Approximation algorithms for partially covering with edges

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Abstract

The edge dominating set (EDS) and edge cover (EC) problems are classical graph covering problems in which one seeks a minimum cost collection of edges which covers the edges or vertices, respectively, of a graph. We consider the generalized partial cover version of these problems, in which failing to cover an edge, in the EDS case, or vertex, in the EC case, induces a penalty. The objective is given a bound on the total amount of penalties that one is permitted to pay, to find a minimum cost cover which respects this bound. We are given an 8/3-approximation for generalized partial EDS This result matches the best known guarantee for the $\{0, 1\}$-EDS problem, a specialization in only which a specified set of edges need to be covered. Moreover, 8/3 corresponds to the integrality gap of the natural formulation of the $\{0, 1\}$-EDS problem. Our techniques can also be used to derive an approximation scheme for the generalized partial edge cover problem, which is NP-hard even though the standard edge cover problem is in P.

Keywords: edge dominating set, edge cover, partial cover, linear program, approximation algorithm

1 Introduction

Edge dominating set. Given a weighted graph $G = (V, E, c)$, the edge dominating set problem (EDS) consists in finding a subset $F \subseteq E$ of edges of minimum cost, $\sum_{e \in F} c_e$, such that for every $e \in E$, the edge dominating set F contains at least one edge adjacent to e. Thus EDS is the problem of covering E using E, and along with edge cover (cov. V using E), vertex cover (cov. E using V), and dominating set (cov. V using V), it is one of the four fundamental covering problem in graphs.

The cardinality case of EDS is NP-hard even for restricted classes of graphs such as planar or bipartite graphs of maximum degree 3 [8, 13] and hard to approximate within any constant factor smaller than 7/6 unless P=NP [4].

EDS was studied at least as early as 1969, when Harary [7] described the first approximation algorithm for this problem by observing that cardinality EDS is equivalent to finding a minimum size maximal matching, which directly yields a linear time 2-approximation. The approximability of the other fundamental covering problems were also classically studied. In fact edge cover can be solved in polynomial time by using the seminal ideas developed by Edmonds [**?**, **?**]. For vertex cover and dominating set, what are widely believed to be the best possible approximation guarantees (unless $\mathbf{P} = \mathbf{NP}$) have been known since the beginnings of the field. Dominating set is equivalent to set cover itself, and vertex cover admits a 2-approximation [1] and has received considerable attention.

A simple 4-approximation for the weighted case of EDS, by reduction to a 2-approximation for vertex cover, has been known for quite some time; however, no improvement was known until

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the work of Carr, Konjevod, Fujito, and Parekh, who gave a 2.1-approximation [3]; moreover, they proved that this is in fact the integrality gap of the natural polyhedral relaxation. EDS in fact generalizes both vertex cover and edge cover; in fact EDS can be cast as the generalization of vertex cover in which pairs of vertices may be selected as well as individual vertices. Since an r-approximation for EDS on bipartite graphs gives an r-approximation for vertex cover on general graphs via a simple linear time reduction [12], the goal was a 2-approximation. Fujito and Nagamochi [6] and, independently, Parekh [12] gave a 2-approximation by using a strengthened polyhedral relaxation for EDS, closing the gap with vertex cover.

More relevant to the work in this paper is the $\{0,1\}$-EDS problem, in which only a specified subset of the edges are required to be covered. Perhaps surprisingly, the best known approximation for this problem is 8/3, and this corresponds to the integrality gap of a natural linear relaxation of the problem [2]. There is evidence to indicate that improving this will require a different approach [2]. The generalized partial edge dominating set (GPEDS) problem, which is the focus of this paper, is a generalization of $\{0,1\}$-EDS.

Partial cover problems. In addition to a weighted graph, an instance of GPEDS also includes a non-negative penalties on each edge and a penalty budget. A solution may not choose to cover an edge but in this case must pay the penalty for it. The problem seeks to find a minimum cost solution among those whose total penalties do not exceed the given budget. We shall describe this problem more precisely in the section. One may analogously consider the partial cover version of any covering problem. Partial cover problems allow one to model instances which may contain outliers by setting the penalities and budget so that all but a few elements are covered. This line of research has found an interesting application of the langrangian relaxation technique from optimization. One might expect that the approximabilities of a covering problem and the associated partial cover problem are related. Such relationships have been observed for a variety of popular combinatorial optimization problems; recently, some of the ideas have been unified [9, 11]. For a more thorough introduction to partial cover problems we ask the reader to consult [9].

