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SPECTRAL PROPERTIES OF THE HERMITIAN AND SKEW-HERMITIAN SPLITTING PRECONDITIONER FOR SADDLE POINT PROBLEMS*

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Abstract. In this paper we derive bounds on the eigenvalues of the preconditioned matrix that arises in the solution of saddle point problems when the Hermitian and skew-Hermitian splitting preconditioner is employed. We also give sufficient conditions for the eigenvalues to be real. A few numerical experiments are used to illustrate the quality of the bounds.

Key words. saddle point problems, iterative methods, preconditioning, eigenvalues

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1. Introduction.

We are given the saddle point problem

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \qquad \text{or} \qquad \mathcal{A}x = b \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$ with $\text{rank}(B) = m \le n$. We assume that the null spaces of A and B have trivial intersection, which implies that $\mathcal{A}$ is nonsingular. We set

$$\mathcal{H} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{S} = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix},$$

so that $\mathcal{A} = \mathcal{H}+\mathcal{S}$. We consider the preconditioner $\mathcal{P} = (2\alpha)^{-1}(\mathcal{H}+\alpha I)(\mathcal{S}+\alpha I)$, with real $\alpha > 0$, and we study the eigenvalue problem associated with the preconditioned matrix, that is,

$$(\mathcal{H} + \mathcal{S})x = \eta(2\alpha)^{-1}(\mathcal{H} + \alpha I)(\mathcal{S} + \alpha I)x. \tag{1.2}$$

This preconditioner has been studied in a somewhat more general setting in [4], motivated by the paper [1]. Letting $D(1, 1) := \{z \in \mathbb{C}\,; |z - 1| < 1\}$, it was shown in [4] that the spectrum of the preconditioned matrix satisfies $\sigma(\mathcal{P}^{-1}\mathcal{A}) \subset \overline{D(1, 1)}\backslash\{0\}$. Furthermore, $\sigma(\mathcal{P}^{-1}\mathcal{A}) \subset D(1, 1)$ if A is positive definite. Some rather special cases (including the case $A = I$) have been studied in [2, 3]. The purpose of this paper is to provide more refined inclusion regions for the spectrum of $\mathcal{P}^{-1}\mathcal{A}$ for saddle point problems of the form (1.1). Most of our bounds are in terms of the extreme eigenvalues and singular values of the blocks A and B, respectively. Although these quantities may be difficult to estimate, our results can be used to explain why small values of α usually give the best results in terms of convergence rates. For instance, we show

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that sufficiently small values of α always result in preconditioned matrices having a real spectrum consisting of two tight clusters.

Throughout the paper, we write M^T for the transpose of a matrix M and u^* for the conjugate transpose of a complex vector u. Also, $A > 0$ ($A \geq 0$) means that matrix A is symmetric positive definite (respectively, semidefinite).

2. Spectral bounds. In this section we provide bounds for the eigenvalues of the preconditioned matrix.

In the following we shall use the fact that A is symmetric positive semidefinite, so that

$$0 \leq \lambda_n \leq \frac{u^* A u}{u^* u} \leq \lambda_1 \qquad \forall\, u \in \mathbb{C}^n,\, u \neq 0, \tag{2.1}$$

where λ_n, λ_1 are the smallest and largest eigenvalues of A, respectively. Moreover, we denote by $\sigma_1, \ldots, \sigma_m$ the decreasingly ordered singular values of B.

The spectrum of the preconditioned matrix can be more easily analyzed by means of a particular spectral mapping, which we introduce next. We shall then derive estimates for the location of the eigenvalues of (1.2).

We first observe that $(\mathcal{H} + \alpha I)(\mathcal{S} + \alpha I) = \mathcal{HS} + \alpha(\mathcal{H} + \mathcal{S}) + \alpha^2 I$. By collecting the terms with $(\mathcal{H} + \mathcal{S})$ we can write the eigenvalue problem (1.2) as

$$\left(1 - \frac{1}{2}\eta\right)(\mathcal{H} + \mathcal{S})x = \frac{\eta\alpha}{2}\left(I + \frac{1}{\alpha^2}\mathcal{HS}\right)x. \tag{2.2}$$

If $1 - \frac{1}{2}\eta = 0$, then $\eta = 2$. For $1 - \frac{1}{2}\eta \neq 0$ we set

$$\theta := \frac{\eta\alpha}{2 - \eta}, \qquad \text{from which} \qquad \eta = 2 - \frac{2\alpha}{\theta + \alpha} = \frac{2\theta}{\theta + \alpha}. \tag{2.3}$$

Therefore, (2.2) can be written as $(\mathcal{H} + \mathcal{S})x = \theta\left(I + \frac{1}{\alpha^2}\mathcal{HS}\right)x$.

By explicitly writing the term $\mathcal{HS}$, the eigenproblem above becomes

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} x = \theta \begin{pmatrix} I & \frac{1}{\alpha^2}AB^T \\ 0 & I \end{pmatrix} x, \qquad \text{or} \qquad \mathcal{A}x = \theta\mathcal{G}x,$$

where

$$\mathcal{G} := \begin{pmatrix} I & \frac{1}{\alpha^2}AB^T \\ 0 & I \end{pmatrix}.$$

The equivalent eigenproblem $\mathcal{G}^{-1}\mathcal{A}x = \theta x$ can be explicitly written as

$$\begin{pmatrix} A + \frac{1}{\alpha^2}AB^TB & B^T \\ -B & 0 \end{pmatrix} x = \theta x. \tag{2.4}$$

Therefore, the two eigenproblems (1.2) and (2.4) have the same eigenvectors, while the eigenvalues are related by (2.3). Our spectral analysis aims at describing the behavior of the spectrum of $\mathcal{G}^{-1}\mathcal{A}$, from which considerations on the spectrum of (1.2) can be derived. In the following, $\Im(\theta)$ and $\Re(\theta)$ denote the imaginary and real part of θ, respectively.

