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Splittings of symmetric matrices and a question of Ortega

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Abstract

A complete answer is given to a problem posed in 1988 by Ortega concerning convergent splittings of symmetric matrices.

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1. The question

For a given matrix $A \in \mathbb{R}^{n\times n}$, a splitting $A = P - Q$ with P nonsingular is said to be *P-regular* if $P + Q$ is positive real, i.e., the symmetric part of $P + Q$ is positive definite. This condition is equivalent to requiring that $x^{\mathrm{T}}(P + Q)x > 0$ for all nonzero $x \in \mathbb{R}^n$.

It is a well-known result that if A is symmetric positive definite and $A = P - Q$ is a P-regular splitting, then the splitting is convergent: that is, $\rho(P^{-1}Q) < 1$, where $\rho(\cdot)$ denotes the spectral radius. This is often referred to as the *P-regular splitting theorem*. Without the terminology of P-regular splitting, the result is due to Weissinger [10]. In [7, pp. 255–256], the following two converses to the P-regular splitting theorem are given.

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Theorem 1.1. *Assume that A is symmetric and nonsingular, $A = P - Q$ is a P-regular splitting, and $\rho(P^{-1}Q) < 1$. Then A is positive definite.*

Theorem 1.2. *Assume that $A = P - Q$ is symmetric and nonsingular, P is symmetric positive definite, and $\rho(P^{-1}Q) < 1$. Then A and $P + Q$ are positive definite.*

In the literature, the P-regular splitting theorem together with its converse Theorem 1.1 is sometimes referred to as the *Householder–John theorem*. Indeed, the result can be found in [4] and [6]. Curiously, in [5, pp. 111] Householder attributes the theorem to Reich [9], although Reich's paper deals exclusively with the special case of Gauss–Seidel's method. Theorem 1.2 appears to be due to Ortega.

It is worth noting that it is not necessary in either Theorem 1.1 or Theorem 1.2 to require that A be nonsingular, since this immediately follows from the assumption that $\rho(P^{-1}Q) < 1$. Also note that in Theorem 1.2, the matrix $P + Q$ is symmetric. (A word of caution on terminology: Ortega uses the phrase *positive definite matrix* for what is called here a positive real matrix. The two notions coincide in the symmetric case.) Ortega [7, p. 256] states: "It is an open question as to whether [Theorem 1.2] holds without the assumption that P is symmetric". In other words, the question is whether $P + Q$ must be positive real and A positive definite if A is symmetric, P positive real, and $\rho(P^{-1}Q) < 1$. To the best of our knowledge, no complete answer to this question has been published so far. In the following section, such a complete answer is provided.

2. The solution

It turns out that if P is nonsymmetric, then $P + Q$ need not be positive real; however, A is necessarily positive definite. A simple counterexample suffices to show that $P + Q$ may fail to be positive real. Let $A = I_2$ (the two-by-two identity matrix) and let

$$P = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}, \qquad Q = P - A = \begin{bmatrix} 0 & 3/2 \\ 0 & 0 \end{bmatrix}.$$

Then P is positive real and $\rho(P^{-1}Q) = 0 < 1$. However,

$$P + Q = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Since the symmetric part of this matrix is indefinite, $P + Q$ is not positive real. It should be noted that a similar counterexample has been used in [1] to point out the incorrectness of a result stated in [11]. The author of [1], however, did not mention that his example also answered part of Ortega's question.

To establish that A must indeed be positive definite, some preliminaries are needed. Recall that a square matrix is *positive stable* if all its eigenvalues have positive real part; note that every positive real matrix is positive stable, but not vice-versa. An important characterization of positive stable matrices is given by Lyapunov's theorem. Here we shall use the following version of Lyapunov's theorem (see [3, pp. 98–99]):

Theorem 2.1. *Let $M \in \mathbb{R}^{n\times n}$ be given. Then M is positive stable if and only if there is a symmetric positive definite matrix G satisfying the equation*

$$GM + M^{\mathrm{T}}G = I_n. \tag{2.1}$$

If M is positive stable, there is precisely one solution G to this equation, and G is symmetric positive definite.

We will further use the following simple result.

Lemma 2.2. *If A is symmetric positive definite and B is positive real, the product AB is positive stable.*

Proof. Since A is symmetric positive definite, it has a symmetric positive definite square root $A^{\frac{1}{2}}$. Now let $B = H + S$ with H symmetric positive definite and S skew-symmetric. Then AB is similar to

$$A^{-\frac{1}{2}}(AB)A^{\frac{1}{2}} = A^{\frac{1}{2}}BA^{\frac{1}{2}} = A^{\frac{1}{2}}HA^{\frac{1}{2}} + A^{\frac{1}{2}}SA^{\frac{1}{2}}.$$

Since $A^{\frac{1}{2}}HA^{\frac{1}{2}}$ is symmetric positive definite and $A^{\frac{1}{2}}SA^{\frac{1}{2}}$ is skew-symmetric, AB is similar to the positive real matrix $A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Hence, AB is positive stable. □

Finally, we note that if P is positive real, so is P^{-1}; see, e.g, [8] or [2, Proposition 1]. We are now in a position to establish the following result.

Theorem 2.3. *Let $A = P - Q$ be symmetric, with P positive real. If $\rho(P^{-1}Q) < 1$, then A is positive definite.*

Proof. Letting $T = P^{-1}Q$, we have that the matrix $P^{-1}A = I_n - T$ is positive stable, since $\rho(T) < 1$. From Lyapunov's Theorem it follows that there exists a symmetric positive definite matrix G such that

$$G(P^{-1}A) + (P^{-1}A)^{\mathrm{T}}G = I_n,$$

or, equivalently,

$$A(GP^{-1})^{\mathrm{T}} + (GP^{-1})A = I_n. \tag{2.2}$$

Now, GP^{-1} is the product of a symmetric positive definite and a positive real matrix; hence, by Lemma 2.2, it is positive stable. But (2.2) shows that A solves equation (2.1) with $M = GP^{-1}$, therefore by Theorem 2.1 it must be positive definite. □

This result gives a complete answer to the question posed by Ortega.

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