this choice of sequences $r(l)$ and $\epsilon(l)$. Given integer l, let a pair $(\mathcal{H}, P)$ be an $(\alpha, \delta, l, r, \epsilon)$-triad, where $r = r(l)$ and $\epsilon = \epsilon(l)$. Further, let $\mathcal{H}'$ be as in Claim 6.1, let

$$V^* = \left\{v \in V : \ deg_{B_{\mathcal{H}'}}(v) \geq \sqrt{\delta}\frac{n}{l^2}\right\},$$
$$G^*_{\mathcal{H}'} = \{e \in G_{\mathcal{H}'} : \ e \cap V^* \neq \emptyset\},$$

and let $P_0 = B_{\mathcal{H}'} \cup D_{\mathcal{H}'} \cup G^*_{\mathcal{H}'}$.

Note that $F_0 \subseteq P_0$. It remains to prove two facts about P_0.

Fact 6.2. $|P_0| \leq 27\sqrt{\delta}n^2/l$

Proof. To prove this fact, note that $|V^*| \leq 2\sqrt{\delta}n/l^2$, and so $|G^*_{\mathcal{H}'}| \leq 2n|V^*| \leq 4\sqrt{\delta}n^2/l^2$. Therefore

$$|P_0| \leq |B_{\mathcal{H}'}| + |D_{\mathcal{H}'}| + |G^*_{\mathcal{H}'}| \leq \delta\frac{n^2}{l^4} + 22\sqrt{\delta}\frac{n^2}{l} + 4\sqrt{\delta}\frac{n^2}{l^2} \leq 27\sqrt{\delta}\frac{n^2}{l}.$$

□

Fact 6.3. *For every edge $e \in P - P_0$ the inequality (4) holds.*

Proof. By symmetry, we will only prove that $\left|Four^+(e, \mathcal{H} - P_0)\right| \geq \left(\alpha^4/2000\right) n^2/l$. Without loss of generality we may assume that $e = xy \in P^{12} - P_0$, where $x \in V_1$. Then, by our choice of δ, the set of vertices z, such that $xyz \in \mathcal{H}$ and $yz, xz \notin P_0$, has size at least

$$\frac{2}{9}\alpha\frac{n}{l^2} - 4\sqrt{\delta}\frac{n}{l^2} > \frac{\alpha}{5}\frac{n}{l^2}, \tag{12}$$

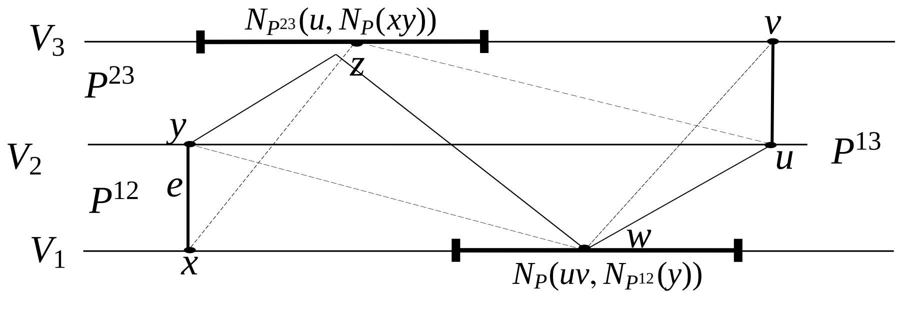
where the deletion takes care of all $z \in V^*$, as well as all z with yz or xz in $B_{\mathcal{H}'}$ (clearly, yz and xz cannot be $\mathcal{H}$-dead). Thus, $xyz \in \mathcal{H} - P_0$. For each such z, the edge yz belongs in turn to at least $\alpha n/(5l^2)$ triplets $yzw \in \mathcal{H}'$ with $w \in V_1 \setminus \{x\}$, $yw \in P^{12} - P_0$ and $zw \in P^{13} - P_0$. So, altogether there are at least $\alpha^2 n^2/(25l^4)$ edges of $P^{13} - P_0$ reached (within $\mathcal{H} - P_0$) in two steps by e. Repeating this argument again we obtain at least $\alpha^4 n^4/(625l^8)$ hyperpaths $xyzwuv \in \mathcal{H} - P_0$ of length four originating at $e = xy$.

