Math 111 Section 006
Worksheet 2: Curve Sketching

1. \( f(x) = x^4 - 8x^2 + 16 \)
   
   (a) Domain: \( \mathbb{R} \)
   
   (b) We can factor \( f(x) = x^4 - 8x^2 + 16 = (x^2 - 4)^2 \), so the \( x \)-intercepts are \( x = \pm 2 \). For the \( y \)-intercept we plug in 0 to get \( f(0) = y = 16 \).
   
   (c) Since \( f \) is defined on all of \( \mathbb{R} \), there are no vertical asymptotes. For the horizontal asymptotes we see that \( \lim_{x \to \pm \infty} (x^4 - 8x^2 + 16) = \infty \), so there are no horizontal asymptotes.
   
   (d) To find the critical points we set \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 0 \). So the critical points are \( x = 0, \pm 2 \).
   
   (e) We check that \( f'(x) \) is positive (so \( f \) is increasing) on \(( -2, 0) \cup (2, \infty) \) and \( f'(x) \) is negative (so \( f \) is decreasing) on \(( -\infty, 2) \cup (0, 2) \).
   
   (f) We use the first derivative test to see that there is a local maximum at \( x = 0 \) and local minimums at \( x = \pm 2 \).
   
   (g) We set \( f''(x) = 12x^2 - 16 = 4(3x^2 - 4) = 0 \) to see that \( f''(x) \) could only change sign at \( x = \pm \sqrt{4/3} \). Now we check the sign of \( f''(x) \) and see that \( f \) is concave up on \(( -\infty, -\sqrt{4/3}) \cup (\sqrt{4/3}, \infty) \) and concave down on \(( -\sqrt{4/3}, \sqrt{4/3}) \). Since the concavity changes at \( x = \pm \sqrt{4/3} \), these are our inflection points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(-\sqrt{4/3})</td>
<td>64/9</td>
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<tr>
<td>( \sqrt{4/3} )</td>
<td>64/9</td>
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</tbody>
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You can see the graph of \( f \) here: ![Graph of f(x) = x^4 - 8x^2 + 16](http://www.wolframalpha.com/input/?i=x%5E4+-+8x%5E2+%2B+16%2C+x+from+-4+to+4)

2. \( f(x) = \frac{x^2 + 1}{x^2 - 9} \)

   (a) Domain: \( x \neq \pm 3 \)

   (b) There are no \( x \)-intercepts, since \( x^2 + 1 \) is always positive. For the \( y \)-intercept we check \( f(0) = -1/9 \).

   (c) For the vertical asymptotes, we need to check what happens to the left and to the right of \( x = \pm 3 \). Note that when we plug in \( x = \pm 3 \) we get 10/0, so we know there will be vertical asymptotes there. To compute the following limits, we note that \( x^2 + 1 \) is always positive, and that \( x^2 - 9 \) factors as \((x + 3)(x - 3)\).

\[
\lim_{x \to 3^-} \frac{x^2 + 1}{x^2 - 9} = \infty \quad \lim_{x \to 3^+} \frac{x^2 + 1}{x^2 - 9} = -\infty \\
\lim_{x \to -3^-} \frac{x^2 + 1}{x^2 - 9} = -\infty \quad \lim_{x \to -3^+} \frac{x^2 + 1}{x^2 - 9} = \infty 
\]
For the horizontal asymptotes we note that \( \lim_{x \to \pm \infty} \frac{x^2 + 1}{x^2 - 9} = 1 \), so we have a horizontal asymptote at \( y = 1 \).

(d) To find the critical points we need to find \( f'(x) \).

\[
f'(x) = \frac{(x^2 - 9)(2x) - (x^2 + 1)(2x)}{(x^2 - 9)^2} = \frac{2x^3 - 18x - 2x^3 - 2x}{(x^2 - 9)^2} = \frac{-20x}{(x^2 - 9)^2}.
\]

The only \( x \)-values for which \( f'(x) \) is undefined are \( x = \pm 3 \), which are not in our domain. So any critical point will come from setting \( f'(x) = 0 \). So the only critical point is \( x = 0 \).

