Rate of Change

1. Let \( s(t) = t + \sin t, \) \( 0 \leq t \leq 2\pi, \) denote the position of a particular at time \( t. \)

   (a) Find the velocity at time \( t. \)

   (b) When is the particle at rest?

   (c) When is the particle moving in the positive direction?

   (d) Find the acceleration at time \( t. \)

   (e) When is the particle speeding up? When is the particle slowing down?

2. Suppose that you have a population which, for examples, doubles every hour, starting with some initial population \( P_0. \) Then \( P(1) = 2P_0, \) \( P(2) = 2(2P_0) = 2^2(P_0), \) \( P(3) = 2^3P_0 \) and so on. Then we get that in general, \( P(t) = 2^tP_0. \) We can model the rate at which this population increases via the derivative. So that in this example, the rate of growth is \( 2^tP_0 \ln 2, \) at the time \( t. \)

   Suppose that a population of bacteria increases by a factor of 10 every hour and starts with 1 bacterium. Find an expression for the number \( n \) of bacteria after \( t \) hours and use it to estimate the rate of growth of the bacteria population after 24 hours.

3. Suppose \( C(x) \) denotes the total cost of production for a given commodity. The cost of producing the 11th item, for example, is \( C(11) - C(10), \) and we can model the cost of producing an extra item by the derivative of the cost function. This is called the marginal cost, and it represents how the cost is changing with respect to the number of items produced.

   The cost function for production of a certain commodity is

   \[
   C(x) = 1000 + 2x + .01x^2.
   \]

   (a) Find \( C'(1000) \) and interpret its meaning.

   (b) Compare \( C'(1000) \) with the cost of producing the 1001st item.
4. A spherical balloon is being inflated. Find a formula for the rate of increase of the volume of the balloon \( V = \frac{4}{3}\pi r^3 \) of the balloon with respect to the radius \( r \). What is the rate of increase when \( r = 1, 2, 3 \)?

**Exponential Growth and Decay**

1. In many natural phenomena, populations will grow or decay at a rate proportional to their size, or in other words, there exists some constant \( k \) such that \( \frac{dP}{dt} = kP \). We call \( k \) the relative growth rate. This type of behavior gives rise to a population function of the form \( P(t) = P_0 e^{kt} \), where \( P_0 \) is the initial population (we will show this later in the class).

The population of the town of Calculus land in 2000 was 1000, and the population in 2010 was 5000.

(a) Assuming the above model for population growth, what is the relative growth rate?

(b) Using this model, estimate the population of Calclus land in 2020.

(c) Estimate in how many years (from 2013), the population will hit 1,000,000.

2. Similarly, we can model decay by \( m(t) = m_0 e^{kt} \), where \( m(t) \) is the mass of a substance at time \( t \), and \( m_0 \) is its initial mass. Recall that the half-life of a substance is the time required for half of any given quantity to decay.

Suppose someone gave you 100mg of Rhenium-182 (which has a half-life of 64 hours) for your wedding, right before you go on an 8 day honeymoon. Upon your return (assuming exactly 8 days have passed since you had 100mg), how much is left?