Problem 1. Given the equation
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \]
with the initial condition
\[ u_0(x) = \begin{cases} 
0 & \text{for } x < 0 \\
x & \text{for } 0 \leq x < 1 \\
1 & \text{for } x \geq 1 
\end{cases} \]

1. Write the conservative form of the equation.
2. Write the Rankine-Hugoniot condition for this equation across a generic discontinuity \([u_l, u_r]\).
3. Draw the characteristic lines in the plane \((x, t)\).
4. Solve the equation.

\[ \begin{align*}
(1) \quad f(u) &= \frac{1}{2} u^2 \\
\text{Conservative form: } \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} &= 0 \\
(2) \quad \text{R.H.: } \frac{ds}{dt} &= \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{1}{2} (u_r + u_l)
\end{align*} \]
\( x = ut + xo = xo(t+1) \)
\( xo = \frac{x}{t+1} = u_o(xo) = u(x,t) \)

\[
U(x,t) = \begin{cases} 
0 & x < 0 \\
\frac{x}{t+1} & 0 \leq x \leq t+1 \\
1 & t+1 \leq x 
\end{cases}
\]
**Problem 2.** Consider a group of cars at a traffic light. The density of the cars at $t=0$ is
\[
\nu_0(x) = \begin{cases} 
0 & \text{for } x < -1 \\
1 & \text{for } -1 < x < 1 \\
0 & \text{for } x > 1
\end{cases}
\]
The conservative form of the equation for $u(x, t)$ is
\[
\frac{\partial u}{\partial t} + \frac{\partial (u(1-u))}{\partial x} = 0.
\]

1. Write the non conservative form of the equation.
2. Write the Rankine-Hugoniot condition for this equation across a generic discontinuity $[u_l, u_r]$.
3. Draw the characteristic lines in the plane $(x, t)$.
4. Solve the equation for $t > 0$ when the traffic light turns on green.

Hint: the solution of the differential equation
\[
\frac{ds}{dt} = \frac{1}{2} \left( 1 + \frac{\beta - \alpha}{\alpha} \right)
\]
is $s(t) = t + 1 + C\sqrt{t}$ where $C$ depend on the initial conditions.

\[\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0\]
\[\frac{ds}{dt} = \frac{f(u_r) - f(u_e)}{u_r - u_e} = \frac{u_r - u_e^2 - u_e + u_e^2}{u_r - u_e} = 1 - (u_r + u_e)\]
For a given $t$ in region (1):

$$U = 0;$$

in region (2):

$$U = 1$$

in region (3) we have a rarefaction wave.

Characteristic line:

$$x = (1 - 2u)t + 1$$

The are all originating from the point $(x = 1, t = 0)$.

From the characteristic equation, we have

$$u = \left( 1 - \frac{x - 1}{t} \right) \frac{1}{2} = \frac{1}{2}$$

Solution:

Point A is in $x = 1 - t$.

Point B is in $x = 1 + t$. 

R.H. says: $\frac{ds}{dt} = 1 - 1 = 0$. Shock wave doesn't move.

Rarefaction wave.
The previous solution holds when point $A$ is in $x > -1$.

So for $1-t > 1 \Rightarrow t \leq 2$.

For $t \geq 2$,

The jump across the shock after $t = 2$

features the two values:

$u_e = 0, \quad u_2 = \text{value of the reflection wave}$

**RH condition**

$$\frac{ds}{dt} = 1 - (u_e u_2) = 1 - \frac{1}{2} \left( t - \frac{x-1}{t} \right) =$$

$$= \frac{1}{2} + \frac{s-1}{2t} \quad \text{with} \quad s(2) = -1 \quad \text{(when the shock sheets)}$$

As suggested,

$$s(t) = 1 + t + (\sqrt{t}) \quad \Rightarrow \quad s(2) = -1$$

$$1 + 2 + \sqrt{2} = -1$$

$$s(t) = 1 + t - \frac{4\sqrt{t}}{\sqrt{2}}$$

$$c = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$$
Solution after \( t = 2 \)

\[
    u = \begin{cases} 
        0, & x < s(t) \\
        \frac{1}{2} \left(1 - \frac{x - 1}{\varepsilon}\right), & s(t) < x < t + 1 \\
        0, & 1 + t < x \\
    \end{cases}
\]

Value here \( u = \frac{1}{2} \left(1 - \frac{x - 1}{\varepsilon}\right) = \frac{1}{2} \left(1 - \frac{t - 2\sqrt{2}}{\varepsilon}\right) \frac{1}{\sqrt{2}} \)

\[
    = \frac{1}{2} \left(\frac{2t\sqrt{2}}{\varepsilon}\right) = \frac{\sqrt{2}}{\varepsilon}
\]

Point A moves forward with velocity \( 1 - \frac{\sqrt{2}}{\varepsilon} \)

Point B moves forward \( u = 1 \)

Summary

\( 0 < t \leq 2 \)

\[
    u = \begin{cases} 
        0, & x < 0 \\
        1, & 0 < x \leq 1 - t \\
        \frac{1}{2} \left(1 - \frac{x - 1}{\varepsilon}\right), & 1 - t < x \leq 1 + t \\
        0, & 1 + t < x \\
    \end{cases}
\]

\( 2 \leq t \)

\[
    u = \begin{cases} 
        0, & x \leq 1 + t - 2\sqrt{2} \varepsilon \\
        \frac{1}{2} \left(1 - \frac{x - 1}{\varepsilon}\right), & 1 + t - 2\sqrt{2} \varepsilon < x \leq 1 + t \\
        0, & 1 < x \\
    \end{cases}
\]
Problem 3. Give the solution to the problem

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad t > 0, x \in \mathbb{R} \]

\[ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad x \in \mathbb{R} \]

explaining the procedure.

**Change variable**

\[ \xi = \alpha x - bt \]
\[ \eta = \alpha x + bt \]

**New equation:**

\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \]

\[ u = \xi(x) + \eta(x) \]

\[ u(x, 0) = f(x) = \xi(x) + \eta(x) \]

\[ \frac{\partial u}{\partial t} = \alpha \frac{\partial \xi}{\partial t} - \alpha \frac{\partial \eta}{\partial t} + \frac{\partial \xi}{\partial t} + \frac{\partial \eta}{\partial t} = 0 \]

\[ \frac{\partial \xi}{\partial t} = \frac{\partial \eta}{\partial t} = 0 \]

\[ \eta - \xi = \int_{x_0}^{x} g(x) \, dx \]

\[ \eta + \xi = f \]

\[ 2 \xi = f + \int_{x_0}^{x} g(x) \, dx \]

\[ 2 \eta = f - \int_{x_0}^{x} g(x) \, dx \]

\[ u = \frac{1}{2} f(x + t) + \int_{-\infty}^{x} g(s) \, ds + \]

\[ \frac{1}{2} f(x - t) + \int_{-\infty}^{x} g(s) \, ds = \]

\[ = \frac{1}{2} (f(x+t) + f(x-t)) + \int_{-\infty}^{x} g(s) \, ds \]