HEREDITARY QUASIRANDOM PROPERTIES OF HYPERGRAPHS

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Abstract. Thomason and Chung, Graham and Wilson were the first to investigate systematically properties of quasirandom graphs. They have stated several quite disparate graph properties—such as having uniform edge distribution or containing a prescribed number of certain subgraphs—and proved that these properties are equivalent in a deterministic sense.

Simonovits and Sós introduced a *hereditary* property (which we call $\mathcal{S}$) stating the following: for a small fixed graph L, a graph G on n vertices is said to have the property $\mathcal{S}$ if for every set $X \subseteq V(G)$, the number of labeled copies of L in $G[X]$ (the subgraph of G induced by the vertices of X) is given by $2^{-e(L)}|X|^{v(L)} + o(n^{v(L)})$. They have shown that $\mathcal{S}$ is equivalent to the other quasirandom properties.

In this paper we give a natural extension of the result of Simonovits and Sós to k-uniform hypergraphs, answering a question of Conlon et al. Our approach yields an alternative, and perhaps simpler, proof of their theorem.

1. Introduction

Given two hypergraphs L and H, an *embedding of L into H* is an injective map $\phi\colon V(L) \to V(H)$ which is also edge preserving, that is, $\phi(e) \in H$ for every edge $e \in L$.

Denote by $\#\textsc{Emb}(L, H)$ the number of embeddings of L into H. In other words, $\#\textsc{Emb}(L, H)$ counts the number of labeled copies of L in H.

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Definition 1.1 (Uniform edge distribution)**.** A k-uniform hypergraph H is called (ξ, d)-*quasirandom* if every vertex set $X \subseteq V(H)$ with $|X| \geq \xi\, |V(H)|$ induces $(d \pm \xi)\frac{|X|^k}{k!}$ edges.

Before discussing k-uniform hypergraphs (k-graphs for short) we shall restrict our attention to graphs.

Several seemingly unrelated properties turned out to be equivalent characterizations of quasirandom graphs. These properties are present in a typical random graph and, moreover, if one of them is present in a deterministic graph then all the others are also present. This equivalence was established in the seminal papers of Thomason [Tho87] and Chung, Graham and Wilson [CGW89]. Below we will call any of these equivalent properties *quasirandom*.

Theorem 1.2. *For any graph L, $d > 0$, $\gamma > 0$, there exists $\xi > 0$ and $n_0 \in \mathbb{N}$ such that for any (ξ, d)-quasirandom graph G on $n \geq n_0$ vertices we have $\#\mathrm{E{\scriptsize MB}}(L, G) = (1 \pm \gamma)d^{e(L)}n^{v(L)}$.*

A natural question is whether the converse of this result is true. Namely, if L is some small fixed graph and G is a graph on n vertices with density d containing $(d^{e(L)} + o(1))n^{v(L)}$ labeled copies of L, is it then true that G is necessarily quasirandom?

While it follows from [CGW89] that when $L = C_4$ the answer is affirmative, it turns out that there are non quasirandom graphs G with the "correct" number of triangles. However, Simonovits and Sós [SS97] showed that, for any 2-graph L, the following *hereditary* property is quasirandom.

Definition 1.3. Given a fixed k-graph L and $d, \gamma, \alpha \in (0, 1)$, a k-graph H on n vertices is said to have the *Simonovits-Sós Property* $\mathcal{S}(L, d, \gamma, \alpha)$ if for every subset $X \subseteq V(H)$ with $|X| \geq \alpha n$, we have $\#\mathrm{E{\scriptsize MB}}\,(L, H[X]) = (d^{e(L)} \pm \gamma)|X|^{v(L)}$.

Remark 1.4. If we take L to be a k-graph on k vertices consisting of a single edge, then $\mathcal{S}(L, d, \xi, \xi)$ is equivalent to (ξ, d)-quasirandomness. Indeed, for this choice of L we have $\#\mathrm{E{\scriptsize MB}}\,(L, H[X]) = k!\, e(X)$ for any $X \subset V(H)$. Therefore, $\mathcal{S}(L, d, \xi, \xi)$ implies that $e(X) = (d \pm \xi)|X|^k/k!$ for all $X \subset V(H)$ with $|X| \geq \xi n$.

It is known that Theorem 1.2 does not generally extend to k-graphs. More explicitly, let L be the (unique) 3-graph on four vertices with two edges and H be the 3-graph with edges corresponding to triangles in the random graph $G(n, 1/2)$. It is simple to check that almost surely H is $(\xi, 1/8)$-quasirandom for any $\xi > 0$. However, H contains $(1/32 + o(1))n^4$ labeled copies of L, which is two times $(d^{e(L)} + o(1))n^{v(L)}$ for the density $d = 1/8$ (see [KNRS]).

Therefore, property $\mathcal{S}$ cannot be quasirandom for k-graphs in general. On the other hand, Kohayakawa et al. [KNRS] extended Theorem 1.2 to *linear* k-graphs L (see Lemma 2.7 and Remark 2.8 below).

Definition 1.5. A k-graph L is *linear* if for every two distinct edges $e, f \in L$, we have $|e \cap f| \le 1$. (In particular, every 2-graph is linear.)

In a recent paper, Conlon et al. [CHPS] extended the results of [CGW89] to k-graphs by finding a number of equivalent hypergraph quasirandom properties analogous to those of [CGW89]. In particular, they considered a hereditary k-graph property (see Definition 2.2 below)—which easily implies property $\mathcal{S}$ (see Remark 2.3)—and established that this new property is quasirandom for k-graphs. They also asked if the property $\mathcal{S}$ is quasirandom for any linear k-graph L.

In this paper we positively answer this question by proving the following theorem.