Let us estimate, by counting repetitions, how many different edges $uv \in P^{23}$, $u \in V_2$, $v \in V_3$, are indeed reached by e in four steps (and in many ways). Consider an auxiliary bipartite graph $C = (X, Y, E_C)$, where $X = E(P^{13})$, $Y = E(P^{23})$, and $\{zw \in X, uv \in Y\} \in E_c$ if $xyzwuv$ is a hyperpath in $\mathcal{H} - P_0$. Hence, $|E_C| \geq \alpha^4 n^4/(625l^8)$.

Every hyperpath $xyzwuv$ must satisfy that

$$z \in N_{P^{23}}(u, N_P(xy)) \text{ and } w \in N_P(uv, N_{P^{12}}(y))$$

(see Figure 5).



F 5. A hyperpath of length four originating at $e = xy$.

Since $F_0 \subseteq P_0$, therefore $xy \notin F_0$, so we have $\epsilon n < |N_{P^{12}}(y)| < (1/l + \epsilon)n$ and $\epsilon n < |N_P(xy)| < (1/l + \epsilon)^2 n$. By Fact 8.3, all but at most ϵn^2 edges uv satisfy $|N_{P^{23}}(u, N_P(xy))| <$

$(1/l+\epsilon)^3 n$ and all but at most $3\epsilon n^2$ edges uv satisfy $|N_P(uv, N_{P^{12}}(y))| < (1/l+\epsilon)^3 n$. If both these sets are greater than ϵn then, by the $(1/l,\epsilon)$-regularity of P^{13}, there are at most $(1/l+\epsilon)^7 n^2$ edges $zw \in P^{13}$ between them. Otherwise, the number of such edges zw is at most $\epsilon n^2 < n^2/l^7$, the last inequality by our assumption on ϵ. Thus, $(1/l+\epsilon)^7 n^2 < \frac{20}{19} n^2/l^7$ is an upper bound on the degree in C of all but at most $4\epsilon n^2$ edges uv whose degree can be even equal to n^2. Denote the set of such edges by Y_1 and set

$$\Delta^* = \max_{e\in Y\setminus Y_1} deg_C(e).$$

Then

$$\Delta^* \le \Delta_0 = \frac{20}{19}\frac{n^2}{l^7}.$$

Let

$$Y_2 = \left\{ uv \in Y : deg_C(uv) \ge \frac{1}{2}\frac{|E_C|}{|P^{23}|} \right\},$$

and $Y_3 = Y \setminus (Y_1 \cup Y_2)$. We have

$$|E_C| \le |Y_1|n^2 + |Y_2|\Delta_0 + |Y_3|\frac{|E_C|}{2|P^{23}|} \le 4\epsilon n^4 + |Y_2|\Delta_0 + \frac{|E_C|}{2}.$$

Therefore, by our choice of ϵ,

$$|Y_2| \ge \left(\frac{\alpha^4 n^4}{1250 l^8} - 4\epsilon n^4\right)\frac{1}{\Delta_0} > \frac{\alpha^4 n^2}{2000 l}$$

Another words, at least $\alpha^4 n^2/2000l$ edges $uv \in P^{23}$ can be reached from e by no less than

$$\frac{1}{2}\frac{|E_C|}{|P^{23}|} > \frac{\alpha^4 n^2}{2500 l^7}$$

hyperpaths of length four, or equivalently, via that many edges $zw \in P^{13}$. It is easy to see that among these edges there is matching of size at least

$$\frac{\alpha^4 n}{5000 l^7}.$$

□

Proof of Claim 6.1.

Given α, let

$$\delta < \frac{\alpha}{9^2},$$

and let $\epsilon_1(l)$ be such that Fact 3.2 holds with above α and δ. Further, let for all l,

$$r(l) = 32 l^3$$

and

$$\epsilon(l) = \min\left(\epsilon_1(l), \frac{\delta}{24 l^3}\right). \tag{13}$$

We will prove Claim 6.1 with this choice of $\delta, r(l)$ and $\epsilon(l)$. Given integer l, let a pair $(\mathcal{H}, P)$ be an $(\alpha,\delta,l,r,\epsilon)$-triad, where $r = r(l)$ and $\epsilon = \epsilon(l)$.

