Part 1

Write a Matlab function \([L, U, g] = lufact(A)\) that computes the LU factorization of an \(n \times n\) matrix \(A\) without pivoting. The function takes \(A\) as input and returns the unit lower triangular factor \(L\), the upper triangular factor \(U\), and the growth factor \(g\) (defined here as the magnitude of the largest number arising in either \(L\) or \(U\), divided by the magnitude of the largest element in \(A\)), or an error message signaling that the factorization breaks down due to division by zero. The matrix \(A\) should not be overwritten by the factors \(L\) and \(U\). Be sure to introduce the necessary checks in the algorithm (for instance, \(A\) must be square).

Use the code to experiment on a large number of matrices (say, up to 100), including randomly generated matrices, Hilbert matrices, diagonally dominant matrices, etc. Number the matrices from 1 to 100 and generate a plot with the matrix index on the horizontal axis and the corresponding growth factor, \(g\), on the vertical axis. How many factorizations failed due to pivot breakdowns?

Finally, monitor the quantity \(\text{norm}(A - L*U, 'inf')\) and report the five largest values observed, together with the corresponding values of \(g\). Discuss your findings.

Part 2

Consider the \(n \times n\) Wilkinson matrix

\[
W_n = \begin{bmatrix}
1 & 0 & 0 & \cdots & 1 \\
-1 & 1 & 0 & \cdots & 1 \\
-1 & -1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & -1 & 1 \\
\end{bmatrix}
\]

1. Compute (by hand) the LU factorization of \(W_5\).
2. “Guess” the LU factorization of \(W_n\), for any \(n\).
3. Write a MATLAB function \(W = \text{wilkin}(n)\) that generates the \(n \times n\) Wilkinson matrix.
4. Perform the following MATLAB experiment:
   (a) Generate \(A = W_{60}\).
   (b) Let \(e \in \mathbb{R}^{60}\) be the column vector with all entries equal to 1. Form \(b = A e\).
   (c) Use the backslash operator to solve \(A x = b\).
   (d) Compare the computed solution with the exact solution, \(x = e\).
5. Repeat the experiment for smaller values of \(n\). What is the largest value of \(n\) for which \(W_n x = b\) can be solved accurately by GEPP when \(b = W_n e\)?
6. Comment on the stability of GEPPS (= GE with Partial Pivoting and Scaling) on this problem. (Note that here PP = no pivoting and that the matrix is already perfectly scaled).
Part 3

Consider GEPP applied to a matrix $A \in \mathbb{R}^{n \times n}$:

$$ A = A^{(0)} \rightarrow A^{(1)} \rightarrow A^{(2)} \rightarrow \cdots \rightarrow A^{(n-1)} = U. $$

Define the growth factor of GE as

$$ \gamma = \frac{\max_{1 \leq k \leq n-1} \beta_k}{\beta_0} \quad \text{where} \quad \beta_k = \max_{1 \leq i,j \leq n} |a^{(k)}_{ij}| \quad \text{for} \quad 0 \leq k \leq n-1. $$

1. Prove that $\gamma \leq 2^{n-1}$ for any matrix $A$.

2. Observe that for the Wilkinson matrix $W_n$ the bound is actually achieved: $\gamma = 2^{n-1}$. Hence, the bound is sharp.

Part 4

Suppose $A$ is a $n \times n$ nonsingular matrix, and that the LU factorization $A = LU$ exists and has been computed. Then the solution to a linear system $Ax = b$ can be computed with $O(n^3)$ operations using forward and backward substitution, for any right-hand side $b$.

1. Suppose that $u$ and $v$ are two given $n$-vectors. We are interested in solving linear systems of the form $\bar{A}\bar{x} = \bar{b}$, where $\bar{A} = A + uv^T$.

   (a) Prove that $\bar{A}$ is nonsingular if and only if $v^T A^{-1} u \neq -1$.

   (b) Show that $\bar{A}^{-1}$ can be expressed in terms of $A^{-1}$, $u$ and $v$ as follows:

   $$ \bar{A}^{-1} = A^{-1} - \alpha A^{-1} uv^T A^{-1} \quad \text{where} \quad \alpha = \frac{1}{v^T A^{-1} u + 1} $$

   (this is known as the Sherman–Morrison formula).

   (c) Assuming that the LU factorization of $A$ is already available, describe an $O(n^2)$ algorithm to solve $\bar{A} \bar{x} = \bar{b}$ for any right-hand side $\bar{b}$.

   Hint: consider first the case $A = I_n$.

2. Describe an efficient method for solving the bordered system

$$ \begin{bmatrix} A & u \\ v^T & \beta \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, $$

where $z$ is unknown and $\beta$ and $c$ are given scalars. When does this system have a unique solution?

Part 5

Any $n \times n$ unit upper triangular matrix $U$ can be written as $U = I_n - N$ where $N = (n_{ij})$ is strictly upper triangular: $n_{ij} = 0$ for all $i \geq j$.

Show that

1. $N$ is nilpotent, that is, there exists a positive integer $p$ such that $N^p = O$, where $O$ denotes the zero matrix;

2. $U^{-1} = I_n + N + N^2 + \cdots + N^{n-1}$. 

2
Part 6

Let $\alpha > 0$. Consider the $n \times n$ upper bidiagonal matrix

$$U_\alpha = \begin{bmatrix}
1 & -\alpha & 0 & \cdots & 0 \\
0 & 1 & -\alpha & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & -\alpha \\
0 & 0 & \cdots & \cdots & 1
\end{bmatrix}.$$

1. Evaluate $\kappa_\infty(U_\alpha) = \|U_\alpha\|_\infty \|U_\alpha^{-1}\|_\infty$ (Hint: use 5(2)).

2. Discuss the conditioning of $U_\alpha$ for different values of $\alpha$ as $n \to \infty$. Be as detailed as possible.

Instructions:

- The assignment should be typed (preferably using LaTeX, but other word processing systems are also acceptable).
- All projects must be received by 5 PM of the due date. Projects that are not turned in by the deadline will not be accepted.
- Remember that collaboration is expected within each team, but strictly forbidden between teams. All suspected violations will be reported to the Honor Council.