LEMMA 2.1. *Assume A is symmetric and positive semidefinite. Let $K := I + \frac{1}{\alpha^2}B^TB$. For each eigenpair $(\eta, [u; v])$ of (1.2), η either is $\eta = 2$ or can be written as $\eta = 2 - \frac{2\alpha}{\alpha+\theta}$, where $\theta \neq 0$ satisfies the following:*

1. *If* $\Im(\theta) \neq 0$, *then*

$$\Re(\theta) = \frac{1}{2}\frac{u^*KAKu}{u^*Ku}, \qquad |\theta|^2 = \frac{u^*KB^TBu}{u^*Ku}. \tag{2.5}$$

2. *If* $\Im(\theta) = 0$, *then*

$$\min\left\{\lambda_n, \frac{\sigma_m^2}{\lambda_1\left(1+\frac{\sigma_m^2}{\alpha^2}\right)}\right\} \leq \theta \leq \rho$$

where $\rho := \lambda_1(1+\frac{\sigma_1^2}{\alpha^2})$.

Proof. The first statement of the lemma was already shown by means of the mapping in (2.3). We are thus left with proving the estimates for θ. First of all, note that $\theta \neq 0$ or else $\eta = 0$, which is not possible since $\mathcal{P}^{-1}\mathcal{A}$ is nonsingular.

Let $x = [u; v] \neq 0$ be the complex eigenvector associated with θ. We explicitly observe that $K = I + \frac{1}{\alpha^2}B^TB$ is symmetric positive definite and that KB^TB is symmetric. We shall make use of the following properties of K,

$$\lambda_{\max}(K) = 1 + \frac{\sigma_1^2}{\alpha^2}, \qquad \lambda_{\min}(K) \geq 1, \tag{2.6}$$

where the inequality becomes an equality whenever B is not square. In addition,

$$\lambda_n \leq \frac{u^*KAKu}{u^*K^2u} \leq \lambda_1, \tag{2.7}$$

and using $KB^TB = \alpha^2(K^2 - K)$,

$$0 \leq \frac{u^*KB^TBu}{u^*K^2u} = \alpha^2\frac{u^*K^2u - u^*Ku}{u^*K^2u} = \alpha^2\left(1 - \frac{u^*Ku}{u^*K^2u}\right) \leq \alpha^2 \quad \forall u \neq 0. \tag{2.8}$$

The two matrix equations in (2.4) are given by

$$\left(A + \frac{1}{\alpha^2}AB^TB\right)u + B^Tv = \theta u, \tag{2.9}$$

$$-Bu = \theta v. \tag{2.10}$$

It must be $u \neq 0$; otherwise (2.10) would imply $\theta = 0$ or $v = 0$, neither of which can be satisfied. For $u \neq 0$ and $v = 0$, from (2.9), θ must satisfy $AKu = \theta u$ and $Bu = 0$. Since K is symmetric and positive definite, we can write $K^{\frac{1}{2}}AK^{\frac{1}{2}}\hat{u} = \theta\hat{u}$, $\hat{u} = K^{\frac{1}{2}}u$, from which it follows that θ is real and satisfies

$$0 < \theta \leq \lambda_1\|K^{\frac{1}{2}}\|^2 = \lambda_1\lambda_{\max}\left(I + \frac{1}{\alpha^2}B^TB\right) = \lambda_1\left(1 + \frac{\sigma_1}{\alpha^2}\right) = \rho.$$

We now assume $u \neq 0 \neq v$. Using (2.10), we write $v = -\theta^{-1}Bu$, which, substituted into (2.9), yields $\theta A(I + \frac{1}{\alpha^2}B^TB)u - B^TBu = \theta^2 u$. By multiplying this equation from the left by u^*K we obtain

$$\theta u^*KAKu - u^*KB^TBu = \theta^2 u^*Ku. \tag{2.11}$$

Let $\theta = \theta_1 + \imath\theta_2$. For A symmetric, the quadratic equation (2.11) has real coefficients so that its roots are given by

$$\theta_\pm = \frac{1}{2}\frac{u^*KAKu}{u^*Ku} \pm \sqrt{\frac{1}{4}\left(\frac{u^*KAKu}{u^*Ku}\right)^2 - \frac{u^*KB^TBu}{u^*Ku}}. \tag{2.12}$$

Eigenvalues with nonzero imaginary part arise if the discriminant is negative.

Case $\theta_2 \neq 0$. It must be

$$(u^*KAKu)^2 - 4(u^*Ku)(u^*KB^TBu) < 0, \tag{2.13}$$

and from (2.12) we get $\theta_1 = \frac{1}{2}\frac{u^*KAKu}{u^*Ku}$. By substituting θ_1 in (2.12), we obtain $\theta_2^2 + \theta_1^2 = \frac{u^*KB^TBu}{u^*Ku}$.

Case $\theta_2 = 0$. In this case, from (2.12) it follows that $\theta = \theta_1 > 0$. For $Bu = 0$, from (2.10) it follows that $v = 0$ ($\theta \neq 0$), and the reasoning for $v = 0$ applies.