We will define a process of deleting hyperedges which after finitely many rounds will arrive at a subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ satisfying the conclusions of Claim 6.1. Recall that for an arbitrary hypergraph $\mathcal{H}$ and a graph G, we denote by $\mathcal{H} - G$ the subhypergraph of $\mathcal{H}$ obtained by removing all hyperedges containing at least one edge of G.

The initial step of the procedure isolates all edges of F_0. Set $\mathcal{H}_1 = \mathcal{H} - F_0$. Clearly, for each $e \in F_0$, we have $\Gamma_{\mathcal{H}_1}(e) = \emptyset$ and so e is $\mathcal{H}_1$-dead.

In each next round we similarly "kill" edges of P which are bad in the current subhypergraph. For technical reasons these rounds take cyclically care of the edges of P^{12}, P^{23}, and P^{13}. For each $s = 1, 4, 7, \dots$, let

$$F_s = \{e \in P^{12} : e \text{ is } \mathcal{H}_s\text{-bad}\}, \quad \mathcal{H}_{s+1} = \mathcal{H}_s - F_s,$$

$$F_{s+1} = \{e \in P^{23} : e \text{ is } \mathcal{H}_{s+1}\text{-bad}\}, \quad \mathcal{H}_{s+2} = \mathcal{H}_{s+1} - F_{s+1},$$

$$F_{s+2} = \{e \in P^{13} : e \text{ is } \mathcal{H}_{s+2}\text{-bad}\}, \quad \mathcal{H}_{s+3} = \mathcal{H}_{s+2} - F_{s+2}.$$

In each operation of the type $\mathcal{H}_{s+1} = \mathcal{H}_s - F_s$ we remove all hyperedges which contain $\mathcal{H}_s$-bad edges e of P^{12}, P^{23} or P^{13}. Thus, those edges become $\mathcal{H}_{s+1}$-dead and therefore will never become bad again. It follows that all sets F_s are disjoint, and, in paticular, for $s \geq 1$, $F_s \cap F_0 = \emptyset$.

Our immediate goal is to estimate $\sum_{s=1}^r |F_s|$. Let us define 3-partite subgraphs Q_s of P, $s = 1, 2, ..., r$, as follows: If $F_s \subset P^{ij}$ then

$$Q_s = F_s \cup \left(P^{ik} - \bigcup_{t=1}^{s} F_t\right) \cup \left(P^{jk} - \bigcup_{t=1}^{s} F_t\right),$$

Set $\mathcal{Q} = (Q_1, ..., Q_r)$.

Observe that for slightly enlarged subgraphs $\overline{Q}_s = F_s \cup P^{ik} \cup P^{jk}$ (where $F_s \subset P^{ij}$), we have, by (13) and the fact that $F_s \cap F_0 = \emptyset$,

$$|Tr(\overline{Q}_s)| \geq |F_s| n \left(\frac{1}{l} - \epsilon\right)^2 \geq \frac{3}{4} \frac{n}{l^2} |F_s|.$$

Trivially,

$$\bigcup_{s=1}^{r} Tr(Q_s) \subseteq \bigcup_{s=1}^{r} Tr(\overline{Q}_s),$$

but the reverse inclusion is also true. Indeed, for $xyz \in Tr(\overline{Q}_s)$ set $t_0 = \min\{t : \{xy, xz, yz\} \cap F_t \neq \emptyset\}$. Then $t_0 \leq s$ and $xyz \in Tr(Q_{t_0})$. Moreover, because the sets F_s are disjoint (and so are $Tr(Q_s)$),

$$|\bigcup_{s=1}^{r} Tr(\overline{Q}_s)| = |\bigcup_{s=1}^{r} Tr(Q_s)| = \sum_{s=1}^{r} |Tr(Q_s)| \geq \frac{1}{3} \sum_{s=1}^{r} |Tr(\overline{Q}_s)|.$$

