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ORIGINAL PAPER

Mauro Picone, Sandro Faedo, and the numerical solution of partial differential equations in Italy (1928–1953)

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Abstract In this paper we revisit the pioneering work on the numerical analysis of partial differential equations (PDEs) by two Italian mathematicians, Mauro Picone (1885–1977) and Sandro Faedo (1913–2001). We argue that while the development of constructive methods for the solution of PDEs was central to Picone's vision of applied mathematics, his own work in this area had relatively little direct influence on the emerging field of modern numerical analysis. We contrast this with Picone's influence through his students and collaborators, in particular on the work of Faedo which, while not the result of immediate applied concerns, turned out to be of lasting importance for the numerical analysis of time-dependent PDEs.

Keywords History of numerical analysis · Istituto per le Applicazioni del Calcolo · Evolution problems · Faedo–Galerkin method · Spectral methods

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1 Introduction

The field of numerical analysis has experienced explosive growth in the last sixty years or so, largely due to the advent of the digital age. Tremendous advances

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have been made in the numerical treatment of differential, integral, and integro-differential equations, in numerical linear algebra, numerical optimization, function approximation, and many other areas. Over the years, numerical analysis has become indispensable for progress in engineering, in the physical sciences, in the biomedical sciences, and increasingly even in the social sciences. Numerical analysis provides one of the pillars on which the broad field of *computational science and engineering* rests: its methods and results underlie the sophisticated computer simulations (invariably involving the solution of large-scale numerical problems) that are currently used to address complex questions which are well beyond the reach of analytical or experimental study.

Largely neglected by professional historians of mathematics, the field of numerical analysis has reached a level of intellectual importance and maturity that demands increased attention to its historical development. In recent years, a few studies on 20th century numerical analysis have begun to appear (for a masterful account of earlier developments, see [40]). Among these, we mention the collections of papers in [62] and [9] (in particular, the general overview in [10]); the recent paper [37] on the history of the Ritz and Galerkin methods; and the richly detailed study [41] of von Neumann and Goldstine's work on matrix computations (see also [39] for a first-hand, non-technical account). Among the histories of scientific institutions devoted to numerical analysis, we mention [48] and [63]. Furthermore, the Society for Industrial and Applied Mathematics (SIAM) maintains a web site dedicated to the history of numerical analysis and scientific computing including articles, transcripts of interviews with leading figures in 20th century numerical analysis, audio files and slides of talks, and additional resources (http://history.siam.org/).

This paper is intended as a contribution to the history of an important chapter of numerical analysis, the numerical solution of partial differential equations, as it developed in Italy during the crucial incubation period immediately preceding the diffusion of electronic computers. This history is inextricably intertwined with that of modern mathematical analysis (in particular, functional analysis and the calculus of variations), but also with the broader political and social issues of the time and, somewhat tangentially, with the early steps in the electronic revolution. Our account will be centered around two protagonists of this era: Mauro Picone and Sandro Faedo. Many studies have already appeared on the remarkable figure of Picone, his influential school of analysis, his pioneering work on constructive methods in analysis, and his most cherished creation, the *Istituto Nazionale per le Applicazioni del Calcolo*.[1] In contrast, there appears to be almost no historical studies devoted to Faedo, apart from a few obituaries, biographical sketches, and the brief discussion in [46]. Because of the importance of Faedo both as a research mathematician and as an institutional leader and organizer, we believe the time has come for an evaluation of his contributions to mathematics and, more broadly, of his influence on Italian science. Here we take a first step in this direction by analyzing some of Faedo's contributions to mathematical research, with an emphasis on his papers on PDEs. As part of our assessment of Faedo's role in the development of numerical methods for PDEs,

[1] Most of the literature on Picone, however, is available only in Italian. An exception is [47].

we contrast his contributions to those of Picone in the same field. The main conclusion reached in this paper is that Faedo, working in the wake of Picone's pioneering researches on the quantitative analysis of PDEs, obtained more important results and had a far more lasting direct influence than his mentor ever had on the subject. It is interesting to note that Faedo seemed to be much less concerned than Picone with the solution of practical problems: his motivation appears to be more on the strictly mathematical side, establishing rigorous results for methods of approximation that result in constructive proofs of existence for the solutions of PDEs. Nevertheless, Faedo was also involved in numerical computations and later on was to play an important role in the establishment of computer science in Italy, both as an academic discipline and in its more applied aspects.