We now assume that $Bu \neq 0$. We have

$$-\theta_1^2 u^*Ku + \theta_1 u^*KAKu = u^*KB^TBu > 0,$$

where the last inequality follows from (2.8). Since $\theta_1 > 0$, the inequality $\theta_1 u^*KAKu - \theta_1^2 u^*Ku > 0$ implies $u^*KAKu - \theta_1 u^*Ku > 0$, hence $\theta_1 < \lambda_1\lambda_{\max}(K) = \rho$.

To prove the lower bound on θ, write the equation (2.9) as $(AK - \theta I)u = -B^Tv$. If θ is an eigenvalue of AK, then $\theta \geq \lambda_n\lambda_{\min}(K) \geq \lambda_n$. Otherwise, $(AK - \theta I)$ is invertible, so that $u = -(AK - \theta I)^{-1}B^Tv$, which, substituted into (2.10), yields

$$B(AK - \theta I)^{-1}B^Tv = \theta v \Leftrightarrow BK^{-1}(A - \theta K^{-1})^{-1}B^Tv = \theta v. \tag{2.14}$$

Let $B^T = [W_1, W_2]\left[\begin{smallmatrix}\Sigma\\ 0\end{smallmatrix}\right]Q^T$ be the singular value decomposition of B^T, and note that

$$K = [W_1, W_2]\begin{pmatrix} I + \frac{1}{\alpha^2}\Sigma^2 & 0 \\ 0 & I\end{pmatrix}[W_1^T, W_2^T]^T,$$
$$BK^{-1} = Q\begin{pmatrix}\Sigma(I + \frac{1}{\alpha^2}\Sigma^2)^{-1} & 0\end{pmatrix}[W_1^T, W_2^T]^T = QD^{-1}\Sigma W_1^T,$$

where $D = I + \frac{1}{\alpha^2}\Sigma^2$. Problem (2.14) can be thus written as $QD^{-1}\Sigma W_1^T(A - \theta K^{-1})^{-1}W_1\Sigma Q^Tv = \theta v$, or, equivalently,

$$\Sigma W_1^T(A - \theta K^{-1})^{-1}W_1\Sigma w = \theta D w, \qquad w = Q^Tv,$$

from which

$$W_1^T(A - \theta K^{-1})^{-1}W_1\widehat{w} = \theta\Sigma^{-1}D\Sigma^{-1}\widehat{w}, \qquad \widehat{w} = \Sigma w. \tag{2.15}$$

We multiply both sides from the left by $\widehat{w}^*$ and we notice that the left-hand side is positive for any $\widehat{w} \neq 0$. If $\theta \geq \lambda_{\min}(AK) \geq \lambda_n$, then λ_n is the sought-after lower bound. Assume now that $\theta < \lambda_{\min}(AK)$. Then, the matrix $A - \theta K^{-1}$ is symmetric and positive definite. Therefore,

$$\begin{aligned}\widehat{w}^*W_1^T(A - \theta K^{-1})^{-1}W_1\widehat{w} &\geq \lambda_{\min}((A - \theta K^{-1})^{-1})\|W_1\widehat{w}\|^2 \\ &= \lambda_{\min}((A - \theta K^{-1})^{-1})\|\widehat{w}\|^2,\end{aligned} \tag{2.16}$$

and we have

$$\lambda_{\min}((A-\theta K^{-1})^{-1}) = \frac{1}{\lambda_{\max}(A-\theta K^{-1})} \geq \frac{1}{\lambda_1 - \theta\lambda_{\min}(K^{-1})}$$
$$= \frac{1}{\lambda_1 - \frac{\theta}{\lambda_{\max}(K)}} = \frac{1}{\lambda_1 - \frac{\theta}{\tau}},$$

where $\tau := \lambda_{\max}(K) = (1+\frac{\sigma_1^2}{\alpha^2})$. This, together with (2.16), provides a lower bound for the left-hand side of (2.15). Using

$$\theta\widehat{w}^*\Sigma^{-1}D\Sigma^{-1}\widehat{w} = \theta\widehat{w}^*\left(\Sigma^{-2}+\frac{1}{\alpha^2}I\right)\widehat{w} \leq \theta\left(\frac{1}{\sigma_m^2}+\frac{1}{\alpha^2}\right)\|\widehat{w}\|^2$$

and recalling that $\lambda_1\tau - \theta > 0$, from (2.15) we obtain

$$\frac{1}{\lambda_1 - \frac{\theta}{\tau}} \leq \theta\left(\frac{1}{\sigma_m^2}+\frac{1}{\alpha^2}\right), \qquad \text{i.e.,} \qquad \frac{\theta^2}{\tau} + \frac{\sigma_m^2\alpha^2}{\alpha^2+\sigma_m^2} \leq \lambda_1\theta.$$

Since $\theta^2 > 0$, we get $\frac{\sigma_m^2\alpha^2}{\alpha^2+\sigma_m^2} \leq \lambda_1\theta$, and the final bound follows. ∎

The quantities in part 1 of the lemma can also be bounded with techniques similar to those for the real case. However, in the next theorem, we derive sharper bounds for complex η than those one would obtain by using estimates for complex θ.