Hence,

$$|\bigcup_{s=1}^{r} Tr(Q_s)| \geq \frac{1}{4} \frac{n}{l^2} \sum_{s=1}^{r} |F_s|. \tag{14}$$

On the other hand, however, by the definition of an $\mathcal{H}_s$-bad edge, for all $s \le r$,

$$|\mathcal{H} \cap Tr(Q_s)| < |F_s|\frac{2}{9}\alpha\frac{n}{l^2},$$

forcing

$$d_{\mathcal{H}}(Q) < \frac{8}{9}\alpha,$$

where $d_{\mathcal{H}}(Q)$ is defined in (1). Therefore, by the (α, δ, r)-regularity of $\mathcal{H}$,

$$|\bigcup_{s=1}^{r} Tr(Q_s)| \le \delta|Tr(P)|, \tag{15}$$

since otherwise $d_{\mathcal{H}}(Q) > \alpha - \delta \ge \frac{8}{9}\alpha$. This inequality together with (13), (14) and Fact 8.6 implies that

$$\sum_{s=1}^{r} |F_s| \le 4|\bigcup_{s=1}^{r} Tr(Q_s)|\frac{l^2}{n} < 8\delta\frac{n^2}{l}. \tag{16}$$

Thus, more than a half of the sets F_s, $s \le r$, have size $|F_s| \le 16\delta n^2/lr$, and so two consecutive sets must be such, that is, there exists an index $s \le r - 2$, such that

$$\max(|F_{s+1}|, |F_{s+2}|) \le 16\delta\frac{n^2}{lr} = \frac{1}{2}\delta\frac{n^2}{l^4}.$$

Let s_0 be the smallest index s with this property.

Without loss of generality we may assume that $F_{s_0} \subset P^{12}$. Set $\mathcal{H}' = \mathcal{H}_{s_0+1}$. Observe that there is no $\mathcal{H}'$-bad edge in the graph P^{12}, while in each P^{23} and P^{13} we have at most $\frac{1}{2}\delta n^2/l^4$ $\mathcal{H}'$-bad edges. In fact, the set of $\mathcal{H}'$-bad edges is the union of F_{s_0+1} ($\mathcal{H}'$-bad edges in P^{23}) and of a subgraph of F_{s_0+2} (F_{s_0+2} may contain $\mathcal{H}_{s_0+2}$-bad edges which were not $\mathcal{H}_{s_0+1}$-bad).

As for the $\mathcal{H}'$-dead edges, these are exactly the edges in $\bigcup_{i=0}^{s_0} F_i$ plus all the edges $e \in P$ which were originally dead, that is, which had $\Gamma_{\mathcal{H}}(e) = \emptyset$. We have already estimated $|\bigcup_{i=1}^{s_0} F_i|$ in (16), while, by (13), $|F_0| < 24\epsilon n^2 < \delta n^2/l$. Finally, by Fact 3.2, there are no more than $21\sqrt{\delta}n^2/l$ originally dead edges. Therefore we have

$$|D_{\mathcal{H}'}| < |\bigcup_{i=1}^{s_0} F_i| + |F_0| + |D_{\mathcal{H}}| < 8\delta\frac{n^2}{l} + \delta\frac{n^2}{l} + 21\sqrt{\delta}\frac{n^2}{l} < 22\sqrt{\delta}\frac{n^2}{l}$$

Hence, Claim 6.1 is proved. □

7. A

In this section we give two immediate applications of Theorem 2.9.

7.1. **Long hyperpaths.** The "Blow-up Lemma" of Komlós, Sárközy and Szemerédi [4] states that with a suitable choice of parameters every s-partite graph G with s-partition $V_1 \cup \cdots \cup V_s$ in which all bipartite subgraphs $G[V_i, V_j]$ are (d, ϵ)-regular contains all bounded degree s-partite graphs G' with s-partition $V_1' \cup \cdots \cup V_s'$, where for all $i = 1, \ldots, s$, $V_i' \subseteq V_i$, $|V_i'| < (1 - f(\epsilon))|V_i|$.

So far no analogous results exist for 3-uniform hypergraphs. As a first step toward a hypergraph "Blow-up Lemma", we derive from Corollary 3.5 a simple consequence which establishes the existence of an almost hamiltonian hyperpath in a quasi-random 3-graph.