Our results. We give an 8/3-approximation for the generalized partial edge dominating set problem. We also consider the generalized partial edge cover problem, which is NP-hard despite the fact that minimum weight edge cover can be solved exactly in polynomial time. This matches the best known approximation guarantee for $\{0,1\}$-EDS as mentioned above. We feel that our technique is interesting since it avoids the standard langrangian relaxation approach, which typically introduces a gap in the approximabilities of the covering problem and associated partial cover problem. For GPEDS, we obtain a ratio matching $\{0,1\}$-EDS by extending the observed connection between the EDS and edge cover problems coupled with a careful examination of the extreme points of the related generalized partial edge cover problem, which will be described in the next section.

2 Generalized partial edge dominating set

Suppose that we are given non-negative edge vectors of costs, $c \in \mathbb{Q}_+^E$ and profits $p \in \mathbb{Q}_+^E$ as well as a minimum profit bound, $\hat{\pi} \in \mathbb{Q}_+$. Selecting the edge e incurs a cost of c_e, and covering an edge e earns a profit of p_e; the *generalized partial edge dominating set* (GPEDS) problem seeks a minimum cost edge dominating set of a selected edge set $D \subseteq E$ with $\sum_{e \in D} p_e \geq \hat{\pi}$. In other words the problem seeks to find a minimum cost edge dominating set of some edge set that earns profit at least $\hat{\pi}$. For a general introduction to partial covering problems see, for instance, [9]. We begin with a linear relaxation for this problem.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{e\in E} c_e x_e \\
\text{subject to} \quad & \sum_{e\in E} p_e \hat{z}_e \geq \hat{\pi} && (1)\\
& x(\nabla(e)) \geq \hat{z}_e \quad \forall e \in E && (2)\\
& 1 \geq x_e \geq 0 \quad \forall e \in E \\
& 1 \geq \hat{z}_e \geq 0 \quad \forall e \in E
\end{aligned}$$

If x_e and $\hat{z}_e$ are restricted to be integers for all $e \in E$, then we have an exact formulation of the GPEDS problem, since for each e, the corresponding inequality (2) ensures that $\hat{z}_e = 1$ if and only if e is covered. The inequality (1) guarantees that the total profit earned by the covered edges is at least $\hat{\pi}$. For our purposes we shall find it convenient to take a different perspective.

Rather than earning a profit for covering an edge, we shall assume that we incur a penalty for not covering an edge. This is illustrated by reformulating the above LP in terms of the variables $z_e = 1 - \hat{z}_e$ for each edge e:

$$\begin{aligned}
\text{Minimize} \quad & \sum_{e\in E} c_e x_e && (GPEDS)\\
\text{subject to} \quad & \sum_{e\in E} p_e z_e \leq \pi := \sum_{e\in E} p_e - \hat{\pi} && (3)\\
& z_e + x(\nabla(e)) \geq 1 \quad \forall e \in E && (4)\\
& 1 \geq x_e \geq 0 \quad \forall e \in E \\
& 1 \geq z_e \geq 0 \quad \forall e \in E
\end{aligned}$$

As our substitution indicates, a way to interpret this formulation is that by default (i.e. $z_e = 0$) we earn the profit for every edge, and $z_e = 1$ if and only if we lose the profit p_e. Thus the problem becomes that of finding a minimum cost solution whose profit loss is at most $\sum_{e\in E} p_e - \hat{\pi}$. In the remainder of our discussion we shall adopt this view and will simply call p_e the *penalty* incurred for not covering edge e.

An outline of our method. Our approach is an extension of the techniques used to derive the currently best known approximation guarantees for the edge dominating set problem [3, 6, 12]. The essential idea used in that context is to use an optimal LP solution to derive a vertex cover $U \subseteq V$. Next an optimal edge cover of the vertices in U is computed. The latter is a problem reducible to maximum weight matching and can be solved in polynomial time. Although the algorithm is quite simple, the analysis appeals to a polyhedral rounding argument and no combinatorial counterpart is known. A 2.1-approximation [3] can be achieved using the natural LP relaxation; however, attaining a 2-approximation requires a more refined approach [6, 12]. More appropriate to the partial EDS problem, however, is an 8/3-approximation for the b-EDS problem, in which each edge must be covered at least b_e times, where b_e is a non-negative integer [2]. This bound corresponds to the integrality gap of the natural LP formulation, and improving this factor seems difficult even for $\{0,1\}$-EDS, the case when $b_e \in \{0,1\}$ for all e [2]. In fact an r-approximation for the generalized partial EDS problem implies an r-approximation for the $\{0,1\}$-EDS problem (by way of an r-approximation for the prize-collecting EDS problem, which is discussed in Section 4).