THEOREM 2.2. *Under the hypotheses and notation of Lemma* 2.1*, the eigenvalues of problem* (1.2) *are such that the following hold:*

1. *If* $\Im(\eta) \neq 0$*, then*

$$\frac{(\alpha+\frac{1}{2}\lambda_n)\lambda_n}{3\alpha^2} < \Re(\eta) < \min\left\{2, \frac{4\alpha}{\alpha+\lambda_n}\right\}, \tag{2.17}$$

$$\frac{\lambda_n^2}{3\alpha^2+\frac{1}{4}\lambda_n^2} < |\eta|^2 \leq \frac{4\alpha}{\alpha+\alpha(1+\frac{\sigma_1^2}{\alpha^2})^{-1}+\lambda_n}. \tag{2.18}$$

2. *If* $\Im(\eta) = 0$*, then* $\eta > 0$ *and*

$$\min\left\{\frac{2\lambda_n}{\alpha+\lambda_n}, \frac{2\frac{\sigma_m^2}{\varrho}}{\alpha+\frac{\sigma_m^2}{\varrho}}\right\} \leq \eta \leq \frac{2\rho}{\alpha+\rho} < 2, \tag{2.19}$$

where $\varrho := \lambda_1(1+\frac{\sigma_m^2}{\alpha^2})$ *and* $\rho := \lambda_1(1+\frac{\sigma_1^2}{\alpha^2})$.

Proof. We have that η is real if and only if θ is real. Assume $\Im(\eta) \neq 0$ and write $\theta = \theta_1 + \imath\theta_2$. Recall that $\tau = (1+\frac{\sigma_1^2}{\alpha^2})$.

Using the definition of θ in (2.3) we obtain

$$\Re(\eta) = 2\frac{\alpha\theta_1+|\theta|^2}{\alpha^2+2\alpha\theta_1+|\theta|^2},$$

that is, $(\alpha^2+2\alpha\theta_1+|\theta|^2)\Re(\eta) = 2\alpha\theta_1+2|\theta|^2$. We substitute the quantities in (2.5) to get $(\alpha^2u^*Ku+\alpha u^*KAKu+u^*KB^TBu)\Re(\eta) = \alpha u^*KAKu+2u^*KB^TBu$. Note that $\alpha^2u^*Ku+u^*KB^TBu = \alpha^2u^*K^2u$. We divide by $u^*K^2u > 0$ to obtain

$$\left(\alpha^2+\alpha\frac{u^*KAKu}{u^*K^2u}\right)\Re(\eta) = \alpha\frac{u^*KAKu}{u^*K^2u}+2\frac{u^*KB^TBu}{u^*K^2u}.$$

We recall that for $\Im(\eta) \neq 0$ relation (2.13) holds, which implies by (2.6) and (2.8)

$$\frac{(u^*KAKu)^2}{(u^*K^2u)^2} < 4\frac{(u^*Ku)}{u^*K^2u}\frac{(u^*KB^TBu)}{u^*K^2u} \leq 4\alpha^2 \tag{2.20}$$

and

$$\frac{(u^*KB^TBu)}{u^*K^2u} > \frac{1}{4}\frac{(u^*KAKu)^2}{(u^*K^2u)^2}\frac{(u^*K^2u)}{u^*Ku} \geq \frac{1}{4}\lambda_n^2. \tag{2.21}$$

Therefore, by applying (2.7), (2.20), and (2.8), we obtain

$$(\alpha^2 + \alpha\lambda_n)\Re(\eta) < \alpha(2\alpha) + 2\alpha^2 \Leftrightarrow \Re(\eta) < \frac{4\alpha}{\alpha + \lambda_n}.$$

By once more applying (2.20), (2.7), and (2.21), we also get

$$(\alpha^2 + \alpha(2\alpha))\Re(\eta) > \alpha\lambda_n + \frac{1}{2}\lambda_n^2 \Leftrightarrow \Re(\eta) > \frac{(\alpha + \frac{1}{2}\lambda_n)\lambda_n}{3\alpha^2},$$

which provide the upper and lower bounds for $\Re(\eta)$.

To complete the proof of the first statement, we write $|\eta|^2$ using (2.3) to obtain

$$(\alpha^2 + 2\alpha\theta_1)|\eta|^2 = (4 - |\eta|^2)|\theta|^2.$$

Substituting (2.5) as before and dividing by u^*K^2u, it yields

$$\left(\alpha^2\frac{u^*Ku}{u^*K^2u} + \alpha\frac{u^*KAKu}{u^*K^2u}\right)|\eta|^2 = (4 - |\eta|^2)\frac{u^*KB^TBu}{u^*K^2u}.$$

Note that $4 - |\eta|^2 > 0$. As before, we bound $|\eta|^2$ from both sides, keeping in mind (2.6), (2.7), (2.8), (2.21), and (2.20), to get

$$\left(\frac{1}{\tau}\alpha^2 + \alpha\lambda_n\right)|\eta|^2 \leq 4\alpha^2 - |\eta|^2\alpha^2 \Leftrightarrow |\eta|^2 \leq \frac{4\alpha}{\alpha + \alpha(1 + \frac{\sigma_1^2}{\alpha^2})^{-1} + \lambda_n},$$

and

$$(\alpha^2 + \alpha(2\alpha))|\eta|^2 > \frac{1}{4}\lambda_n^2(4 - |\eta|^2) \Leftrightarrow |\eta|^2 > \frac{\lambda_n^2}{3\alpha^2 + \frac{1}{4}\lambda_n^2}.$$

This completes the proof of the first part.

Assume now that η is real. Then, from the corresponding bound for real θ in Lemma 2.1 and the fact that $\eta = \phi(\theta) = \frac{2\theta}{\alpha+\theta}$ is a strictly increasing function of its argument, we obtain the desired bounds on η. ∎

A few comments are in order. We start by noticing that, in general, real eigenvalues η may well cover the whole open interval $(0, 2)$, depending on the parameter α. Our numerical experiments show that these bounds are indeed sharp for several values of α (cf. section 4).