Proposition 7.1. *For all real* $\alpha \in (0,1)$ *and for all* $\delta < (\delta_0/4)^4$*, where* δ_0 *is as in Theorem 3.5, there exist two sequences* $r(l)$ *and* $\epsilon(l)$ *such that for all* $\mathcal{H}$*,* P *and integers* l *if a pair* $(\mathcal{H}, P)$ *is an* $(\alpha, \delta, l, r(l), \epsilon(l))$*-triad with* $|V| = n$ *sufficiently large, then there is in* $\mathcal{H}$ *a hyperpath of length at least* $(1-\delta^{\frac{1}{4}})n$*.*

Proof. Given α, let $\delta < (\delta_0/4)^4$ and $r = r(l)$ i $\epsilon_1(l)$ be ensured by Theorem 3.5. Set $\epsilon = \epsilon(l) = \delta^{\frac{1}{4}}\epsilon_1(l)$. Observe, that

$$27\sqrt{4\delta^{\frac{1}{4}}} < 27\sqrt{\delta_0} < \frac{\alpha^4}{2000}. \tag{17}$$

Let a pair $(\mathcal{H}, P)$ be an $(\alpha, \delta, l, r, \epsilon)$-triad. Suppose, that no hyperpath in $\mathcal{H}$ is longer than $(1-\delta^{\frac{1}{4}})n$. For a hyperpath Q, let $\mathcal{H}'_Q$ be the subhypergraph of $\mathcal{H}$, obtained by deleting from $\mathcal{H}$ all, but the last four vertices of the path Q (if $|V(Q)| < 4$, then we set $\mathcal{H}'_Q = \mathcal{H}$).

Let us fix an arbitrary edge $e = \{x, y\} \in P - R_0$ and let Q be the longest hyperpath in $\mathcal{H}$ originating at e (in the cyclic order $V_1 \to V_2 \to V_3 \to V_1$) and such that its other endpair $f \in P - R_0(\mathcal{H}'_Q)$. It follows trivially from the definition of the set $R_0(\mathcal{H})$ that Q has at least four vertices. Let us denote the last four vertices of Q by $x_{-3}, x_{-2}, x_{-1}, x_0$.

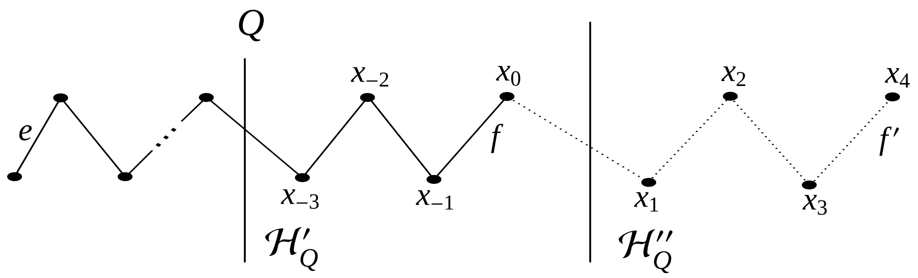
Since $|V(Q)| < (1-\delta^{\frac{1}{4}})n$, the subhypergraph $\mathcal{H}''_Q = \mathcal{H} - V(Q)$ has at least $\delta^{\frac{1}{4}}n$ vertices. Moreover, since Q traverses the sets V_1, V_2, V_3 in the cyclic order, the sizes of the sets $V(\mathcal{H}''_Q) \cap V_i$, $i = 1, 2, 3$, differ from each other by at most one. Hence (see [8], Fact 4.2), the pair $(\mathcal{H}''_Q, P[V(\mathcal{H}''_Q)])$ is an $(\alpha, 4\delta^{\frac{1}{4}}, l, r, \epsilon/\delta^{\frac{1}{4}})$-triad. Note that $|V(\mathcal{H}'_Q)| = |V(\mathcal{H}''_Q)| + 4$, $4\delta^{\frac{1}{4}} < \delta_0$ and $\epsilon/\delta^{\frac{1}{4}} = \epsilon_1(l)$. Therefore, by Theorem 3.5,

$$|R_0(\mathcal{H}''_Q)| \le 27\sqrt{4\delta^{\frac{1}{4}}}\frac{\lceil |V(\mathcal{H}''_Q)|/3\rceil^2}{l} \le 27\sqrt{4\delta^{\frac{1}{4}}}\frac{\lfloor |V(\mathcal{H}'_Q)|/3\rfloor^2}{l}.$$