As with the aforementioned approach, we shall appeal to a generalization of the edge cover problem in designing our approximation algorithm. Suppose we are given an optimal solution $(x^*, z^*) \in GPEDS$. We shall use this solution to assign the responsibility of covering edges and paying penalties to a set $U^* \subseteq V$ of vertices. More precisely we seek to identify a function $\phi : E \rightarrow V$, where for each edge e we say e is assigned to the vertex $\phi(e)$. For each $v \in V$, we will denote by $\phi^{-1}(v)$ the set of edges assigned to v. In particular, by (4) we have that for each edge uv,

$$z^*_{uv} + x^*(\delta(u)) + x^*(\delta(v)) = z^*_{uv} + x^*(\nabla(uv)) + x^*_{uv} \geq 1 + x^*_{uv} \geq 1.$$

Thus we must have

$$z^*_{uv}/2 + \max\{x^*(\delta(u)),\, x^*(\delta(v))\} \geq 1/2. \tag{5}$$

We set $\phi(uv)$ to an endpoint for which one of the above holds, breaking ties arbitrarily. We let $U^* = \bigcup_{e\in E} \phi(e)$. Our immediate goal is to use ϕ to define an instance of the *generalized partial edge cover problem* such that feasible solutions for this instance are also feasible solutions for our generalized partial EDS instance.

Generalized partial edge cover. Given a graph (V,E), edge costs $\bar{c} \in \mathbb{R}^E_+$, vertex penalties $\bar{p}^* \in \mathbb{R}^V_+$, and a maximum penalty bound $\bar{\pi} \in \mathbb{R}_+$, the *generalized partial edge cover* (GPEC) problem seeks to find minimum cost edge cover of the vertices in a selected vertex set $U \subseteq V$ with $\sum_{v\in V-U} \bar{p}^*_v \leq \bar{\pi}$. As for the GPEDS problem, this problem can be reformulated in terms of earning profit. Although the edge cover problem and partial edge cover problem (i.e. $\bar{\pi}_v = 1$ for all $v \in V$) are both solvable optimally in polynomial time, GPEC is NP-hard as it generalizes the knapsack problem. This is one of the reasons that our approach requires more care than that for the $\{0,1\}$-EDS problem. We outline a polynomial time approximation scheme for GPEC in Section 3. Below we give a linear relaxation for GPEC that will be essential to the analysis of our algorithm.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{e\in E} \bar{c}_e \bar{x}_e && (GPEC(\bar{c},\bar{p},\bar{\pi}))\\
\text{subject to} \quad & \sum_{v\in V} \bar{p}_v \bar{z}_v \leq \bar{\pi} && (6)\\
& \bar{z}_v + \bar{x}(\delta(v)) \geq 1 \quad \forall v \in V && (7)\\
& 1 \geq \bar{x}_e \geq 0 \quad \forall e \in E &&\\
& 1 \geq \bar{z}_e \geq 0 \quad \forall e \in E &&
\end{aligned}$$

Analogously to $GPEDS$, $\bar{z}_v = 1$ indicates that the vertex v does not need to be covered and will incur a penalty $\bar{p}^*_v$.

Constructing a feasible GPEC solution. We are almost in a position to describe our algorithm; however, before proceeding we derive an instance of GPEC and corresponding feasible solution to $GPEC(\bar{c},\bar{p},\bar{\pi})$ from our $GPEDS$ solution, (x^*, z^*) and assignment ϕ. We let $\bar{\pi}^* = \pi$, $\bar{c}^* = c$,

$$\bar{p}^*_v = 0 \qquad \forall v \in V \setminus U^*, \tag{8}$$

$$\bar{z}^*_v = 1 \qquad \forall v \in V \setminus U^*, \tag{9}$$

$$\bar{p}^*_v = \sum_{e\in\phi^{-1}(v)} p_e \qquad \forall v \in U^*, \tag{10}$$

$$\bar{z}^*_v = \min_{e\in\phi^{-1}(v)} z^*_e \qquad \forall v \in U^*, \text{ and} \tag{11}$$

$$\bar{x}^*_e = \min\{2x^*_e, 1\} \quad \forall e \in E. \tag{12}$$

Recall that $V \setminus U^*$ is precisely the set of vertices v for which $\phi^{-1}(v) = \emptyset$. The motivation for this construction is holding each $v \in U^*$ responsible for each $e \in \phi^{-1}(v)$. This is particularly apparent in (10), which models the notion that if v is not covered then a penalty must be paid for all $e \in \phi^{-1}(v)$.