Although much less sharp in general, we also found the bounds for eigenvalues with nonzero imaginary part of interest. The lower estimate for $|\eta|$ indicates that nonreal eigenvalues are not close to the origin, especially for small α. In addition, they are located in a section of an annulus as in Figure 2.1. We will see in Theorem 3.1

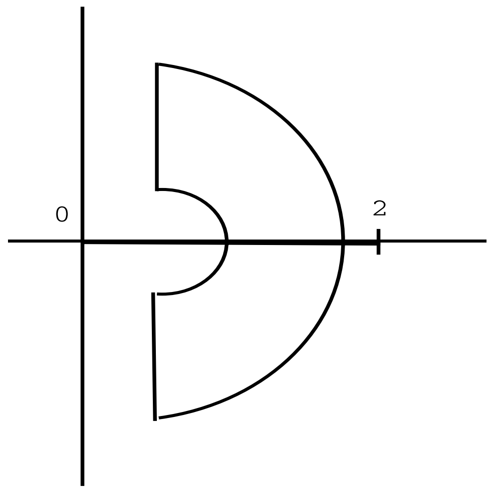

FIG. 2.1. *Inclusion region for the typical spectrum of the preconditioned matrix.*

that complex eigenvalues cannot arise for values of α smaller than one half the smallest eigenvalue of A.

Remark 2.1. We note that when A is positive definite, selecting $\alpha = \lambda_n$ provides constant bounds for the cluster of eigenvalues with nonzero imaginary part. Indeed, substituting $\alpha = \lambda_n$ in (2.17) and in (2.18) we obtain

$$\frac{1}{2} \le \Re(\eta) < 2 \quad \text{and} \quad \frac{4}{13} \le |\eta|^2 \le \frac{4(\lambda_n^2 + \sigma_1^2)}{3\lambda_n^2 + 2\sigma_1^2} \le \frac{4(\lambda_n^2 + \sigma_1^2)}{2\lambda_n^2 + 2\sigma_1^2} = 2.$$

For $\alpha \approx \lambda_n$ we expect to obtain similar bounds. This complex clustering seems to be relevant in the performance of the preconditioned iteration; cf. section 4.

3. Conditions for a real spectrum and clustering properties. We next show that under suitable conditions, the spectrum of the nonsymmetric preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ is real. We stress the fact that a real spectrum is a welcome property, because it enables the efficient use of short-recurrence Krylov subspace methods such as Bi-CGSTAB; see, e.g., [11, p. 139].

THEOREM 3.1. *Assume the hypotheses and notation of Lemma* 2.1 *hold and assume in addition that A is symmetric positive definite. If $\alpha \le \frac{1}{2}\lambda_n$, then all eigenvalues η are real.*

Proof. We prove our assertion for the eigenvalues θ, from which the statement for η will follow. Let $x = [u; v]$ be an eigenvector associated with θ. For $u \neq 0, v = 0$ we already showed that the spectrum is real, while $u = 0$ implies $v = 0$, a contradiction. We now assume $u \neq 0 \neq v$.

The eigenvalues θ of (2.4) are the roots of equation (2.11), which can be expressed as in (2.12). These are all real if the discriminant is nonnegative. Equivalently,

$$\theta \in \mathbb{R} \quad \text{if} \quad (u^*KAKu)^2 \ge 4(u^*Ku)\,(u^*KB^TBu) \quad \forall u \neq 0.$$

Since $u^*K^2u > 0$ for $u \neq 0$, we write the problem above as

$$\theta \in \mathbb{R} \quad \text{if} \quad \frac{(u^*KAKu)^2}{(u^*K^2u)^2} \ge 4\frac{u^*Ku}{u^*K^2u}\,\frac{u^*KB^TBu}{u^*K^2u} \quad \forall u \neq 0.$$

We have $\frac{(u^*KAKu)^2}{(u^*K^2u)^2} \geq \lambda_n^2$, and $\frac{u^*Ku}{u^*K^2u} \leq \lambda_{\min}(K)^{-1} \leq 1$; see (2.6). Therefore, using (2.8), if $\alpha \leq \frac{1}{2}\lambda_n$, we have

$$\frac{(u^*KAKu)^2}{(u^*K^2u)^2} \geq \lambda_n^2 \geq 4 \cdot 1 \cdot \alpha^2 \geq 4\frac{u^*Ku}{u^*K^2u}\,\frac{u^*KB^TBu}{u^*K^2u} \quad \forall u \neq 0. \tag{3.1}$$

The discriminant is nonnegative, therefore all roots of (2.12) are real, and so are the eigenvalues θ. □

The smallest eigenvalue of A can be increased by suitable scalings, thus enlarging the interval of α values leading to a real spectrum. Note, however, that multiplying (1.1) by a positive constant ω is equivalent to applying the Hermitian/skew-Hermitian splitting preconditioner with parameter $\hat{\alpha} := \sqrt{\omega}\alpha$ to the original, unscaled system.

Under additional assumptions on the spectrum of the block matrices, it is possible to provide a less strict condition on α. This is stated in the following corollary.