Theorem 1.6 (Main result)**.** *Let L be a linear k-graph with at least one edge, $\xi > 0$ and $d > 0$ be given. There exists constants $n_0 \in \mathbb{N}$, $\gamma > 0$ and $\alpha > 0$ such that every k-graph H on $n \ge n_0$ vertices satisfying $\mathcal{S}(L, d, \gamma, \alpha)$ is (ξ, d)-quasirandom.*

The main tools used in the proof of Theorem 1.6 are the weak regularity lemma for k-graphs (Lemma 2.5) and its associated counting lemma for linear k-graphs (Lemma 2.7). Using this counting lemma, we show that any k-graph satisfying $\mathcal{S}$ must admit a regular partition for which almost all regular k-tuples have density close to d (Theorem 3.2). A standard argument establishes that the existence of a regular partition of this kind implies quasirandomness (Claim 3.1).

The main idea of the proof given here, which is based on the Ramsey Theorem, is different from that of [SS97]. In fact, our proof allows for a natural extension from graphs to k-graphs. We give a full outline of our proof strategy for the graph case (that is, $k = 2$) in Section 4. The proof for general k is along the same lines but needs somewhat heavier notation and the general form of the Ramsey Theorem.

2. Preliminaries

In this section we include the definitions and notation necessary to the tools used in our proof.

Definition 2.1. Let L be a linear hypergraph with $V(L) = [\ell]$ and H be an arbitrary hypergraph. Given disjoint sets $V_1, \dots, V_\ell \subset V(H)$, a *partite embedding* of L into $H[V_1, \dots, V_\ell]$ is an embedding $\phi\colon [\ell] \to V(H)$ such that $\phi(i) \in V_{\sigma(i)}$, for some $\sigma \in S_\ell$, where S_ℓ is the set of all permutations of $[\ell]$.

A partite embedding is called an *ordered embedding* if it satisfies $\phi(i) \in V_i$ for all $i \in [\ell]$ (that is, the corresponding permutation is the identity).

We denote by $\#\textsc{Part}(L, H[V_1, \dots, V_\ell])$ the number of partite embeddings and by $\#\textsc{Ord}(L, H[V_1, \dots, V_\ell])$ the number of ordered embeddings of L into $H[V_1, \dots, V_\ell]$.

It is clear from the above definitions that we have

$$\#\text{PART}(L, H[V_1, \dots, V_\ell]) = \sum_{\sigma \in S_\ell} \#\text{ORD}(\sigma(L), H[V_1, \dots, V_\ell]), \tag{1}$$

where $\sigma(L)$ is a hypergraph with edges $\{\sigma(e) : e \in L\}$.

From now on, we fix $k \geq 2$ and a linear k-graph L with $V(L) = [\ell]$ having at least one edge.

In [CHPS], the following property was proved to be quasirandom.

Definition 2.2. For any $d > 0$, $\alpha > 0$ and $\gamma > 0$, a k-graph is said to have the *Ordered Partite Property* $\mathcal{O}(L, d, \gamma, \alpha)$ if for all choices of pairwise disjoint sets $V_1, \dots, V_\ell$, with $|V_i| \geq \alpha n$ for all i, we have

$$\#\text{ORD}(L, H[V_1, \dots, V_\ell]) = (d^{e(L)} \pm \gamma) \prod_{i=1}^{\ell} |V_i|.$$

Remark 2.3. The property $\mathcal{O}$ implies $\mathcal{S}$ in the following sense: for any $\alpha, \gamma > 0$ there is $\gamma' > 0$ and $n_0 \in \mathbb{N}$ such that if H is a k-graph on $n \geq n_0$ vertices satisfying $\mathcal{O}(L, d, \gamma', \alpha)$ then H also satisfies $\mathcal{S}(L, d, \gamma, \ell\alpha)$.

Let us give a brief informal argument. Suppose that $X \subset V(H)$ is an arbitrary set of size ℓm with $m = \alpha n$. Let $X = V_1 \cup \dots \cup V_\ell$ be a random partition of X with each $|V_i| = m$. Given any embedding ϕ of L into $H[X]$, the probability that $\phi(i) \in V_i$ for all i is given by

$$p = \frac{\binom{\ell(m-1)}{m-1}\binom{(\ell-1)(m-1)}{m-1} \cdots \binom{2(m-1)}{m-1}}{\binom{\ell m}{m} \cdots \binom{2m}{m}} = \frac{m^\ell}{\ell m(\ell m - 1) \cdots (\ell m - \ell + 1)}.$$

Notice that $(1 - \ell/m)\ell^\ell \leq (1 - 1/m)^\ell \ell^\ell \leq 1/p \leq \ell^\ell$. By the first moment method there are two partitions $X = V_1 \cup \dots \cup V_\ell = W_1 \cup \dots \cup W_\ell$ satisfying

$$\begin{aligned} \#\text{ORD}(L, H[W_1, \dots, W_\ell]) &\leq p \times \#\text{EMB}\,(L, H[X]) \\ &\leq \#\text{ORD}(L, H[V_1, \dots, V_\ell]). \end{aligned}$$

Because both the left and right hand side of the above inequalities are given by $(d^{e(L)} \pm \gamma')m^\ell$, it follows that

$$\begin{aligned} \#\text{EMB}\,(L, H[X]) &= (d^{e(L)} \pm \gamma')m^\ell/p \\ &= (d^{e(L)} \pm \gamma')m^\ell(1 \pm \ell/m)\ell^\ell \\ &= (d^{e(L)} \pm \gamma)|X|^\ell, \end{aligned}$$

if n_0 is sufficiently large and γ' is sufficiently small. Since X was arbitrary, H satisfies $\mathcal{S}(L, d, \gamma, \ell\alpha)$.

The following lemma was proved for $k = 2$ in [Sha]. The same proof works for k-graphs. For completeness, we include it here.