Lemma 1. *The solution $(\bar{x}^*, \bar{z}^*)$ is feasible for $GPEC(c, \bar{p}^*, \pi)$.*

Proof. We consider each of the inequalities of $GPEC(c, \bar{p}^*, \pi)$ in turn. The non-negativity constraints and upper bounds are clearly satisfied by $(\bar{x}^*, \bar{z}^*)$.

The penalty constraint, (6) is satisfied since

$$
\begin{aligned}
\sum_{v \in V} \bar{p}_v^* \bar{z}_v^* &= \sum_{v \in U^*} \bar{p}_v^* \bar{z}_v^* && [\text{ by (8) }] \\
&= \sum_{v \in U^*} \left(\sum_{e \in \phi^{-1}(v)} p_e \right) \bar{z}_v^* && [\text{ by (10) }] \\
&= \sum_{e \in E} p_e \bar{z}_{\phi(e)}^* \\
&\leq \sum_{e \in E} p_e z_e^* && [\text{ by (11) }] \\
&\leq \pi. && [\text{ by } (x^*, z^*) \in GPEDS \text{ }]
\end{aligned}
$$

If $v \in V \setminus U^*$, then the corresponding cover constraint, (7) is clearly satisfied since $\bar{z}_v^* = 1$. If $v \in U^*$ then note that by our selection of U^*, and in particular (5), we have $z_e^* + 2x^*(\delta(v)) \geq 1$, $\forall e \in \phi^{-1}(v)$, hence by (11) we must have,

$$\bar{z}_v^* + 2x^*(\delta(v)) \geq 1.$$

If $x_e = 1$ for some $e \in \delta(v)$ then the constraint is clearly satisfied, otherwise we have $\bar{x}^*(\delta(v)) = 2x^*(\delta(v))$, demonstrating the feasibility of $(\bar{x}^*, \bar{z}^*)$. □

A simple corollary of this lemma is that optimal solution to $GPEC(c, \bar{p}^*, \pi)$ has low cost relative to the cost of (x^*, z^*).

Corollary 2. *For any feasible solution $(x^*, z^*) \in GPEDS$, if $\bar{w}$ is the optimal value of $GPEC(c, \bar{p}^*, \pi)$, then*

$$\bar{w} \leq 2 \sum_{e \in E} c_e x_e^*.$$

In light of this corollary, constructing a feasible integral solution to $GPEDS$ with low cost relative to $\bar{w}$ suffices in approximating GPEDS. We, in an analogous manner to Lemma 1, in fact show that any (integral) feasible solution to $GPEC(c, \bar{p}^*, \pi)$ is in fact feasible for GPEDS. Armed with this fact, we may reduce approximating GPEDS to approximating $GPEC(\bar{c}, \bar{p}, \bar{\pi})$ and paying a factor of 2 in the approximation guarantee.

Lemma 3. *If $(\bar{x}, \bar{z})$ is a solution for $GPEC(c, \bar{p}^*, \pi)$, then $(\bar{x}, z)$ is feasible for $GPEDS$, where*

$$z_e = \bar{z}_{\phi(e)} \quad \forall e \in E.$$

Proof. It suffices to show that the vector $(\bar{x}, \bar{z})$ satisfies constraints (3) and (4) of $GPEDS$. The penalty constraint, (3) is satisfied since, analogously to the proof of Lemma 1,

$$\sum_{e \in E} p_e z_e = \sum_{e \in E} p_e \bar{z}_{\phi(e)} = \sum_{v \in U^*} \left(\sum_{e \in \phi^{-1}(v)} p_e \right) \bar{z}_v = \sum_{v \in U^*} \bar{p}_e^* \bar{z}_v \leq \pi.$$

The fact that the cover constraint, (4) for an edge e is satisfied is readily demonstrated:

$$z_e + \bar{x}(\nabla(e)) \geq \bar{z}_{\phi(e)} + \bar{x}(\delta(\phi(e))) \geq 1.$$

□

Algorithm. Unfortunately, $GPEC(\bar{c}, \bar{p}, \bar{\pi})$ generalizes the knapsack problem and does not have a bounded integrality gap; however, as noted above, the approximation guarantee of our algorithm will depend on the approximabiity of the GPEC problem relative to the optimal value of $GPEC(\bar{c}, \bar{p}, \bar{\pi})$. We will, however, prove the following in the next section.