Corollary 3.2. *Under the hypotheses and notation of Theorem* 3.1*, assume that* $4\sigma_1^2 - \lambda_n^2 > 0$*. If* $\alpha \leq \frac{\lambda_n\sigma_1}{\sqrt{4\sigma_1^2-\lambda_n^2}}$ *then all eigenvalues* η *are real.*

Proof. Using (2.8), we can write

$$\frac{u^*KB^TBu}{u^*K^2u} = \alpha^2\left(1 - \frac{u^*Ku}{u^*K^2u}\right) \leq \alpha^2\left(1 - \frac{1}{1+\frac{\sigma_1^2}{\alpha^2}}\right) = \alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2}.$$

Therefore, if $\lambda_n^2 \geq 4\alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2}$, the bound equivalent to (3.1) follows. Moreover, we note that under the assumption that $4\sigma_1^2 - \lambda_n^2 > 0$,

$$\lambda_n^2 \geq 4\alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2} \Leftrightarrow \alpha^2 \leq \frac{\lambda_n^2\sigma_1^2}{4\sigma_1^2-\lambda_n^2}. \qquad \square$$

It is interesting to observe that if $\sigma_1^2 = \lambda_1$, the condition $4\sigma_1^2 - \lambda_n^2 > 0$ corresponds to the inequality

$$\frac{\lambda_1}{\lambda_n} > \frac{1}{4}\lambda_n,$$

which is easily satisfied since usually λ_n is small and λ_1 is much bigger than λ_n. Note that such a setting is very common in the Stokes problem, where A is a discretization of a (vector) Laplacian and BB^T can also be regarded as a discrete Laplacian.

The following result shows that the eigenvalues form two tight clusters as $\alpha \to 0$. This is an important property from the point of view of convergence of preconditioned Krylov subspace methods. This result extends and sharpens the clustering result obtained in [3] (using different tools) for the special case of Poisson's equation in saddle point form.

Proposition 3.3. *Assume A is symmetric and positive definite. For sufficiently small $\alpha > 0$, the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ cluster near zero and two. More precisely, for small $\alpha > 0$, $\eta \in (0, \varepsilon_1) \cup (2 - \varepsilon_2, 2)$, with $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_1, \varepsilon_2 \to 0$ for $\alpha \to 0$.*

Proof. We assume α is small, and in particular $\alpha \leq \frac{1}{2}\lambda_n$; therefore all eigenvalues are real. Let $[u; v]$ be an eigenvector of (2.4) and let $\theta_\pm$ be the roots of equation (2.11). These are given by (2.12). Collecting u^*Ku and dividing and multiplying (2.12) by $u^*K^2u > 0$, we obtain

$$\theta_\pm = \frac{u^*K^2u}{u^*Ku}\left(\frac{1}{2}\frac{u^*KAKu}{u^*K^2u} \pm \sqrt{\frac{1}{4}\left(\frac{u^*KAKu}{u^*K^2u}\right)^2 - \frac{u^*Ku}{u^*K^2u}\,\frac{u^*KB^TBu}{u^*K^2u}}\right) \equiv \frac{u^*K^2u}{u^*Ku}\nu_\pm.$$

We recall the bounds in (2.7) and (2.8), while $1 \le \frac{u^*K^2u}{u^*Ku} \le (1+\frac{\sigma_1^2}{\alpha^2})$ for any $u \ne 0$, with $(1+\frac{\sigma_1^2}{\alpha^2}) = O(\alpha^{-2})$ as $\alpha \to 0$. Moreover, $0 \le \frac{u^*Ku}{u^*K^2u}\frac{u^*KB^TBu}{u^*K^2u} \le \alpha^2$, so that $\frac{u^*Ku}{u^*K^2u}\frac{u^*KB^TBu}{u^*K^2u} \to 0$ as $\alpha \to 0$. We thus have $\nu_+ \to \frac{u^*KAKu}{u^*K^2u}$ as $\alpha \to 0$. Since $\frac{u^*KAKu}{u^*K^2u}$ is bounded independently of α, we also obtain

$$\nu_- = O\left(\frac{u^*Ku}{u^*K^2u}\frac{u^*KB^TBu}{u^*K^2u}\right) \qquad \text{for} \quad \alpha \to 0.$$

Therefore, $\theta_+ = O(\frac{u^*K^2u}{u^*Ku}) = O(\alpha^{-2})$ as $\alpha \to 0$, whereas $\theta_- = O(\frac{u^*KB^TBu}{u^*K^2u}) = O(\alpha^2)$ as $\alpha \to 0$. It thus follows that

$$\eta_+ = 2 - \frac{2}{1+\frac{\theta_+}{\alpha}} \to 2 \qquad \text{and} \qquad \eta_- = 2 - \frac{2}{1+\frac{\theta_-}{\alpha}} \to 0 \qquad \text{for} \quad \alpha \to 0. \qquad \square$$

We mention that the dependency of the "optimal" value of α on the mesh size h has been discussed, using Fourier analysis, in [3] for the case of Poisson's equation in first order system form, and in [5] for the case of the Stokes problem. In the first case one can choose α so as to have h-independent convergence, whereas in the second case there is a moderate growth in the number of iterations as $h \to 0$.

It is important to remark that the occurrence of a gap in the spectrum for small α can be deduced from known results for overdamped systems. Indeed, equation (2.11) stems from the quadratic eigenvalue problem

$$\theta^2 Ku - \theta KAKu + KB^TBu = 0.$$

The eigenproblem above has $2n$ eigenvalues, $n-m$ of which are zero, corresponding to the dimension of the null space of KB^TB. The remaining $n+m$ eigenvalues coincide with the eigenvalues of our problem (2.4). By introducing $\widetilde{\theta} = -\theta$, we obtain the quadratic symmetric eigenproblem (see [6])

$$\widetilde{\theta}^2 Ku + \widetilde{\theta} KAKu + KB^TBu = 0, \qquad K > 0, \quad KAK > 0, \quad KB^TB \ge 0.$$

It can be shown (see, e.g., [6, Theorem 13.1]) that if the discriminant is positive—that is, if $(u^*KAKu)^2 - 4(u^*Ku)(u^*KB^TBu) > 0$ for any $u \ne 0$—then all eigenvalues $\widetilde{\theta}$ are real and nonpositive. Moreover, the spectrum is split in two parts, each of which contains n eigenvalues.[1]

In our context, and in light of Proposition 3.3, the result above implies that m eigenvalues η will cluster towards zero, while n eigenvalues η will cluster around 2, for sufficiently small α.