On the other hand, by the definition of $R_0(\mathcal{H}'_Q)$, we know that the edge $f = \{x_{-1}, x_0\}$ reaches in four steps at least

$$\frac{\alpha^4}{2000}\frac{\lfloor |V(\mathcal{H}'_Q)|/3\rfloor^2}{l} > 27\sqrt{4\delta^{\frac{1}{4}}}\frac{\lfloor |V(\mathcal{H}'_Q)|/3\rfloor^2}{l} + 2n \ge |R_0(\mathcal{H}''_Q)| + 2n$$

other edges of $P[V(\mathcal{H}'_Q)]$ (the term $2n$ takes care of all edges with at least one endpoint in x_{-2} or x_{-3}; the first inequality follows from (17) for large n). Therefore, there is at least one edge $f' = \{x_3 x_4\} \in P[V(\mathcal{H}''_Q)] - R_0(\mathcal{H}''_Q)$, reached by f in $\mathcal{H}'_Q$ by at least three (in fact, many more) internally disjoint hyperpaths of length four of the form $x_{-1}x_0x_1x_2x_3x_4$. Thus, for at least one of them $\{x_1, x_2, x_3, x_4\} \cap \{x_{-3}, x_{-2}\} = \emptyset$, and we may extend Q by adding the vertices x_1, x_2, x_3, x_4 – a contradiction with the maximality of Q (see Figure 6). □



F 6. A hyperpath Q originating at e.

Similarly, one can prove that for most pairs of edges of P there is a path of length at least $(1-\delta^{\frac{1}{4}})n$ between them.

7.2. **Approximate decomposition into small diameter subhypergraphs.** It is easy to see that for every n-vertex graph and for every $\epsilon > 0$ one can partition $E(G) = E_0 \cup \cdots \cup E_k$, where $k \leq 1/\epsilon$, so that $|E_0| \leq \epsilon n^2$ and for each $1 \leq i \leq k$ the diameter of the subgraph $G_i = G[E_i]$ is at most $3/\epsilon$ (see [7]). Thus, in a sense, every graph can be decomposed into a bounded number of "small worlds" provided a small set of edges can be ignored. Here both bounds, on the diameter and on the number of subgraphs G_i depend linearly on $1/\epsilon$. Using the Szemerédi Regularity Lemma [9] and Corollary 8.5(b), one may put the cap of four on the diameter, at the cost of letting k, the number of subgraphs in the partition, to be an enormous constant.

Proposition 7.2. [7] *For all $\epsilon > 0$ there exist integers K and N such that for all n-vertex graphs G, where $n \geq N$, there is a partition $E(G) = E_0 \cup E_1 \cup \cdots \cup E_k$, where $k \leq K$, and $|E_0| \leq \epsilon n^2$, and for each $1 \leq i \leq k$, the diameter of the subgraph $G_i = G[E_i]$ is at most four.*

An analogous result for 3-uniform hypergraphs follows from our Theorem 2.9 and the Hypergraph Regularity Lemma in [1].

Theorem 7.3. *For all $\xi > 0$ there exist integers K and N such that for all n-vertex 3-uniform hypergraphs $\mathcal{H}$, where $n \geq N$, there is a partition $\mathcal{H} = \mathcal{H}_0 \cup \cdots \cup \mathcal{H}_k$ where $k \leq K$ and $|\mathcal{H}_0| \leq \xi n^3$ and for each $1 \leq i \leq k$, every two pairs of vertices with positive degree in $\mathcal{H}_i$ are connected by a hyperpath in $\mathcal{H}_i$ of length at most twelve.*