Theorem 4. *Given an extreme point solution, $(\bar{x}^*, \bar{z}^*)$ of $GPEC(\bar{c}, \bar{p}, \bar{\pi})$, an integral solution of cost at most*

$$\frac{4}{3} \sum_{e \in E} \bar{c}_e \bar{x}^*_e + \bar{c}_{\max}$$

can be constructed in linear time, where $\bar{c}_{\max} = \max_{e \in E} \bar{c}_e$.

A standard technique for eliminating a dependence on $\bar{c}_{max}$ is to preprocess the instance, by guessing the cost of the most expensive edge(s) picked by an optimal solution. Here we use a refined analysis suggested by Julian Mestre in the context of more general work linking the approximability of generalized partial covering problems with that of the corresponding prize collecting problem [11, 9]. For our purposes it suffices to guess the cost of the most expensive edge chosen by an optimal solution to the GPEDS problem. We may now present our algorithm.

Algorithm 1 8/3-Approximation algorithm for GPEDS

1: set $D = E$
2: **for** $\hat{e} \in E : p_{\hat{e}} \leq \pi$ **do**
3:   set $c'_e = \begin{cases} 0 & e = \hat{e} \\ \infty & c_e > c_{\hat{e}} \\ c_e & \text{otherwise} \end{cases}$
4:   find an optimal solution (x^*, z^*) of $GPEDS(c', p, \pi)$
5:   **if** $\sum_{e \in E} c'_e x^*_e < \infty$ **then**
6:     construct $\bar{p}^*$ and ϕ from (x^*, z^*)
7:     set $E' = \{e \in E \mid x^*_e > 0\}$ and $G' = (V, E')$
8:     find an optimal extreme point solution $(\bar{x}^*, \bar{z}^*)$ of $GPEC(c', \bar{p}^*, \pi)$ on G'
9:     use $(\bar{x}^*, \bar{z}^*)$ to construct an integral solution $(\hat{x}, \hat{z})$ for $GPEC(c', \bar{p}^*, \pi)$ on G'
10:     **if** $\sum_{e \in E'} c_e \hat{x}_e < \sum_{e \in D} c_e$ **then**
11:       set $D = \{e \in E' \mid \hat{x}_e = 1\}$
12:     **end if**
13:   **end if**
14: **end for**
15: **return** D

Analysis. The core of the algorithm is simple; however, we must take a bit of care to ensure that the preprocessing step mentioned above is implemented correctly. The algorithm clearly runs in polynomial time as Theorem 4 ensures that line 10 runs in linear time.

We assume that the optimal solution contains at least one edge, which can easily be verified in linear time. We prove that that the algorithm always returns a feasible solution.

Claim 5. *If Algorithm 1 ever reaches line 7, then it must return a feasible solution D.*

Proof. Since the construction preceding Lemma 1 sets $\bar{x}^*_e = 0$ whenever $x^*_e = 0$, line 8 does not alter this construction. Hence by Lemma 1, $GPEC(c', \bar{p}^*, \pi)$ has some feasible solution, and $(\bar{x}^*, \bar{z}^*)$ must exist. By Theorem 4, $(\hat{x}, \hat{z})$ exists, and it is feasible by Lemma 3. □

Claim 6. *If the optimal solution to GPEDS(c, p, π) contains at least one edge, then Algorithm 1 reaches line 7.*

Proof. Consider the iteration of the for loop in which $\hat{e}$ is the costliest edge in some optimal solution. In this case $GPEDS(c', p, \pi)$ is certainly feasible; moreover, the optimal solution (x^*, z^*) found in line 5 must have cost, $\sum_{e\in E} c'_e x^*_e < \infty$ since there is an optimal solution to GPEDS(c, p, π) that uses no edges with $c'_e = \infty$. $\square$

Theorem 7. *Algorithm 1 is an 8/3-approximation for the instance GPEDS(c, p, π).*