4. Numerical experiments. In this section we present the results of a few numerical tests aimed at assessing the tightness of our bounds. The first problem we consider is a saddle point system arising from a finite element discretization of a model Stokes problem (leaky-lid driven cavity). This problem was generated using the IFISS software written by Howard Elman, Alison Ramage, and David Silvester [9]. Here $n = 578$, $m = 254$, $\lambda_n = 0.0763666$, $\lambda_1 = 3.949253$, $\sigma_1 = 0.247606661$, and $\sigma_m = 0.005319517$. Note that the B matrices (discrete divergence operators) generated by this software are rank deficient; we obtained a full rank matrix by dropping the two first rows of B.

[1]Note that in the statement of Theorem 13.1 in [6], matrix KB^TB is required to be positive definite rather than just semidefinite. However, the result is still true under the weaker assumption $KB^TB \ge 0$; see also the treatment in [10] and references therein.

TABLE 4.1
Real bounds in (2.19) vs. actual eigenvalues, Stokes problem.

α	Lower bound	$\eta_{\min}$	$\eta_{\max}$	Upper bound
0.001	0.00048902	0.00050629	1.9999	1.9999
0.01	0.00111635	0.00169724	1.9999	1.9999
0.1	0.00014289	0.00022355	1.9929	1.9929
0.2	0.00007160	0.00011205	1.9608	1.9608
0.3	0.00004775	0.00007473	1.9134	1.9135
0.4	0.00003582	0.00005606	1.8633	1.8635
0.5	0.00002866	0.00004485	1.8150	1.8154
0.6	0.00002388	0.00003738	1.7696	1.7702
0.7	0.00002047	0.00003204	1.7271	1.7278
0.8	0.00001791	0.00002803	1.6871	1.6880
0.9	0.00001592	0.00002492	1.6494	1.6504
1.0	0.00001433	0.00002243	1.6137	1.6147
2.0	0.00000717	0.00001121	1.3327	1.3344
5.0	0.00000287	0.00000449	0.8826	0.8838

TABLE 4.2
Bounds in (2.19) vs. actual real eigenvalues, groundwater flow problem.

α	Lower bound	$\eta_{\min}$	$\eta_{\max}$	Upper bound
0.001	0.181813	0.181818	2.000000	2.000000
0.01	0.285713	0.310869	1.999893	1.999971
0.05	0.064515	0.070481	1.985944	1.996341
0.1	0.032786	0.035865	0.137154	1.971127
0.3	0.011049	0.012099	0.047856	1.437903
0.5	0.006644	0.007277	0.028988	0.722331
1.0	0.003327	0.003645	0.014599	0.145003
3.0	0.001110	0.001217	0.004890	0.011648
5.0	0.000666	0.000730	0.002937	0.005078

In Table 4.1 we compare the lower and upper bounds given in Theorem 2.2 with the actual values of the smallest and largest eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$, which in this case are all real. One can see that the upper bound is always very tight and that the lower bound is fairly tight, especially for small values of α. For $\alpha \approx 0.01$ or smaller, the eigenvalues form two tight clusters near 0 and 2, containing m and n eigenvalues, respectively, as predicted by Proposition 3.3.

Next, we consider a saddle point system arising from the discretization of a groundwater flow problem using mixed-hybrid finite elements [7]. In the example at hand, $n = 270$, $m = 207$, $n + m = 477$, and $\mathcal{A}$ contains 1,746 nonzeros. Here we have $\lambda_n = 0.0017$, $\lambda_1 = 0.010$, $\sigma_1 = 2.611$, and $\sigma_m = 0.19743$.

In this case there are nonreal eigenvalues (except for very small α). In Table 4.2 we compare the lower and upper bounds given in Theorem 2.2 with the actual values of the smallest and largest *real* eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ while in Tables 4.3 and 4.4 we provide the analogous results for the real part and modulus of the nonreal eigenvalues.

One can see that the location of the real eigenvalues is well detected with our bounds. In particular, the lower bound is very sharp, whereas the upper bound gets looser when the whole spectrum becomes complex ($\alpha \geq 0.05$), providing again good estimates for large values of α. The lower bounds suggest that the leftmost cluster will not be too close to zero, particularly for α between 10^{-3} and 10^{-2}, and it turns out that these values of α yield the best results (see below).