The remainder of the paper is organized as follows. In Section 2 we set the stage by briefly reviewing the main contributions to the numerical solution of differential equations up to the late 1920s. Section 3 recounts Picone's ascent to a leadership position in applied mathematics in Italy, while Section 4 reviews his contributions to the numerical solution of PDEs. A biographical sketch of Faedo is given in Section 5, with his work on numerical PDEs being analyzed in Section 6. Section 7 is concerned with the immediate impact of Faedo's work and with related contributions to numerical PDEs by other mathematicians (both Italian and from other countries). A critical assessment of the influence and legacy of Picone and Faedo's work in numerical PDEs is provided in Section 8. Section 9 contains concluding remarks.

2 Early contributions to numerical solution methods for differential equations

The need for approximate solution procedures for differential equations was already clear to the founders of the subject. In the late 1600s, both Newton and Leibniz sought to approximate the solution of "inverse tangent problems" by means of power series expansions [12, Chapter 12]. *Euler's (forward) method* [22], dating back to 1768, is perhaps the first discretization-based method specifically introduced to compute approximate solutions to differential equations. Euler's simple idea, moreover, was seized by other mathematicians who were motivated by theoretical concerns rather than numerical ones. Cauchy, around the year 1820, was to give an existence proof for the solution of the first-order initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0\,, \tag{2.1}$$

with f and $\frac{\partial f}{\partial y}$ assumed continuous in a neighborhood of (x_0, y_0). Cauchy's proof is based on Euler's method, indefinitely refined (i.e., taking the limit as the discretization parameter tends to zero). This may be the first instance of a constructive existence proof for a differential equation problem, where the constructive procedure itself originates from a numerical algorithm. Cauchy's proof remained unpublished and went largely unnoticed, even after its publication by Coriolis (1837) and by Moigno (1844); see [12, Chapter 12] and [40, Chapter 5.9] for details and for the original references. Although Cauchy's theorem was to be overshadowed by the more complete and general results of Lipschitz (1868), of Peano (1886), and especially of Picard (1890), to whom the definitive treatment of the question of existence and uniqueness of solutions for the initial value problem (2.1) is due,[2] Cauchy's name is often found together with Euler's in connection with both numerical methods and existence proofs for (2.1); see, e.g., [8, 14, 75].

On the numerical side proper, Euler's rather crude method was to give way to more sophisticated procedures developed during the late 19th century and early 20th century. The methods of Adams, Runge, Heun[3] and Kutta were all developed in the years between 1883 and 1901, often motivated by specific questions arising in physics, celestial mechanics, exterior ballistics, and engineering. Moulton's improvement on Adams' method came somewhat later, around 1925, as did the method of Milne. These methods, all based on finite differences, are still widely used for the numerical solution of initial value problems and are part of the standard curriculum of most numerical analysis courses.

The numerical treatment of second-order, two-point boundary value problems has a somewhat more convoluted history. Initially, the problem presented itself as a byproduct of classical problems in the calculus of variations. It was again Euler, in 1744, to make use of discretization to reduce the simple variational problem

$$\int_a^b F(x, y, y')\, dx = \min \tag{2.2}$$

to an ordinary (finite-dimensional) minimum problem of the form

$$\sum_{i=0}^{m} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x = \min \tag{2.3}$$

for the values $y_0, y_1, \ldots, y_{m+1}$ of the unknown function $y = y(x)$ at the end of the subintervals [21] (see also [17, pages 176–177] and [37]). Dividing the expression on the left-hand side of (2.3) by Δx, setting the partial derivatives with respect to y_i of the resulting expression equal to zero (for $i = 0, 1, \ldots, m+1$) and taking the limit as $\Delta x \to 0$ yields the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0\,,$$

[2] As is well known, Picard's method of proof is also constructive, being based on successive approximations, a method already used by Cauchy [40, page 294]. In modern books, Picard's approach is often presented as an example of fixed-point iteration and as a special case of the *contraction mapping principle* for complete metric spaces, due to Banach [2]. The usefulness of Picard's method in numerical analysis, however, has been limited, except as a linearization technique.

[3] As pointed out in [40], Heun's method was the first algorithm implemented on the ENIAC for the numerical integration of differential equations.

the *Euler–Lagrange equation*[4] corresponding to the variational problem (2.2). The minimizer of the integral in (2.2) is usually sought among the class of functions that are continuously differentiable in $[a, b]$ and assume prescribed values at the endpoints: $y(a) = A$, $y(b) = B$, thus leading to a two-point boundary value problem for the corresponding Euler–Lagrange equation.