Proof. To derive the approximation bound, we again focus on the iteration through the for loop in which $\hat{e}$ is the most expensive edge in some optimal solution. Thus we have

$$\sum_{e\in E} c'_e x^*_e \le OPT(c', p, \pi) = OPT(c, p, \pi) - c_{\hat{e}}, \tag{13}$$

where the inequality follows from the optimality of (x^*, z^*), and the equality follows from the definition of $\hat{e}$. Our ultimate goal is to bound $\sum_{e\in E'} c'_e \hat{x}_e$ in terms of $OPT(c, p, \pi)$. Since we are already equipped with (13), it suffices to bound $\sum_{e\in E'} c'_e \bar{x}^*_e$ in terms of $\sum_{e\in E} c'_e x^*_e$ and to bound $\sum_{e\in E'} c'_e \hat{x}_e$ in terms of $\sum_{e\in E'} c'_e \bar{x}^*_e$. Corollary 2 aids with the former:

$$\sum_{e\in E'} c'_e \bar{x}^*_e \le 2 \sum_{e\in E'} c'_e x^*_e = 2 \sum_{e\in E} c'_e x^*_e, \tag{14}$$

where the equality follows since $x^*_e = 0$ for all $e \in E \setminus E'$. The final inequality we will need follows from Theorem 4:

$$\sum_{e\in E'} c'_e \hat{x}_e \le \frac{4}{3} \sum_{e\in E'} c'_e \bar{x}^*_e + \max_{e\in E'} c'_e, \tag{15}$$

where the final additive term follows from the fact that lines 9 and 10 find solutions on the graph G' rather than G. We can combine these inequalities to obtain the desired result.

$$\begin{aligned}
\sum_{e\in E'} c'_e \hat{x}_e &\le \frac{4}{3} \sum_{e\in E'} c'_e \bar{x}^*_e + \max_{e\in E'} c'_e \\
&\le \frac{8}{3} \sum_{e\in E} c'_e x^*_e + \max_{e\in E'} c'_e && [\text{ by (14) }] \\
&\le \frac{8}{3} (OPT(c, p, \pi) - c_{\hat{e}}) + \max_{e\in E'} c'_e && [\text{ by (13) }]
\end{aligned}$$

We are almost finished with our proof, yet we must account for the fact that the solution $\hat{x}$ may select the edge $\hat{e}$ at a cost of $c'_{\hat{e}} = 0$. That is, the bound we seek is on $\sum_{e\in E} c_e \hat{x}_e$ rather than $\sum_{e\in E'} c'_e \hat{x}_e$:

$$\sum_{e\in E} c_e \hat{x}_e \le \sum_{e\in E'} c'_e \hat{x}_e + c_{\hat{e}} \le \frac{8}{3} OPT(c, p, \pi) - \frac{5}{3} c_{\hat{e}} + \max_{e\in E'} c'_e \le \frac{8}{3} OPT(c, p, \pi).$$

$\square$

In the next section, we look at the GPEC problem in more detail and prove Theorem 4, completing our result.

3 Generalized partial edge cover

As noted in the previous section, the GPEC problem is NP-hard as it generalizes the knapsack problem, as a knapsack instance may be readily modeled by a GPEC instance on a star in which the leaf edges correspond to the knapsack items. This is interesting since instances of GPEC in which the penalties are uniform may be solved in polynomial time, since one may then model the penalties by treating the instance as an edge cover instance augmented with a single vertex that has a upper bound on the number of edges chosen incident to it. This type of problem can in fact be reduced to a standard matching problem [5].

We note that coupled with the ideas we present in this section, an approximation scheme may be derived for GPEC by analyzing the structure of adjacent vertices of the edge cover polytope and applying the standard langrangian relaxation approach, which for instance is outlined in [9]. We omit the details. Our focus in this section is proving Theorem 4. In particular we prove a slightly stronger version of it.

Theorem 8. *Given an extreme point solution, $(\bar{x}^*, \bar{z}^*)$ of $GPEC(\bar{c}, \bar{p}, \bar{\pi})$, an integral solution of cost at most*

$$\frac{4}{3}\sum_{e\in E} \bar{c}_e \bar{x}^*_e + \bar{c}_{\max\min}$$

can be constructed in linear time, where $\bar{c}_{\max\min} = \max_{v\in V} \min_{e\in\delta(v)} \bar{c}_e$.