Lemma 2.4. *Suppose that a k-graph H on n vertices satisfies $\mathcal{S}(L, d, \gamma, \alpha)$ for some $d, \alpha \in (0, 1)$. For any disjoint $V_1, \dots, V_\ell \subset V(H)$ with $|V_i| = M \geq$*

αn for all i, we have

$$\#\textsc{Part}(L, H[V_1, \ldots, V_\ell]) = \big(\ell! d^{e(L)} \pm (2\ell)^\ell \gamma\big) M^\ell.$$

Proof. The result follows from the inclusion-exclusion principle. Notice that

$$\begin{aligned}\#\textsc{Part}(L, H[V_1, \ldots, V_\ell]) &= \sum_{I \subseteq [\ell]} (-1)^{\ell - |I|} \#\textsc{Emb}\left(L, H\left[\bigcup_{i \in I} V_i\right]\right) \\ &= \sum_{I \subseteq [\ell]} (-1)^{\ell - |I|} (d^{e(L)} \pm \gamma)(|I|\, M)^\ell \\ &= \left\{ d^{e(L)} M^\ell \sum_{I \subseteq [\ell]} (-1)^{\ell - |I|} |I|^\ell \right\} \pm (2\ell)^\ell \gamma M^\ell.\end{aligned}$$

Since $\sum_{I \subset [\ell]} (-1)^{\ell - |I|} |I|^\ell = \ell!$ the lemma is proved. (This identity can be proved by by enumerating all maps $\phi\colon [\ell] \to [\ell]$ and including/excluding those with $\phi([\ell]) \subseteq I$ for $I \subseteq [\ell]$.) ∎

One of the main tools used in this paper is the *weak hypergraph regularity lemma* (see Lemma 2.5). This result is a straightforward extension of Szemerédi's regularity lemma [Sze78] for graphs. Before stating this lemma, we must introduce some definitions.

Given a k-graph H and disjoint sets $V_1, \ldots, V_k \subset V(H)$, the *density* of the k-tuple $\{V_1, \ldots, V_k\}$ is given by

$$d_{\{V_1, \ldots, V_k\}} = d(V_1, \ldots, V_k) = \frac{e(V_1, \ldots, V_k)}{|V_1| \cdot |V_2| \cdots |V_k|},$$

where $e(V_1, \ldots, V_k) = \#\{e \in H \ : \ |e \cap V_i| = 1 \text{ for all } i = 1, \ldots, k\}$. We say that the k-tuple $\{V_1, \ldots, V_k\}$ is *ε-regular* if, for all choices of $W_i \subseteq V_i$, with $|W_i| \geq \varepsilon\, |V_i|$, for all $i \in [k]$,

$$|d(V_1, \ldots, V_k) - d(W_1, \ldots, W_k)| \leq \varepsilon.$$

The weak hypergraph regularity lemma can be stated as below. Its proof is identical to the original proof of Szemerédi [Sze78].

Lemma 2.5 (Weak Hypergraph Regularity Lemma)**.** *For all $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ there exists $n_0 = n_{L2.5}(\varepsilon, t_0), T = T_{L2.5}(\varepsilon, t_0) \in \mathbb{N}$ such that the following holds.*

Given any k-graph H on $n \geq n_0$ vertices, there exists a partition $V(H) = V_1 \cup \cdots \cup V_t$, $t_0 \leq t \leq T$, with the properties

(i) $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$;
(ii) at least $(1 - \varepsilon)\binom{t}{k}$ tuples $e \in \binom{[t]}{k}$ are such that $\{V_i\}_{i \in e}$ is ε-regular.

Definition 2.6. Given a k-graph H with an ε-regular partition $V(H) = V_1 \cup \cdots \cup V_t$, we define the *reduced k-graph* $\mathcal{R}$ corresponding to this partition as the k-graph containing all ε-regular k-tuples. In particular, $V(\mathcal{R}) = \{V_1, \ldots, V_t\}$ and $|\mathcal{R}| \geq (1 - \varepsilon)\binom{t}{k}$.

For simplicity, we always assume that the number of vertices in the hypergraph H is a multiple of t and thus every regular class V_i has the same number of vertices. Indeed, we may simply add $r = t - (n \mod t)$ isolated vertices to H in order to have $t \mid n$. As $t \ll n_0 \leq n$, these new vertices have a negligible impact on the property $\mathcal{S}$.

Lemma 2.7 (Counting lemma for linear k-graphs [KNRS])**.** *For all $\gamma > 0$ there is $0 < \varepsilon = \varepsilon_{L2.7}(\gamma) < \gamma$ and $m = m_{L2.7}(\gamma) \in \mathbb{N}$ such that the following holds.*

Let H be a k-graph and $V_1, \ldots, V_\ell \subset V(H)$ be disjoint sets having $|V_i| = M \geq m$. Suppose that, for every $e \in L$, the k-tuple $\{V_i\}_{i \in e}$ is ε-regular. Then

$$\#\textsc{Ord}(L, H[V_1, \ldots, V_\ell]) = \left(\prod_{e \in L} d_{\{V_i\}_{i \in e}} \pm \gamma\right) M^\ell. \tag{2}$$

Remark 2.8. One may use the above lemma to obtain the number of labeled copies of any small linear k-graph in any quasirandom k-graph (namely, an extension of Theorem 1.2 to k-graphs follows as a corollary of Lemma 2.7). This follows from the fact that in a (ξ, d)-quasirandom k-graph H on n vertices, every k-tuple of disjoint sets $V_1, \ldots, V_k$, with $|V_i| \geq \xi n$, has density $d \pm \xi \cdot (2k)^k$ (see Lemma 2.4). As a result, if ξ is small enough, all the regular tuples in an ε-regular partition of H have density close to d. Using Lemma 2.7 this is enough to give a tight estimate of $\#\textsc{Emb}(L, H)$ for any small linear k-graph L.