Although Euler introduced the discretized problem (2.3) in order to derive the differential equation that the minimizing function must satisfy, his approach also suggests a numerical method to compute approximate solutions for a broad class of second-order two-point boundary value problems. The approach consists in reformulating the problem (when possible) as a minimization problem for an integral of the type (2.2), which is then reduced to a standard (finite-dimensional) minimization problem (2.3) by discretization. This problem can then be solved numerically, leading to an approximate solution. For instance, the computed values y_i $(0 \leq i \leq m+1)$ can be linearly interpolated to yield a piecewise linear approximate solution. As observed by Courant and Hilbert [17, page 177],

> This method may be regarded as a special case of the Ritz method, with suitable piecewise linear coordinate functions.

Thus, we see that such a numerical approach consists in *reversing* the procedure used by Euler in 1744 to go from the minimum problem (2.2) to the corresponding Euler–Lagrange equation.

The numerical solution of boundary value problems for *partial* differential equations did not attract much attention until the early 20th century. The first "modern" methods, based either on finite differences or on the variational approach, appear in the period going from 1908 (publication of Ritz's method [76]) to 1928 (publication of the revolutionary paper [16] by Courant, Friedrichs and Lewy). Concerning the method of finite differences, we mention here the paper [74], which also contains the classical *Richardson iterative method* for solving the linear algebraic system arising from finite difference discretizations, and the work by Phillips and Wiener [64] on the finite difference approximation of the Dirichlet problem for Laplace's equation. None of these authors, however, achieved results comparable, for importance and thoroughness, to those found in the "CFL" paper [16]; the true value of this paper for numerical analysis was first understood and exploited by John von Neumann during his time as a consultant for the Manhattan Project at Los Alamos during World War II [39].

Methods of numerical approximations based on Fourier series expansions were also used to solve boundary value problems arising in engineering and physics early in the 20th century; see, for example, Somigliana and Vercelli [81], Timoshenko [83], and Brillouin [11].

The motivation for this early work on constructive methods for PDEs is twofold: on one hand, we register an increasing need for practical algorithms for solving

[4]Euler himself was not entirely satisfied with his derivation of this equation, and in 1755 enthusiastically endorsed the much more satisfactory derivation communicated to him by Joseph-Louis Lagrange, then only 19 years old.

scientific and technical problems, and on the other there is increased interest in developing *constructive* proofs of existence for the solution of functional equations. Concerning this last aspect, an important precedent was provided by the classical *alternierendes Verfahren* (alternating method) introduced by Hermann A. Schwarz already in 1870 to establish the existence of solutions of elliptic boundary value problems [78]. Another important factor was the foundation and rapid development of the theory of (linear) integral equations by Volterra, Fredholm, Hilbert, Schmidt and others in the last part of the 19th century and at the beginning of the 20th. The methods developed by Volterra and Fredholm were based on the following principle: first, the integral equations were discretized using simple quadrature rules, thus reducing the problem to a *finite* ($n \times n$) system of linear algebraic equations. The unique solvability of these systems followed from well-known results, and the solutions could be expressed in terms of determinants, as in Cramer's rule. The existence of solutions to the original integral equations was then established by taking the limit as $n \to \infty$. It is clear that this approach suggests an obvious method for obtaining approximate solutions, using discretization to reduce the infinite-dimensional problem to a finite-dimensional one, with n chosen large enough to achieve the desired accuracy in an appropriate metric (typically, the uniform convergence norm).

Another major development which will prove very influential, and not just on the subsequent history of the numerical analysis of PDEs, is the development, originating with Hilbert, of the *direct methods* in the calculus of variations. This story is well known and has been told many times. Hilbert's work [49] was motivated in part by Weierstrass' famous counterexample to what was then known as *Dirichlet's principle*, namely, the statement that the *Dirichlet integral*

$$J[u] = \iint_S \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \, dx\, dy\,,$$

where S is a bounded region in the plane, always attains a minimum value on the class of all differentiable functions assuming specified values on the boundary ∂S. This function is necessarily harmonic, and thus solves the corresponding boundary value problem for the Laplace equations. In [49], Hilbert was able to make the principle rigorous, thus re-establishing the usefulness and full power of the principle, at least for functions of two variables. Hilbert's "direct method" consisted of the following steps [7]. First, a "minimizing sequence" of admissible functions $\{u_n\}$ is constructed, i.e., a sequence with the property that $\lim_{n\to\infty} J[u_n] = \inf J[u]$. Second, suitable restrictions are imposed on the class of admissible functions in order to ensure the existence of a subsequence $\{u_{n_k}\}$ uniformly convergent to a limiting function u_*. Finally, one shows that for any convergent sequence of admissible functions, the inequality