Sketch of proof: Given $\xi > 0$, set $t_0 = 8/\xi$, $\alpha = \xi/8$ and let $\delta_0 > 0$ and sequences $r(l), \epsilon_2(l)$ be as in Theorem 2.9 with above α. Further, let N_1 be the smallest natural number, for which Theorem 2.9 holds. Set

$$\delta = \min\left(\delta_0, \left(\frac{\xi}{16}\right)^2\right)$$

and apply the Hypergraph Regularity Lemma of [1] with $\delta, \epsilon_1 = \delta^4, \epsilon_2(l)$ and $r(l)$ to get T_0, L_0 i N_0. Set

$$K = \binom{T_0}{3} L_0^3 \text{ and } N = \max\left(N_0, \frac{16T_0}{\xi}, N_1 T_0\right)$$

and let $\mathcal{H}$ be an arbitrary 3-uniform hypergraph with $n \geq N$ vertices and $|\mathcal{H}| \geq \xi n^3$ triplets.

By the Hypergraph Regularity Lemma, $\mathcal{H}$ admits an auxilary vertex partition $V(\mathcal{H}) = V_0 \cup V_1 \cup \cdots \cup V_t$, where $t_0 \le t < T_0$, $|V_0| < t$ and $|V_1| = |V_2| = \cdots = |V_t|$, and for each pair i, j, $1 \le i < j \le t$, partitions of the complete bipartite graphs $K(V_i, V_j) = \bigcup_{a=0}^{l} P_a^{ij}$, where $1 \le l < L_0$, such that most triplets of $\mathcal{H}$ belong to $(\alpha, \delta, l, r(l), \epsilon_2(l))$-triads $(\mathcal{H}', P)$, where $\mathcal{H}' = \mathcal{H} \cap Tr(P)$ and $P = (P_a^{hi}, P_b^{hj}, P_c^{ij})$, $1 \le i < j < h \le t$, $1 \le a, b, c \le l$.

Let $(\mathcal{H}'_s, P_s)$, $s = 1, \ldots, k \le \binom{t}{3} l^3 < K$, be all such triads. For each $s = 1, \ldots, k$, let $(P_s)_0$ be the subgraph of P_s guaranteed by Theorem 2.9, and set $\mathcal{H}_s = \mathcal{H}'_s - (P_s)_0$. Then, each pair of edges of $P_s - (P_s)_0$, that is each pair of edges of P_s with positive degree in $\mathcal{H}_s$ is connected in $\mathcal{H}_s$ by a hyperpath of length at most twelve.

Let us set $\mathcal{H}_0 = \mathcal{H} \setminus \bigcup_{s=1}^{k} \mathcal{H}_s$. To complete the proof of Theorem 7.3, it remains to show that $|\mathcal{H}_0| \le \xi n^3$. We omit the details of tedious but straightforward calculations. □

8. A – ϵ-

Let $G = (V, E)$ be a graph, where V and E are the vertex-set and the edge-set of G. Throughout the paper we often identify G with its set of edges and therefore write $|G|$ instead of $|E|$. When U, W are subsets of V, we define

$$e_G(U, W) = \{\{x, y\} \in E : x \in U, y \in W\}.$$

For nonempty and disjoint U and W,

$$d_G(U, W) = \frac{e_G(U, W)}{|U||W|}$$

is the *density* of the graph G between U and W, or simply, the density of the pair (U, W).

Definition 8.1. Given $\epsilon > 0$, a bipartite graph G with bipartition (V_1, V_2), where $|V_1| = n$ and $|V_2| = m$, is called *ϵ-regular* if for every pair of subsets $U \subseteq V_1$ and $W \subseteq V_2$, $|U| > \epsilon n, |W| > \epsilon m$, the inequalities

$$d - \epsilon < d_G(U, W) < d + \epsilon$$

hold for some real number $d > 0$. We may then also say that G, or the pair (V_1, V_2), is *(d, ϵ)-regular*.

Let a graph $G = (V, E)$ be given. We write $N_G(v)$ for the set of neighbors of $v \in V$ in the graph G. The size of $N_G(v)$ is $|N_G(v)| = deg_G(v)$, the *degree* of v. We set $N_G(xy) = N_G(x) \cap N_G(y)$ as the set of common neighbors of $x, y \in V$ in G. For a set $U \subset V$, we write $N_G(v, U)$ for the set of neighbors of v in U and $N_G(xy, U)$ for the set of common neighbors of x and y in U. The size of $N_G(v, U)$ is $|N_G(v, U)| = deg_G(v, U)$.