The above counting lemma will be used in conjunction with the property $\mathcal{S}$ to obtain a sufficiently regular partition $\mathcal{P}$ in which the densities of regular k-tuples must satisfy the identity (3) below. We will show that in order to satisfy (3) most of the densities $d(V_{i_1}, \ldots, V_{i_k})$, with $\{V_{i_1}, \ldots, V_{i_k}\}$ a regular k-tuple in $\mathcal{P}$, are close to d.

Lemma 2.9. *For any $\gamma > 0$, $d > 0$ and $t_0 \in \mathbb{N}$ there exists $0 < \varepsilon = \varepsilon_{L2.9}(\gamma) < \gamma$, $n_0 = n_{L2.9}(\gamma, t_0) \in \mathbb{N}$, and $T = T_{L2.9}(\gamma, t_0) = T_{L2.5}(\varepsilon, t_0)$ such that the following holds.*

Suppose that H is a k-graph on $n \geq n_0$ vertices satisfying the property $\mathcal{S}\Big(L, d, \frac{\gamma}{2 \cdot (2\ell)^\ell}, \alpha\Big)$, where $\alpha = 1/T$.

Then there exists an ε-regular partition $V(H) = V_1 \cup \cdots \cup V_t$, $t_0 \leq t \leq T$, with reduced k-graph $\mathcal{R}$ satisfying the following: for every ℓ-clique in $\mathcal{R}$, say $\{V_1, \ldots, V_\ell\}$, we have

$$\sum_{\sigma \in S_\ell} \prod_{e \in \sigma(L)} d_{\{V_i\}_{i \in e}} = \ell! d^{e(L)} \pm \gamma. \tag{3}$$

Remark 2.10. Any ε-regular partition with at least t_0 classes satisfies the conclusion of Lemma 2.9. However, in order to simplify the exposition we chose to encapsulate the regularity lemma inside Lemma 2.9.

Proof of Lemma 2.9. Let $\varepsilon = \varepsilon_{L2.7}\big(\gamma/(2\ell!)\big)$ and $m = m_{L2.7}\big(\gamma/(2\ell!)\big)$. From Lemma 2.5, obtain $T = T_{L2.5}(\varepsilon, t_0)$, $n_0 = \max\{n_{L2.5}(\varepsilon, t_0), Tm\}$ and $\alpha = 1/T$.

Applying Lemma 2.5 to H we obtain an ε-regular partition $V(H) = V_1 \cup \cdots \cup V_t$, $t_0 \le t \le T$. Notice that by our choice of parameters we have $M = |V_i| = n/t \ge n/T = \alpha n \ge m$ for all i.

Suppose that every k-tuple in $\{V_1, \dots, V_\ell\}$ is ε-regular. The choice of ε, equation (1) and the counting Lemma 2.7 ensure that

$$\begin{aligned}
\#\textsc{Part}(L, H[V_1,\dots,V_\ell]) &\overset{(1)}{=} \sum_{\sigma\in S_\ell} \#\textsc{Ord}(\sigma(L), H[V_1,\dots,V_\ell]) \\
&\overset{L2.7}{=} \sum_{\sigma\in S_\ell}\Big(\prod_{e\in\sigma(L)} d_{\{V_i\}_{i\in e}} \pm \frac{\gamma}{2\ell!}\Big)M^\ell \qquad (4)\\
&= \Big\{\Big(\sum_{\sigma\in S_\ell}\prod_{e\in\sigma(L)} d_{\{V_i\}_{i\in e}}\Big) \pm \frac{\gamma}{2}\Big\}M^\ell.
\end{aligned}$$

On the other hand, from Lemma 2.4 and the property $\mathcal{S}\Big(L, d, \frac{\gamma}{2\cdot(2\ell)^\ell}, \alpha\Big)$ we obtain

$$\#\textsc{Part}(L, H[V_1,\dots,V_\ell]) = (\ell! d^{e(L)} \pm \gamma/2)M^\ell. \qquad (5)$$

The lemma follows from equations (4) and (5). ■

In view of Lemma 2.9 we will deal primarily with cliques in the reduced k-graph of a sufficiently regular partition. The next lemma establishes the abundance of large cliques in (reduced) k-graphs which are almost complete; in particular, most edges are contained in some large clique.

Lemma 2.11. *For every $s \in \mathbb{N}$ and $\delta > 0$ there exists $\varepsilon = \varepsilon_{L2.11}(s,\delta) < \delta$ such that the following holds.*

Suppose that $\mathcal{R}$ is a k-graph on $t \ge s$ vertices and $|\mathcal{R}| \ge (1-\varepsilon)\binom{t}{k}$.

Then there are at least $(1-\delta)\binom{t}{k}$ edges $e \in \mathcal{R}$ for which there exists a set $S \subset V(\mathcal{R})$, with $|S| = s$, such that $e \in S$ and $\mathcal{R}[S]$ is a complete k-graph.

Proof. Let $\varepsilon = \varepsilon_{L2.11}(s,\delta) = \delta\cdot\binom{s}{k}^{-1}$ and suppose that $\mathcal{R}$ is a k-graph on $t \ge s$ vertices with $|\mathcal{R}| \ge (1-\varepsilon)\binom{t}{k}$. If we sample an s-subset S of $V(\mathcal{R})$ randomly and uniformly, we have

$$p = \mathbf{P}[S \text{ is not a clique in } \mathcal{R}] \le \mathbf{E}\Big[\Big|\binom{S}{k} \setminus \mathcal{R}\Big|\Big] = \mathbf{E}\Big[\sum_f \mathbf{1}[f \subset S]\Big],$$

where the sum is over all $f \in \binom{V(\mathcal{R})}{k} \setminus \mathcal{R}$. By linearity of expectation, the right-hand side is upper bounded by $\varepsilon\binom{s}{k} = \delta$.