Proof. We will find it convenient to extend the graph G by adding a vertex t which represents the penalty constraint (6), identifying each penalty variable $\bar{z}_v$ with an edge variable $\bar{x}_{vt}$. Let E' refer to the edges of this extended graph. We shall refer to (6) as the *degree constraint of vertex t*, $\sum_{e\in\delta(t)} \bar{p}_e \bar{x}_e \le \bar{\pi}$, where $\bar{p}_{vt} := \bar{p}_v$ for all $v \in V' := V \cup \{t\}$. The *degree constraint of vertex v*, for $v \in V$, is simply the corresponding constraint of (7).

Structure of an extreme point. Consider the extreme point solution $\bar{x}^*$, extended over E' as described above. If the degree constraint of t is not tight, then $\bar{x}^*$ is fully determined by tight degree constraints among (7) and tight upper or lower bounds. It is well known that in this case that $\bar{x}^*$ consists integral edges and vertex disjoint odd cycles with each component at a value of 1/2 [5]. An integral solution of cost at most $4/3 \sum_{e\in E} \bar{c}_e \bar{x}^*_e$ can be constructed in linear time by subdividing the cheapest edge for each odd 1/2-cycle and then selecting the cheaper of the two edge disjoint matchings that cover each of resulting even cycles. Thus we assume the degree constraint of t is tight.

Let $E^* = \{e \in E' \mid 0 < \bar{x}^*_e < 1\}$, and for an $S \subseteq V$, let $\delta^*(S) = \delta(S) \cap E^*$, the edges of E^* crossing S. We denote by V^* the set of vertices in V' whose degree constraints are tight; the vertices of $V \setminus V^*$ are called *slack* vertices.

Our approach. Our aim is to first show that t must occur in a component of (V', E^*) with a simple structure. We will then show how such a component may be augmented by paying a cost at most $\bar{c}_{\max\min}$ so that the resulting fractional solution on can be decomposed into a convex combination of integral solutions. Observe that we need not concern ourselves with the other components of (V', E^*) since the argument we used in the case when the degree constraint of t is not tight applies to each of these components.

Let T be the component of (V', E^*) that contains t. Since $\bar{x}^*$ is an extreme point, the edges $E^*(T)$ are defined by the tight degree constraints among $V(T)$. Thus we must have that $|E^*(T)| \le |V(T) \cap V^*|$. This implies that

- T contains at most one slack vertex,
- T contains at most one cycle, and

- T may not contain both one slack vertex and one cycle.

Moreover, no vertex $l \in V^* \setminus \{t\}$ can be a leaf in T (i.e. $|\delta^*(l)| = 1$) by the definition of E^* and V^*. Thus T must take one of the following forms:

1. a path between t and some slack vertex, s,
2. a cycle, or
3. a cycle with a path between some vertex on the cycle and t,

where in each case all vertices other than s must have tight degree constraints.

Next we show that in each of the cases above, we can find an integral solution of cost at most

$$\sum_{e \in E^*(T)} \bar{c}_e \bar{x}^*_e + \max_{v \in V(T)} \min_{e \in \delta(v)} \bar{c}_e :$$

1. Since each degree constraint other than s is tight, the path, P, must alternate between edges of $\bar{x}^*$ value α and $1-\alpha$ for some $0 < \alpha < 1$. Let $q \in V^*$ be the neighbor of t on the path and assume that $\bar{x}^*_{qt} = \alpha$. Note that we only need to ensure that the internal vertices of P are covered.

   We construct an integral solution as follows. In order to ensure that the degree constraint of t is satisfied, we set $\bar{x}^*_{qt} = 0$. However, this leaves q fractionally uncovered, so we augment $\bar{x}^*$ by including a minimal cost edge incident upon q, rq, at a fractional value of α. This increases the cost of the fractional solution by at most $\alpha \cdot \bar{c}_{rq} < \max_{v \in V(P)} \min_{e \in \delta(v)} \bar{c}_e$. The benefit of this augmentation is that we may now decompose the path P into a convex combination of two alternating matchings, both of which cover the internal vertices of P, hence the cheaper of the two must cost less than $\sum_{e \in E^*(P)} \bar{c}_e \bar{x}^*_e$.

   One might be concerned with how this effects the vertex r. Note that if r is on P then it is covered, and if it is not on P then it is covered by our previous argument which pays a total of 4/3 the fractional cost without assuming the existence of the augmented $\bar{x}^*_{rq}$ edge.