Table 4.3
Bounds in (2.17) *vs. actual real part of nonreal eigenvalues, groundwater flow problem.*

α	Lower bound	$\min \Re(\eta)$	$\max \Re(\eta)$	Upper bound
0.001	–	–	–	–
0.01	–	–	–	–
0.05	0.011296	1.823080	1.962387	2.000000
0.1	0.005602	1.571808	1.975776	2.000000
0.3	0.001857	0.608980	1.966375	2.000000
0.5	0.001113	0.274840	1.924906	2.000000
1.0	0.000556	0.078255	1.742401	2.000000
3.0	0.000185	0.009779	0.862083	2.000000
5.0	0.000111	0.003810	0.428775	2.000000

Table 4.4
Bounds in (2.18) *vs. actual modulus of nonreal eigenvalues, groundwater flow problem.*

α	Lower bound	$\min \lvert\eta\rvert$	$\max \lvert\eta\rvert$	Upper bound
0.001	–	–	–	–
0.01	–	–	–	–
0.05	0.019244	1.860113	1.963349	1.967129
0.1	0.009622	1.753875	1.977199	1.982111
0.3	0.003207	1.093125	1.979200	1.981669
0.5	0.001924	0.731979	1.959713	1.962379
1.0	0.000962	0.386709	1.865509	1.881779
3.0	0.000321	0.131260	1.312480	1.596393
5.0	0.000192	0.078883	0.925533	1.496510

Concerning nonreal eigenvalues, we observe that our bounds are generally not very sharp. The real part of the eigenvalues changes considerably as α varies, clustering on different regions of the interval $(0, 2)$. Our lower bounds on $\Re(\eta)$ are rather loose, although they get better for larger values of α; conversely, the upper bounds are tight for small α and loose for large α.

We conclude this section with the results of a few experiments that illustrate the convergence behavior of (full) GMRES [8] with Hermitian/skew-Hermitian splitting preconditioning; we refer to [4] for more extensive experimental results. The purpose of these experiments is to investigate the influence of the eigenvalue distribution, and in particular of the clustering that occurs as $\alpha \to 0$, on the convergence of GMRES. We also monitor the conditioning of the eigenvectors of the preconditioned matrix for different values of α.

In Table 4.5 we report a sample of results for both the Stokes and the groundwater flow problem, for different values of α (from tiny to fairly large). Here $\kappa_2(V) := \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)}$ denotes the spectral condition number of the matrix of (normalized) eigenvectors of $\mathcal{P}^{-1}\mathcal{A}$, and "Its" denotes the corresponding number of preconditioned GMRES iterations (matrix-vector products) needed to reduce the initial residual by at least six orders of magnitude. For the Stokes problem, the condition number of the eigenvector matrix of the unpreconditioned $\mathcal{A}$ is $\kappa_2(V) = 6.94$. Without preconditioning, full GMRES converges in 199 iterations. For the (unpreconditioned) groundwater flow problem, it is $\kappa_2(V) = 1.37$ and GMRES stagnates.

Note that for both problems, the best results (in terms of GMRES iterations) are obtained for $\alpha = 0.005$, with generally good convergence behavior for α between 10^{-6} and 10^{-2}. Good performance is observed in particular for $\alpha \approx \lambda_n$, for which nonreal eigenvalues, when they occur, lie in a small region in the disc $D(1, 1)$ (cf. Remark 2.1).

Table 4.5
Conditioning of the eigenvectors and iteration count.

	Stokes		Groundwater flow	
α	$\kappa_2(V)$	Its	$\kappa_2(V)$	Its
10^{-12}	1.28E+18	> 200	4.31E+09	25
10^{-9}	1.31E+10	45	1.01E+08	17
10^{-6}	4.51E+08	41	1.41E+17	17
10^{-5}	3.30E+04	40	5.69E+00	17
10^{-4}	9.65E+03	40	1.23E+01	17
10^{-3}	1.48E+03	40	8.01E+00	13
0.005	1.16E+04	38	1.31E+03	11
0.01	1.18E+03	38	1.57E+04	13
0.03	7.63E+02	40	1.32E+01	17
0.05	2.68E+02	44	6.79E+01	19
0.07	2.26E+02	48	1.91E+01	20
0.1	6.05E+01	54	1.37E+01	26
0.3	3.55E+01	76	2.76E+00	67
0.5	4.38E+01	88	1.92E+00	109
0.7	2.88E+01	97	8.87E+00	> 200
1.0	1.77E+01	108	1.56E+00	> 200
5.0	3.33E+01	157	1.20E+00	> 200
10.0	6.44E+00	174	1.90E+00	> 200

The convergence rate remains fairly stable even for smaller values of α, but eventually it starts deteriorating as α approaches zero. It is likely that this is due to the fact that the preconditioner (and with it, the preconditioned matrix) becomes singular as $\alpha \to 0$. On the other hand, as $\alpha \to \infty$ the preconditioned matrix tends to the unpreconditioned one and the preconditioner becomes ineffective. Note that somewhat better results can be obtained by a suitable diagonal scaling of $\mathcal{A}$ (see [4]); however, no scaling was used here.

For both problems, $\kappa_2(V)$ appears to be very sensitive to changes in α, at least when α is small. This is in stark contrast with the rather smooth variation in the number of GMRES iterations. Overall, the condition number of the eigenvector matrix does not seem to have much influence on the convergence of GMRES.

5. Conclusions. In this paper we have provided bounds and clustering results for the spectra of preconditioned matrices arising from the application of the Hermitian/skew-Hermitian splitting preconditioner to saddle point problems. Numerical experiments have been used to illustrate the capability of our estimates to locate the actual spectral region. We have also shown that for small α, all the eigenvalues are real and fall in two clusters, one near 0 and the other near 2. Our bounds are especially sharp precisely for these values of α, which are those of practical interest. Indeed, our analysis suggests that the “best” value of α should be small enough so that the spectrum is clustered, but not so small that the preconditioned matrix is close to being singular. Numerical experiments confirm this, and it appears that when A is positive definite, $\alpha \approx \lambda_n(A)$ is generally a good choice.

Finally, we found a connection with the quadratic eigenvalue problems arising in the theory of overdamped systems; it is possible that exploitation of this connection may lead to further insight into the spectral properties of preconditioned saddle point problems.

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