Now consider the incidence graph of edges in $\mathcal{R}$ versus s-cliques of $\mathcal{R}$. Namely, we set $\mathcal{C}$ to be the collection of all s-cliques in $\mathcal{R}$ and define a bipartite graph B with classes $\mathcal{R}$ and $\mathcal{C}$ for which $(e,S) \in B \subseteq \mathcal{R}\times\mathcal{C}$ if and only if $e \in S$.

Notice that $|\mathcal{C}| = (1-p)\binom{t}{s}$ and that the degree of $S \in \mathcal{C}$ in the graph B is $\binom{s}{k}$. On the other hand, for any $e \in \mathcal{R}$ its degree in B is upper bounded by $\binom{t-k}{s-k}$. Therefore, there must be at least

$$(1-p)\frac{\binom{t}{s}\binom{s}{k}}{\binom{t-k}{s-k}} = (1-p)\binom{t}{k} \geq (1-\delta)\binom{t}{k}$$

edges $e \in \mathcal{R}$ contained in some s-clique. ∎

3. Proof of Theorem 1.6

Before we give a proof of Theorem 1.6 we will state two auxiliary results—Claim 3.1 and Theorem 3.2—from which our main result follows. Claim 3.1 establishes a connection between regular partitions and quasirandomness.

Claim 3.1. *For any $\xi, d > 0$ there exists $\delta > 0$ and $t_0 \in \mathbb{N}$ such that the following holds.*

Suppose that H is a k-graph with a δ-regular partition $\mathcal{P}$ having $t \geq t_0$ classes. Moreover, assume that at least $(1-\delta)\binom{t}{k}$ δ-regular k-tuples in $\mathcal{P}$ have density $d \pm \delta$.

Then H is (ξ, d)-quasirandom.

The above claim was observed in [SS91] for graphs. We omit the proof of Claim 3.1 since it is essentially the same as that of [SS91].

Theorem 3.2. *For any $\delta > 0$, $d > 0$, $t_0 \in \mathbb{N}$ there exists $\alpha = \alpha_{T3.2}(\delta, d), \gamma = \gamma_{T3.2}(\delta, d) \in (0,1)$, $0 < \varepsilon = \varepsilon_{T3.2}(\delta, d) < \delta$ and $n_0 = n_{T3.2}(\delta, d, t_0) \in \mathbb{N}$ such that the following holds.*

Suppose that H is a k-graph on $n \geq n_0$ vertices satisfying $\mathcal{S}(L, d, \gamma, \alpha)$.

Then there exists an ε-regular partition $\mathcal{P}$ of H with $t \geq t_0$ classes such that at least $(1-\delta)\binom{t}{k}$ k-tuples in $\mathcal{P}$ are ε-regular and have density $d \pm \delta$.

Now we will conclude the proof of Theorem 1.6. For a given linear k-graph L and values of $\xi, d > 0$, we obtain by Claim 3.1, $\delta > 0$ and $t_0 \in \mathbb{N}$. From Theorem 3.2 we then obtain α, γ, n_0 and ε.

Consider a k-graph H on $n \geq n_0$ satisfying $\mathcal{S}(L, d, \gamma, \alpha)$. Applying Theorem 3.2 to H we obtain an ε-regular partition $\mathcal{P}$ such that at least $(1-\delta)\binom{t}{k}$ k-tuples in $\mathcal{P}$ are ε-regular and have density $d \pm \delta$. Since $\varepsilon < \delta$, the partition $\mathcal{P}$ is also δ-regular. Claim 3.1 thus ensures that H is (ξ, d)-quasirandom. Therefore Theorem 1.6 follows.

4. A Ramsey-type argument (proof of Theorem 3.2)

First we will outline our strategy for proving Theorem 3.2. To further simplify the presentation, we will focus only on graphs (that is, the $k = 2$ case). More specifically, we will show that a graph H satisfying property $\mathcal{S}$ for suitable parameters admits a sufficiently regular partition for which most of the regular pairs have density close to d.

Given δ, d and t_0, we will choose parameters $\delta_1, \delta_0, \gamma, \varepsilon$ and α satisfying

$$\delta \gg \delta_1 \gg \delta_0 \gg \gamma \gg \varepsilon \gg \alpha. \tag{6}$$

We we will also choose $t_1 \gg s \gg \ell$, $t_1 \geq t_0$, and n_0 and consider a k-graph H on $n \geq n_0$ vertices satisfying $\mathcal{S}\Big(L, d, \frac{\gamma}{2\cdot(2\ell)^\ell}, \alpha\Big)$.

From Lemma 2.9 we obtain an ε-regular partition $V(H) = V_1 \cup V_2 \cup \cdots \cup V_t$, $t \geq t_1$ such that for any ℓ-clique in its reduced graph $\mathcal{R}$, say $\{V_1, \ldots, V_\ell\}$,

$$\sum_{\sigma \in S_\ell} \prod_{e \in \sigma(L)} d_{\{V_i\}_{i \in e}} = \ell! d^{e(L)} \pm \gamma \tag*{[3]}$$

is satisfied.

Moreover, since we choose $t_1 \geq s$ and $\varepsilon \leq \varepsilon_{L2.11}(s, \delta)$ it follows by Lemma 2.11 that at least $(1-\delta)\binom{t}{2}$ edges of $\mathcal{R}$ are contained in some s-clique of $\mathcal{R}$.

Suppose that $S = \{V_1, V_2, \ldots, V_s\}$ is a clique in $\mathcal{R}$ (by possibly reordering the elements in $\{V_i\}_{i=1}^t$). We will show that $d(V_1, V_2) = d \pm \delta$. Since at least $(1-\delta)\binom{t}{2}$ edges of $\mathcal{R}$ are contained in an s-clique, Theorem 3.2 follows.