2. We subdivide this case depending on whether the cycle, C, is even or odd. If the cycle is even, then for the same reason as above, it must alternate between edges of $\bar{x}^*$ value α and $1-\alpha$. Let p and q be the neighbors of t on C, with $\bar{x}^*_{pt} = \alpha$ and $\bar{x}^*_{qt} = 1-\alpha$. Furthermore, suppose without loss of generality that $\bar{p}_{pt} \le \bar{p}_{qt}$.

   Considering the degree constraint of t, we have that

   $$\alpha \cdot \bar{p}_{pt} + (1-\alpha) \cdot \bar{p}_{qt} = \bar{\pi} - \sum_{e \in \delta(t) \setminus E^*} \bar{p}_e \bar{x}^*_e,$$

   thus we note that setting $\bar{x}^*_{qt} = 0$ and $0 \le \bar{x}^*_{pt} \le 1$ satisfies the degree constraint of t. In particular we do set $\bar{x}^*_{qt} = 0$; however, we leave the value of $\bar{x}^*_{pt}$ at α. Analogously to the path case above, we let rq be a minimal cost edge incident upon q and set $\bar{x}^*_{rq} = 1-\alpha$. As above, we have an alternating path between r and t which fractionally covers all internal vertices, completing the even cycle case.

   C is odd. Selecting p and q as above, in this case we may assume $\bar{x}^*_{pt} = \bar{x}^*_{qt} = \alpha$ and that C alternates at all vertices other than t. If $\alpha \ge 1/2$, then we have

   $$\frac{1}{2} \cdot \bar{p}_{pt} + \frac{1}{2} \cdot \bar{p}_{qt} \le \alpha \cdot \bar{p}_{pt} + \alpha \cdot \bar{p}_{qt} = \bar{\pi} - \sum_{e \in \delta(t) \setminus E^*} \bar{p}_e \bar{x}^*_e,$$

   hence the same argument as for the even cycle case applies.

   On the other hand, let a and b be the non t neighbors of p and q respectively. If $\alpha < 1/2$, we have that $\bar{x}^*_{ap} = \bar{x}^*_{bq} \ge 1/2$. Thus if e_p and e_q are minimal cost edges incident to p

and q respectively, our augmentation sets $\bar{x}^*_{e_p} = \bar{x}^*_{e_q} = \alpha$ incurring an additional cost of at most

$$\alpha \cdot \bar{c}_{e_p} + \alpha \cdot \bar{c}_{e_q} < \max_{v \in V(C)} \min_{e \in \delta(v)} \bar{c}_e,$$

yet again giving us a path (or possibly a cycle) which fractionally covers all vertices of C other than t and can be decomposed into a convex combination of two integral covers of $V(C) \setminus \{t\}$.

3. By doubling the edges on the path between t and the cycle, one may think of this case analogously to the second. We leave the details to the reader.

□

4 Prize collecting edge dominating set

The prize collecting edge dominating set problem is similar to the generalized partial edge dominating set problem in that penalties are incurred for not covering edges; however, instead of a bound on the total amount of penalties that may be payed, the sum of the penalties are simply added to the objective function. A linear relaxation for this problem is given below.

$$\begin{array}{lrl}
\text{Minimize} & \sum_{e \in E} c_e x_e + \sum_{e \in E} p_e z_e & \\
\text{subject to} & z_e + x(\nabla(e)) \geq 1 & \forall e \in E \\
& 1 \geq x_e \geq 0 & \forall e \in E \\
& 1 \geq z_e \geq 0 & \forall e \in E
\end{array}$$

Although an approximation algorithm for the generalized partial version of a covering problem can be used to solve the prize collecting version with essentially the same approximation ratio, we note that we can in fact use the construction used for Lemma 1 to give a direct rounding algorithm. The constructed solution is feasible for the corresponding prize collecting edge cover relaxation. The prize collecting edge cover problem can be solved exactly in polynomial time by reducing it to the standard edge cover problem using the same construction used in the beginning of the proof of Theorem 8. The analysis is similar to that presented in Section 2.

5 Conclusion

Our approach extends the polyhedral rounding technique that has successfully been applied to variants of the edge dominating set problem to obtain an 8/3 approximation for the generalized partial edge dominating set. In order to circumvent the NP-hardness of the generalized partial edge cover problem, we required guessing the most expensive edge chosen by an optimal solution. An obvious open problem is to use a more refined approach such as that introduced by Julian Mestre for the partial vertex cover problem [10]. Although improving the factor of 8/3 is another open problem; however, as this would imply an better approximation for $\{0,1\}$-EDS, perhaps the latter problem should be addressed first.

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