(e) Notice that \((x^2 - 9)^2\) is always positive, so the sign can only change at the critical point \( x = 0 \). We see that \( f \) is increasing on \((-\infty, 0)\) and \( f \) is decreasing on \((0, \infty)\).

(f) Since \( f \) is increasing then decreasing at \( x = 0 \), it corresponds to a local maximum. There are no other critical points, so \( f \) has no local minimums.

(g) To figure out the concavity, we find \( f''(x) \).

\[
f''(x) = \frac{(x^2 - 9)^2(-20) - (20x)(2(x^2 - 9)(2x))}{(x^2 - 9)^4} = \frac{(x^2 - 9)((x^2 - 9)(-20) + 80x^2)}{(x^2 - 9)^3} = \frac{60(x^2 + 3)}{(x^2 - 9)^3}.
\]

Since \( f''(x) \) is never 0, the only \( x \)-values where the concavity can change are \( x = \pm 3 \), when the denominator changes sign. These are not inflection points since they are not in the domain, so there are no inflection points, but the concavity does change at the vertical asymptotes. \( f \) is concave up on \((-\infty, 3) \cup (3, \infty)\) and concave down on \((-3, 3)\).

(h) The only relevant point is \((0, -1/9)\).

You can see the graph of \( f \) here: [http://www.wolframalpha.com/input/?i=%28x%5E2+%2B+1%29%2F%28x%5E2+-+9%29+x+from+-10+to+10](http://www.wolframalpha.com/input/?i=%28x%5E2+%2B+1%29%2F%28x%5E2+-+9%29+x+from+-10+to+10)

3. \( f(x) = \frac{e^x}{x} \)

(a) Domain: \( x \neq 0 \)

(b) \( f \) has no \( x \) or \( y \) intercepts.

(c) Since plugging in \( x = 0 \) gives \( 1/0 \), we know that \( f \) has a vertical asymptote there. We check that

\[
\lim_{x \to 0^-} \frac{e^x}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{e^x}{x} = \infty.
\]

For the horizontal asymptotes we check what happens as \( x \to \pm \infty \). We see that

\[
\lim_{x \to \infty} \frac{e^x}{x} = \lim_{x \to \infty} \frac{e^x}{1} = \infty
\]

by l'Hôpital’s rule, and

\[
\lim_{x \to -\infty} \frac{e^x}{x} = 0
\]

since \( e^x \) approaches 0 as \( x \) approaches \( -\infty \) and a limit type of \( 0/\infty \) returns 0.
(d) For the critical points we see that
\[ f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x - 1)}{x^2} \]
is undefined only at \( x = 0 \) and equals 0 at \( x = 1 \), since \( e^x > 0 \) for all \( x \). So the only critical point is \( x = 1 \).

(e) Since \( e^x > 0 \) and \( x^2 > 0 \), \( f'(x) < 0 \) when \( x < 1 \) and \( f'(x) > 0 \) when \( x > 1 \). So \( f \) is decreasing on \((-\infty, 0) \cup (0,1)\) and increasing on \((1,\infty)\).

(f) Then the first derivative test tells us that \( f \) has a local minimum at \( x = 1 \).

(g) To figure out the concavity we see that
\[ f''(x) = \frac{x^2(e^x + e^x(x - 1)) - e^x(x - 1)(2x)}{x^4} = \frac{x^2e^x + x^3e^x - x^2e^x - 2xe^x + 2xe^x}{x^4} \]
\[ = \frac{xe^x(x^2 - 2x + 2)}{x^4} = \frac{e^x(x^2 - 2x + 2)}{x^3}. \]