A pair in S with density ρ is classified as δ_0-*dense* if $\rho > d + \delta_0$, δ_0-*sparse* if $\rho < d - \delta_0$ and δ_0-*balanced* if $\rho = d \pm \delta_0$. Clearly, this classification is a three-coloring of the pairs in S.

If there is an ℓ-clique $\{V_1, \ldots, V_\ell\}$ in S for which every pair is δ_0-dense then (3) fails. Indeed, the left-hand side of (3) would be at least $\ell!(d+\delta_0)^{e(L)}$ which, due to our choice of parameters (see (6)), is larger than $\ell! d^{e(L)} + \gamma$, implying a contradiction with (3). Similarly, an ℓ-clique in which every pair is δ_0-sparse would also fail to satisfy (3). By the Ramsey Theorem for graphs and three colors, there exists a large clique S_0 in $\{V_3, \ldots, V_s\}$, say $S_0 = \{V_3, \ldots, V_r\}$, such that every pair in S_0 is δ_0-balanced.

Next we claim that if $|S_0| = r-2 \geq 5(\ell-2)$ then there exists $\ell-2$ classes in S_0, say $\{V_3, \ldots, V_\ell\}$, such that both $d(V_1, V_j) = d \pm \delta_1$ and $d(V_2, V_j) = d \pm \delta_1$ for all $j = 3, \ldots, \ell$. Otherwise, there exists $4(\ell-2)+1$ classes $V_j \in S_0$ that do not form a δ_1-balanced pair with either V_1 or V_2. Therefore, one of V_1 or V_2, say V_1, does not form a δ_1-balanced pair with at least $2(\ell-2)+1$ classes $V_j \in S_0$. Consequently, there are either $\ell-1 = (\ell-2)+1$ classes $V_j \in S_0$ forming δ_1-dense pairs with V_1 or $\ell-1$ classes $V_j \in S_0$ forming δ_1-sparse pairs with V_1.

We will demonstrate that the existence of any collection with $\ell-1$ classes forming δ_1-dense pairs with V_1 contradicts (3). Indeed, consider any such collection of classes together with V_1 in (3). Every term in the sum (3) would at least $(d-\delta_0)^{e(L)}$ and there is at least one term of this sum which would be larger than $(d+\delta_1)(d-\delta_0)^{e(L)-1}$ (since L contains at least one edge e, there must be some $\sigma \in S_\ell$ such that 1 is a vertex of $\sigma(e)$). Given our choice of $\delta_1 \gg \delta_0 \gg \gamma$, the sum must be larger than $\ell! d^{e(L)} + \gamma$. Similarly, any collection with $\ell-1$ classes forming δ_1-sparse pairs with V_1 contradicts (3).

Summarizing, we have argued that all pairs in $\{V_1, \ldots, V_\ell\}$, except possibly, $\{V_1, V_2\}$ have density close to d (in fact, they are δ_1-balanced). Then the only way to satisfy (3) is by having $d(V_1, V_2)$ close to d as well. Indeed, if, say $d(V_1, V_2) > d + \delta$, then every term in the sum on the left-hand side of (3) would be at least $(d - \delta_1)^{e(L)}$ and there is at least one term which is larger than $(d + \delta)(d - \delta_1)^{e(L)-1}$. Given our choice of $\delta \gg \delta_1 \gg \gamma$, the sum must be larger than $\ell! d^{e(L)} + \gamma$, contradicting (3).

We have just proved that any pair in the reduced $\mathcal{R}$ which is contained in an s-clique must have density $d \pm \delta$. Recalling that we choose our parameters so that at least $(1 - \delta)\binom{t}{2}$ pairs have this property, this concludes the proof of Theorem 3.2 (for graphs).

In the argument for the general case we require a more general form of the Ramsey theorem.

Definition 4.1 (Ramsey numbers for k-graphs)**.** Let $R_k(a_1, \ldots, a_j)$ denote the smallest number R such that any j-coloring of the edges of the complete k-graph on R vertices induces, for some $i \in [j]$, a complete i-colored k-graph of size a_i.

The following estimate

$$(d \pm z)^a = d^a \pm 2^a z, \quad a \in \mathbb{N},\ |z| < d \leq 1 \tag{7}$$

will be useful in the computations below. To prove it, observe that for $|z| < d \leq 1$ and $a \in \mathbb{N}$ we have

$$|(d \pm z)^a - d^a| \leq \sum_{i=1}^{a} \binom{a}{i} d^{a-i} |z|^i \leq 2^a |z|.$$

Proof of Theorem 3.2. First we introduce a large auxiliary constant s (depending on k and ℓ only) that we define later. Given δ, d and t_0 we will choose the following parameters. Set

$$\gamma = \gamma(\delta, d) = \min\left\{\varepsilon_{L2.11}(s, \delta), \delta \cdot \left(\frac{d^{e(L)-1}}{\ell!\, 2^{e(L)+1}}\right)^{k+1}\right\}, \tag{8}$$

define $\gamma_{T3.2}(\delta, d) = \frac{\gamma}{2 \cdot (2\ell)^\ell}$ and $\varepsilon = \varepsilon_{3.2}(\delta, d) = \varepsilon_{L2.9}(\gamma) < \min\{\gamma, \delta\}$. We also set $t_1 = \max\{t_0, s\}$ and let $\alpha = \alpha_{L2.9}(\gamma, t_1)$, $n_0 = n_{L2.9}(\gamma, t_1)$.

Let H be a k-graph on $n \geq n_0$ vertices satisfying $\mathcal{S}(L, d, \gamma_{T3.2}(\delta, d), \alpha)$. Applying Lemma 2.9 to H we obtain an ε-regular partition $\mathcal{P}$ with $t \geq t_1$ classes. Let $\mathcal{R}$ denote the reduced k-graph corresponding to the ε-regular partition $\mathcal{P}$. By construction, we have $|\mathcal{R}| \geq (1 - \varepsilon)\binom{t}{k}$.

