Math 111 - Quiz 3
Name: Solutions

September 21, 2012

Instructions: Show all of your work and mark your answers clearly.

1. (5 points) Prove that \( \lim_{x \to 0} x^4 \cos(2/x) = 0. \)

\[-1 \leq \cos\left(\frac{2}{x}\right) \leq 1, \ so \]
\[-x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4 \]

\( \lim_{x \to 0} (-x^4) = \lim_{x \to 0} (x^4) = 0, \ so \)

\( \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0 \) by the Squeeze Theorem.

2. (5 points) Find an equation of the tangent line to the curve \( y = \sqrt{x} \) at the point \( (1,1) \).

\[ y = f(x) \]

\[ f'(1) = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \left( \frac{\sqrt{x} - 1}{x - 1} \right) \left( \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \right) \]

\[ = \lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2} \]

Tangent Line: \( y - 1 = \frac{1}{2} (x - 1) \).
3. (10 points)

(a) From the graph of \( f \), state the numbers at which \( f \) is discontinuous and explain why.

(b) For each of the numbers stated in part (a), determine whether \( f \) is continuous from the right, or from the left, or neither.

(a) \( f \) is discontinuous at
- \( a = -4 \) because \( f(-4) \) is not defined
- \( a = -2, 2, 4 \) because \( \lim_{x \to a} f(x) \) does not exist

\[ \lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x), \quad \text{and} \quad \lim_{x \to 4^-} f(x) \text{ does not exist} \]

(b) \( f \) is left-continuous at \( x = -2 \)

\( f \) is right-continuous at \( x = 2 \) and \( x = 4 \)

\( f \) is neither left nor right continuous at \( x = -2 \)
4. (Optional Bonus - 3 points) Show that the equation $x^3 - x + 1 = 0$ has a real root.

Let $f(x) = x^3 - x + 1$.

Then $f(-2) = (-2)^3 - (-2) + 1 = -5$ and $f(-1) = (-1)^3 - (-1) + 1 = 1$.

So $f(-2) < 0 < f(-1)$.

Since $f$ is continuous on $(-\infty, \infty)$, and in particular on $[-2, -1]$, by the Intermediate Value Theorem, $f$ has a root (on the interval $(-2, -1)$).