$$J\left[\lim_{n\to\infty} u_n\right] \le \lim_{n\to\infty} J[u_n]$$

holds. This last property is essentially the requirement that the functional J be *lower semi-continuous* on the class of admissible functions [38, Chapter 8]. This concept, initially introduced by René–Louis Baire for functions of a real variable, was extended to functionals by Leonida Tonelli, who made it the cornerstone of the direct methods of the calculus of variations [84].

The reason why this approach is called "direct" is that it completely avoids the need to consider the associated Euler–Lagrange equation. Indeed, Hilbert's point of view was precisely the opposite: starting with the given elliptic boundary value problem, find (when possible) a corresponding functional; using his direct method, establish the existence of a minimizing sequence, and determine a suitable class of functions in which the minimizing sequence converges. Finally, show that the minimizer is indeed a solution to the original boundary value problem. Although it will take several decades and the work of a large number of mathematicians to perfect the variational approach to the existence theory of elliptic PDEs, Hilbert's approach is still largely in use today. Moreover, the direct methods of the calculus of variations can be seen as precursors of modern techniques for the numerical solution of PDEs. In particular, through the works of Ritz, Bubnov, Galerkin, Trefftz, Krylov, and many others,[5] the direct approach to the calculus of variations eventually led to the finite element method; we mention here the early contribution by Courant [15], and the recent paper [37] for a detailed account.

In summary, it can be safely stated that while by the late 1920s the numerical methods for initial value problems for ordinary differential equations (ODEs) were in a relatively advanced stage of development, research on numerical methods for PDEs was still in its infancy. Although Ritz's method was widely known by then, its practical usefulness was limited to relatively simple self-adjoint elliptic boundary value problems. The method of Galerkin, which has much broader applicability, was not yet widely known, and finite difference methods had been used primarily as a tool to establish existence results for the simplest types of elliptic, parabolic and hyperbolic PDEs [16]. The usefulness of approximation methods based on Fourier series was limited to very special cases, essentially simple PDEs posed on simple geometries. It is in this context that Mauro Picone and his collaborators began their research activity on the numerical analysis of PDEs.

3 Mauro Picone and the *Istituto Nazionale per le Applicazioni del Calcolo*

Mauro Picone was born in Palermo, Sicily, on May 2, 1885 and died in Rome on April 11, 1977 (Fig. 1). His father was a mining engineer who worked in the sulfur mines of Sicily. After the collapse of this industry (due to the discovery of vast sulfur deposits in the United States), in 1889 he left his hometown of Lercara Friddi, near Palermo,

[5] At least passing mention should be made of the fundamental contributions made in the period 1927–1930 by physicists like D. R. Hartree in the United Kingdom, V. Fock in the Soviet Union, E. Hylleraas in Norway, and J. C. Slater in the United States. These scientists developed computational techniques for solving quantum-mechanical problems based on variational principles closely related to Ritz's method and carried out extensive calculations; see [39, pages 99–105].

Fig. 1 Mauro Picone

to become a teacher of technical subjects in secondary schools. The family moved first to Arezzo and then to Parma, where young Mauro attended high school. In 1903 Mauro won the competitive examination for admission to the prestigious *Scuola Normale Superiore* in Pisa, where he received his training under the supervision of outstanding mathematicians like Ulisse Dini and Luigi Bianchi. After receiving his degree in 1907, he remained in Pisa as Dini's assistant until 1913, when he moved to the Politecnico (Technical University) of Turin as an assistant to the chairs of Rational Mechanics and Analysis, then covered by Guido Fubini. He remained in Turin until he received his summons to serve as an officer in the artillery corps of the Italian Army after the outbreak of World War I. As Picone himself will remark in many of his later writings, the time he spent at the front of military operations completely changed his views on mathematics. Indeed, it is during this period of his life that Picone worked out a personal vision of mathematics in which the applied and computational aspects of the subject attain an importance not inferior to the theoretical ones. For the time, this was a revolutionary view, and one that was shared by very few other mathematicians.