By ensuring that $\gamma \leq \varepsilon_{L2.11}(s, \delta)$, we have $\varepsilon \leq \gamma \leq \varepsilon_{L2.11}(s, \delta)$. Hence, from Lemma 2.11, we conclude that at least $(1 - \delta)\binom{t}{k}$ k-tuples $e \in \mathcal{R}$ are such that there exists $e \subset S \subset V(\mathcal{R})$, $|S| = s$, for which $\mathcal{R}[S]$ is a complete k-graph. Let

$$\mathcal{R}_s = \{e \in \mathcal{R} : \text{ there exists an } s\text{-clique } S \subset V(\mathcal{R}) \text{ with } e \subset S\}.$$

Claim 4.2. *For any $e \in \mathcal{R}_s$ we have $d_e = d \pm \delta$.*

Since $|\mathcal{R}_s| \geq (1-\delta)\binom{t}{k}$, Theorem 3.2 immediately follows from Claim 4.2. From now on, fix any edge $e \in \mathcal{R}_s$ and a clique $S \supset e$ with s elements. We now proceed to show that $d_e = d \pm \delta$.

Consider the sequence $\delta = \delta_k \gg \delta_{k-1} \gg \cdots \gg \delta_0$ (terms of a geometric progression) defined by

$$\delta_i = \delta \cdot \Big(\frac{d^{e(L)-1}}{\ell!\, 2^{e(L)+1}}\Big)^{k-i}. \tag{9}$$

Also define the sequence $\{s_i\}_{i=0}^{k-1}$ by setting $s_{k-1} = \ell$ and

$$s_i = R_{k-i-1}\big(\underbrace{\ell, \ell, \ldots, \ell}_{2\binom{k}{i+1}\text{ times}}, s_{i+1}\big) \tag{10}$$

for all $i = 0, 1, \ldots, k-2$. Set $s = R_k(\ell, \ell, s_0) + k$. (Notice that $s = s(k, \ell)$ as required.)

We will construct sets $S \supset S \setminus e \supset S_0 \supset S_1 \supset \cdots \supset S_{k-1} \supset S_k = \emptyset$ such that $|S_i| = s_i$ and

(‡) **any tuple** $f \in \binom{S_i \cup e}{k}$**, with** $|f \cap e| = i$**, has density** $d_f = d \pm \delta_i$

for all $i = 0, \ldots, k$. In particular, we have $d_e = d \pm \delta_k = d \pm \delta$.

Let $V(\mathcal{R}) = \{V_1, \ldots, V_t\}$ be the collection of regular classes in $\mathcal{P}$. Since $\mathcal{P}$ was obtained from Lemma 2.9, we have, for any ℓ-clique, say $\{V_1, \ldots, V_\ell\}$ in $\mathcal{R}$,

$$\sum_{\sigma \in S_\ell} \prod_{f \in \sigma(L)} d_{\{V_i\}_{i \in f}} = \ell! d^{e(L)} \pm \gamma. \tag{3}$$

Denote by $2^e = \{A_1, A_2, \ldots, A_{2^k}\}$ the collection of all subsets of e. Let $\Sigma = \{+, -\}$. For $i = 0, 1, \ldots, k-1$, define the coloring $\chi_i \colon \binom{S \setminus e}{k-i} \to \{\sim\} \cup \big(\binom{e}{i} \times \Sigma\big)$ as follows. Given a $(k-i)$-tuple B in $S \setminus e$, if $d_{A \cup B} = d \pm \delta_i$ for all $A \in \binom{e}{i}$ then we set $\chi_i(B) = \sim$. Otherwise we let j be the minimum number such that $A_j \in \binom{e}{i}$ satisfies $|d_{A_j \cup B} - d| > \delta_j$ and set

$$\chi_i(B) = \begin{cases} (A_j, +) & \text{if } d_{A_j \cup B} > d + \delta_j, \\ (A_j, -) & \text{if } d_{A_j \cup B} < d - \delta_j. \end{cases}$$

Let us consider the 3-coloring $\chi_0 \colon \binom{S \setminus e}{k} \to \{\sim, (\emptyset, +), (\emptyset, -)\}$. Since $|S \setminus e| = s - k = R_k(\ell, \ell, s_0)$, if we show that there are neither ℓ-cliques colored $(\emptyset, +)$ nor ℓ-cliques colored $(\emptyset, -)$ in S then we infer by the Ramsey Theorem that there must exist a set $S_0 \subset S$, with $|S_0| = s_0$, such that every k-tuple in S_0 is colored $\sim$. In particular, every k-tuple in $\binom{S_0}{k}$ has density $d \pm \delta_0$ and thus S_0 satisfies (‡).

Suppose, for the sake of contradiction, that we may find an ℓ-clique in $\binom{S \setminus e}{k}$, say $\{V_1, \ldots, V_\ell\}$, that is colored $(\emptyset, +)$ under χ_0. Because of the coloring, every k-tuple in $\{V_i\}_{i=1}^{\ell}$ has density $> d + \delta_0$. Therefore,

$$\sum_{\sigma \in S_\ell} \prod_{f \in \sigma(L)} d_{\{V_i\}_{i \in f}} > \ell!(d+\delta_0)^{e(L)} \geq \ell! d^{e(L)} + \ell! \delta_0 d^{e(L)-1} \overset{(8)}{>} \ell! d^{e(L)} + \gamma,$$

since $\gamma \leq \delta_0 d^{e(L)-1}/2$. However, this is a contradiction with (3) and hence no such ℓ-clique exists. Similarly, we may show that there are no ℓ-cliques colored $(\emptyset, -)$ under χ_0. Hence, we may obtain a set $S_0 \subset S \setminus e$ satisfying (‡).

