Exercise 1. [50pt] Find the general solution to the equation
\[ \cos^2(x) \sin(x) \frac{dy}{dx} - \cos^3(x) y = 1. \]
Specify the interval of definition of the solution passing in \( y(1) = 1 \).

Assume that \( x \neq k \frac{\pi}{2} \) with \( k \in \mathbb{Z} \). Divide by \( \cos^2(x) \sin(x) \). We obtain
\[ \frac{dy}{dx} \frac{\cos(x)}{\sin(x)} y = \frac{1}{\cos^2(x) \sin(x)}. \]
Integrating factor:
\[ \frac{d\mu}{dx} = -\frac{\cos(x)}{\sin(x)} \mu. \]
One possible solution:
\[ \log |\mu| = -\log |\sin(x)| \Rightarrow \mu = (\sin(x))^{-1}. \]
\[ \frac{d((\sin(x))^{-1}y)}{dx} = \frac{1}{\cos^2(x) \sin^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x) \sin^2(x)} = \frac{1}{\sin^2(x)} + \frac{1}{\cos^2(x)} = \csc^2(x) + \sec^2(x). \]
Remind that \( \int \csc^2(x) = -\cot(x) + C \), and \( \int \sec^2(x) = \tan(x) + C \) so that
\[ \sin(x) y = -\cot(x) + \tan(x) + C \Rightarrow y(x) = -\frac{\cos(x)}{\sin^2(x)} + \frac{1}{\cos(x)} + \frac{C}{\sin(x)}. \]
The solution for \( y(1) = 1 \) is obtained for \( C = \frac{\sin^3(1) \cos(1)}{\sin^4(1) - \cos^4(1)} \). Point \( x = 1 \) belongs to the interval \( (0, \pi/2) \), where the solution is defined.
Exercise 2. [50pt] Find the continuous solution to the initial value problem
\[
\frac{dy}{dx} + y = \begin{cases} 
1 & \text{for } x \in (0, 1] \\
-1 & \text{for } x \in (1, \infty) 
\end{cases}, \quad y(0) = 1.
\]

Draw the solution and specify the transient term.

Integrating factor: \( \mu = e^x \).

Problem:
\[
d(e^xy) = \begin{cases} 
e^x & \text{for } x \in (0, 1] \\
- e^x & \text{for } x \in (1, \infty) 
\end{cases}
\]

For \( 0 \leq x \leq 1 \):
\[
y(x) = 1 + Ce^{-x}.
\]

For the initial condition, \( y(0) = 1 + C = 1 \Rightarrow C = 0 \)

Not surprisingly we have the equilibrium solution \( y(x) = 1 \) for \( 0 \leq x \leq 1 \).

For \( x > 1 \) we have
\[
y(x) = -1 + Ce^{-x}.
\]

For the continuity, we look for a solution s.t. \( y(1) = 1 \). This implies \( y = 2e^{1-x} - 1 \).

The solution tends to -1, the term \( 2e^{1-x} \) is the transient term.