Definition 8.2. Let $G = (V_1 \cup V_2, E)$ be a (d, ϵ)-regular bipartite graph, where $|V_1| = |V_2| = n$. We say, that a vertex $x \in V_i$, $i = 1, 2$, is *typical in G*, if the following inequalities hold

$$n(d - \epsilon) < deg_G(x) < (d + \epsilon)n.$$

Further, let $G = G^{12} \cup G^{23} \cup G^{13}$ be a 3-partite graph with partition (V_1, V_2, V_3), where $|V_1| = |V_2| = |V_3| = n$, and each graph G^{ij} is (d, ϵ)-regular, $1 \le i < j \le 3$. We call a pair of vertices $(x, y) \in V_i \times V_j$ *typical* if it satisfies inequalities

$$n(d - \epsilon)^2 < |N_G(xy)| < n(d + \epsilon)^2.$$

The next fact is well-known and follows immediately from Definition 8.1 (see e.g. [6]).

Fact 8.3. *For all real $\epsilon > 0$ and $d > 0$, and for all integers n and m, the following holds. Let G be a (d, ϵ)-regular bipartite graph with a bipartition (V_1, V_2), where $|V_1| = n, |V_2| = m$. Further, let $A \subseteq V_2$, $|A| > \epsilon m$. Then all but at most ϵn vertices $x \in V_1$ satisfy*

$$deg_G(x, A) < (d + \epsilon)|A|, \tag{18}$$

and all but at most ϵn vertices $x \in V_1$ satisfy

$$deg_G(x, A) > (d - \epsilon)|A|. \tag{19}$$

In particular, if $|V_1| = |V_2| = n$, then for each $i \in \{1, 2\}$, all but at most $2\epsilon n$ vertices $x \in V_i$ are typical in G.

Corollary 8.4. *For all $\epsilon > 0$ and $d > 2\epsilon$ and for all integers n, the following holds. Let $G = G^{12} \cup G^{23} \cup G^{13}$ be a 3-partite graph with partition (V_1, V_2, V_3), where $|V_1| = |V_2| = |V_3| = n$ and each graph G^{ij} is (d, ϵ)-regular, $1 \leq i < j \leq 3$. Then all but at most $4\epsilon n^2$ pairs of vertices $(x, y) \in V_i \times V_j$ are typical.*

Another simple consequence of Fact 8.3 deals with the distances in a quasi-random bipartite graph (see [6] and [7]).

Corollary 8.5. *Let B be a (d, ϵ)-regular bipartite graph with bipartition (V_1, V_2), where $|V_1| = |V_2| = n$.*

(a) *If $d > 2\epsilon$ then all pairs of vertices of B of degree at least ϵn can be connected by paths of length at most four.*
(b) *If $d > 4\epsilon$ then by removing from B at most $2\epsilon n$ vertices (those of degree less than $3\epsilon n < (d - \epsilon)n$), we obtain a subgraph with diameter four.*

Finally, we state another well-known result which tightly estimates the size of $Tr(G)$, the set of triangles in a quasi-random 3-partite graph G (see, e.g., [8]).

Fact 8.6. *Let $G = G^{12} \cup G^{23} \cup G^{13}$ be a 3-partite graph, where all three bipartite graphs G^{ij} are (d, ϵ)-regular, $1 \leq i < j \leq 3$. If $d > 2\epsilon$ then*

$$(d^3 - 10\epsilon) < \frac{|Tr(G)|}{|V_1||V_2||V_3|} < (d^3 + 10\epsilon).$$

In particular, if $\epsilon < 0.1d^3$ then $|Tr(G)| < 2d^3|V_1||V_2||V_3|$.

R

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A. M U P ´, P
E-mail address: joaska@amu.edu.pl

E U , A , GA
E-mail address: rodl@mathcs.emory.edu

A. M U P ´, P
E-mail address: rucinski@amu.edu.pl

R U , N B
E-mail address: szemered@cs.rutgers.edu