Here it gets a little tricky, because you can’t immediately factor \( x^2 - 2x + 2 \). We can check whether it has any roots by taking the discriminant, which \((-2)^2 - 4(1)(2) = -4\). Since the discriminant is negative, it doesn’t have any roots, and since it takes positive values, it must be entirely above the \( x \)-axis. Another way to see this is to graph it, because you have the skills to do that! Notice that its derivative is \( 2x - 2 \) which equals 0 when \( x = 1 \). Plugging in \( x = 1 \), we see that we get \( 1 - 2 + 2 = 1 \), so the vertex of the graph (the absolute minimum), occurs above the \( x \)-axis, so the parabola lies entirely above the \( x \)-axis. Now, back to the matter at hand. This tells us that the numerators is always positive and that there are therefore no inflection points. The only \( x \)-value at which \( f''(x) \) can change sign is \( x = 0 \), when the denominator changes sign. We see that \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \), so \( f \) is concave up on \((0,\infty)\) and concave down on \((-\infty, 0)\).

(h) The only relevant point is \((1, e)\).

You can see the graph of \( f \) here: [http://www.wolframalpha.com/input/?i=e%5Ex%2Fx%2C+x+from+-2+to+5](http://www.wolframalpha.com/input/?i=e%5Ex%2Fx%2C+x+from+-2+to+5)

Note: it’s not important to get the shape exactly right, what is important is to see the local minimum at \( x = 1 \) and that it’s concave up after \( x = 0 \).

4. \( f(x) = \frac{\ln x}{x} \)

(a) Domain: \((0,\infty)\)

(b) \( f \) has no \( y \)-intercept, and the only \( x \)-intercept occurs at \( x = 1 \), when \( \ln x = 0 \).

(c) We know we have a vertical asymptote at \( x = 0 \), because when we plug in \( x = 0 \) we get \( 1/0 \). We only need to consider what happens to the right of 0 since our function is not defined to the left. We see that
\[ \lim_{x \to 0^+} \frac{\ln x}{x} = -\infty \]
since ln \( x \) becomes negative as we approach \( x = 0 \) from the right. For the horizontal asymptotes, we only need to consider what happens as \( x \to \infty \). We use l'Hôpital's rule to determine the limit, since it is of the form \( \frac{\infty}{\infty} \).

\[
\lim_{{x \to \infty}} \frac{\ln x}{x} = \lim_{{x \to \infty}} \frac{1/x}{1} = 0.
\]

(d) For the critical points we consider

\[
f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.
\]

The derivative is defined on the entire domain, and \( 1 - \ln x = 0 \) when \( \ln x = 1 \), which occurs at \( x = e \). So we have a critical point at \( x = e \).

(e) We see that \( f'(x) > 0 \) for \( x < e \), for example by plugging in \( x = 1 \), and that \( f'(x) < 0 \) for \( x > e \), for example by plugging in \( x = e^2 \). So \( f \) is increasing on \((0, e)\) and decreasing on \((e, \infty)\).

(f) Therefore by the first derivative test, \( f \) has a local max at \( x = e \).

(g) Now we consider

\[
f''(x) = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{-x - 2x(1 - \ln x)}{x^4} = \frac{x(-1 - 2 + 2 \ln x)}{x^4} = \frac{2 \ln x - 3}{x^3}.
\]

We see that \( f''(x) \) is defined on the entire domain and equals zero when \( \ln x = 3/2 \), which occurs at \( x = e^{3/2} \) by definition. To check concavity we see that \( f''(x) < 0 \) when \( x < e^{3/2} \), for example by plugging in \( x = 1 \), and that \( f''(x) > 0 \) for \( x > e^{3/2} \), for example by plugging in \( x = e^2 \). So \( f \) is concave down on \((0, e^{3/2})\) and concave up on \((e^{3/2}, \infty)\).

You can see the graph of \( f \) here: [http://www.wolframalpha.com/input/?i=lnx%2Fx%2C+x+from+0+to+20%2C+y+from+-5+to+1](http://www.wolframalpha.com/input/?i=lnx%2Fx%2C+x+from+0+to+20%2C+y+from+-5+to+1)