Suppose that the sets $S_i \subset \cdots \subset S_0 \subset S \setminus e \subset S$ were already constructed and satisfy (‡). Let us construct the set $S_{i+1} \subset S_i$. Consider the coloring χ_{i+1} induced on $\binom{S_i}{k-i-1}$. Suppose that $A \in \binom{e}{i+1}$ is such that there exists an $(\ell - i - 1)$-set K, say $K = \{V_{i+2}, \ldots, V_\ell\}$ and $A = \{V_1, \ldots, V_{i+1}\}$, such that every $(k-i-1)$-tuple in K is colored $(A, +)$ under χ_{i+1}.

The tuples $f \in \binom{K \cup A}{k}$ that do not contain A must intersect e in at most i elements since $K \subset S_i \subset S \setminus e$. Therefore, since $K \subset S_i$ and S_i satisfies (‡), we must have $d_f = d \pm \delta_i$. On the other hand, if $A \subset f$ then the coloring of the tuples of K under χ_{i+1} indicates that $d_f > d + \delta_{i+1}$.

Set $x = \#\big\{\sigma \in S_\ell : \{1, \ldots, i+1\} \subset f \text{ for some } f \in \sigma(L)\big\} \geq 1$. We have

$$\begin{aligned}
\ell! d^{e(L)} + \gamma &\overset{(3)}{\geq} \sum_{\sigma \in S_\ell} \prod_{f \in \sigma(L)} d_{\{V_i\}_{i \in f}} \\
&> (\ell! - x)(d - \delta_i)^{e(L)} + x(d + \delta_{i+1})(d - \delta_i)^{e(L)-1} \\
&\overset{(7)}{\geq} (\ell! - x)(d^{e(L)} - 2^{e(L)}\delta_i) + x(d+\delta_{i+1})d^{e(L)-1} - x(d+\delta_{i+1})2^{e(L)-1}\delta_i \\
&\geq \ell! d^{e(L)} + x\delta_{i+1} d^{e(L)-1} - \ell! 2^{e(L)} \delta_i \\
&\overset{(9)}{\geq} \ell! d^{e(L)} + \delta_{i+1} d^{e(L)-1}/2,
\end{aligned}$$

which is a contradiction since $\gamma \leq \delta_0 d^{e(L)-1}/2 < \delta_{i+1} d^{e(L)-1}/2$ (see equations (8) and (9)).

Similarly, we may show that there cannot be an $(\ell - i - 1)$-set in which every tuple is colored $(A, -)$ under χ_{i+1}. By the definition of s_i and s_{i+1}, there exists a set $S_{i+1} \subset S_i$, with $|S_{i+1}| = s_{i+1}$, in which every tuple is colored $\sim$ under χ_{i+1}. This means that for any $f \in \binom{S_{i+1} \cup e}{k}$, with $|f \cap e| = i+1$, we have $d_f = d \pm \delta_{i+1}$. Therefore the set S_{i+1} satisfies the requirements in (‡).

It follows by induction that we can construct $S_{k-1} \subset \cdots \subset S_0 \subset S \setminus e$ satisfying (‡). We are now going to show that $d_e = d \pm \delta$. Consider the classes belonging to e together with $\ell - k$ classes from the set S_{k-1}. By possibly re-labeling the elements of $V(\mathcal{R})$ we may assume that the obtained set is $\{V_1, \ldots, V_\ell\} \subset S_{k-1} \cup e$ and that $e = \{V_1, \ldots, V_k\}$.

Set $x = \#\{\sigma \in S_\ell : \{1,\dots,k\} \in \sigma(L)\} \geq 1$. Similarly as before, we have

$$\begin{aligned}\sum_{\sigma\in S_\ell}\prod_{f\in\sigma(L)} d_{\{V_i\}_{i\in f}} &= (\ell! - x)(d \pm \delta_{k-1})^{e(L)} + x d_e (d \pm \delta_{k-1})^{e(L)-1} \\ &\overset{(7)}{=} (\ell! - x)(d^{e(L)} \pm 2^{e(L)}\delta_{k-1}) + x d_e (d^{e(L)-1} \pm 2^{e(L)-1}\delta_{k-1}) \\ &= \ell! d^{e(L)} + x(d_e - d) d^{e(L)-1} \pm \ell!\, 2^{e(L)}\delta_{k-1} \\ &\overset{(9)}{=} \ell! d^{e(L)} + x\Big(d_e - d \pm \frac{\delta}{2}\Big) d^{e(L)-1}\end{aligned}$$

From (3) and $\gamma \leq \delta_0 d^{e(L)-1}/2 < \delta d^{e(L)-1}/2$ (see equations (8) and (9)) we conclude that $d_e = d \pm \delta$. Therefore, Claim 4.2 is proved and Theorem 3.2 follows. ∎

References

[CGW89] Fan R. K. Chung, Ronald L. Graham, and Richard M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989. 1, 1, 1

[CHPS] David Conlon, Hiệp Hàn, Yury Person, and Mathias Schacht. Weak quasi-randomness for uniform hypergraphs. Submitted. 1, 2

[KNRS] Yoshiharu Kohayakawa, Brendan Nagle, Vojtěch Rödl, and Mathias Schacht. Weak regularity and linear hypergraphs. J. Comb. Theory, Ser. B, to appear. 1, 2.7

[Sha] Asaf Shapira. Quasi-randomness and the distribution of copies of a fixed graphs. Combinatorica, to appear. 2

[SS91] Miklós Simonovits and Vera T. Sós. Szemerédi's partition and quasirandomness. *Random Structures Algorithms*, 2(1):1–10, 1991. 3

[SS97] Miklós Simonovits and Vera T. Sós. Hereditarily extended properties, quasi-random graphs and not necessarily induced subgraphs. *Combinatorica*, 17(4):577–596, 1997. 1, 1

[Sze78] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978. 2

[Tho87] Andrew Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987. 1