In July 1916, after a brief training, Picone, with the rank of Junior Lieutenant, was sent to the war front in the Trentino mountains. Almost immediately upon arriving, Picone received from his Commander the task of recomputing the firing tables to be used by the heavy gun batteries in the high mountain setting, the existing tables being designed only for guns firing across a plain.[6] After working feverishly on the assigned task, Picone was able to deliver the new tables in just one month.

[6] As Picone recalled years later, the use of inadequate firing tables would frequently result in disaster, with the ordnance falling on Italian troops instead of the enemy lines.

He lived this success, which resulted in a promotion to artillery Captain, not so much as a personal achievement but as a triumph of mathematics.[7] Here are Picone's own words in an autobiographical essay [73] published five years before his death:

> One can imagine, after this success of Mathematics, under how different a light the latter appeared to me. I thought: but, then, Mathematics is not only beautiful, it can also be useful.[8]

The formative years in Pisa and the war experience are the two poles around which Picone's personality and scientific activity develop. Picone owes to the former the extraordinary concern for expository rigor and for the utmost generality of the achieved results which pervade his vast scientific production (over 360 papers and about 18 monographs and university textbooks covering a broad array of topics in pure and applied analysis), and to the latter the deeply held conviction that one cannot ignore the need for constructive and numerically feasible tools for the solution of concrete, real-world problems posed by the applied sciences. At the end of the war, Picone was charged with teaching Analysis courses at the University of Catania. After a brief parenthesis at the University of Cagliari, Picone returned to Catania as a full professor of Mathematical Analysis in 1921. In 1924 he moved to the more prestigious University of Pisa, and from there to the University of Naples.

Picone's Neapolitan period goes from 1925 until 1932, when he transferred to the University of Rome, where he will spend the rest of his career until his retirement in 1960. For several years (to be precise, since his military service in WWI), Picone nurtured the dream of creating a research institute devoted to numerical analysis and its applications. Quoting again from [73]:

> The idea dawned on me, since those first years of found again peace (alas, how short lived!), of creating an Institute where mathematicians, equipped with the most powerful computing tools, might collaborate with practitioners of experimental and applied sciences, in order to obtain the concrete solution of their problems of numerical evaluation. I thought that mathematical ingenuity, provided it is based on sound analytical foundations, is capable of the greatest achievements in the fascinating problems that Natural Science poses to our intellect; but if one did not want the whole enterprise to end, as LEONARDO [DA VINCI] says, "in words", it was indispensable to provide the mathematician with a powerful organization of means in order to achieve the numerical evaluation of the quantities arising in the problems being studied. Whence the utilization of computing machines also by the mathematician, and the conception of laboratories also for the mathematician, who could no longer be regarded as the abstract, isolated thinker who only needs paper and pencil for his work. The mathematician had to leave his cloistered office and join the crowd of

[7] For a detailed description of Picone's contributions to ballistics, see [82].

[8] Our translation, here as for all the quotes from the original Italian in the rest of the paper. The original reads: "Si può immaginare, dopo questo successo della Matematica, sotto quale diversa luce questa mi apparisse. Pensavo: ma, dunque, la Matematica non è soltanto bella, può essere anche utile."

> those who strive to uncover the mysteries of Nature and to conquer its hidden treasures.[9]

The dream became reality in 1927, thanks to a grant from the Banco di Napoli, made possible by the intervention of the economist Luigi Amoroso, a close friend of Picone's and a fellow alumn of the Scuola Normale in Pisa. The *Istituto di Calcolo*, initially attached to the chair of Mathematical Analysis at the University of Naples, grew into the *Istituto Nazionale per le Applicazioni del Calcolo* after its move to Rome in 1932 and its organization as an institute of the *Consiglio Nazionale delle Ricerche*, then chaired by Guglielmo Marconi. Under Picone's leadership, the INAC quickly became one of the first and most prominent research institutes in the world specifically devoted to numerical analysis, in the modern sense of the phrase; see, e.g., [10, 19, 34, 48]. As Picone himself wrote in [73], establishing the INAC required great perseverance and political acumen on his part, also in view of the staunch opposition he encountered by a majority of Italian mathematicians. Picone was undeterred by a contrary vote of the Italian Mathematical Society (UMI), and could count on the verbal support he received from eminent scientists like Luigi Bianchi, Guido Castelnuovo, Ludwig Prandtl, Arnold Sommerfeld and Vito Volterra [73].

Picone was, among other things, an excellent talent scout, and was very good at identifying and attracting promising students. Once he had become convinced that a budding mathematician had the necessary attributes, he did everything in his power to encourage and promote the young researcher's work. And his power was considerable: Picone was highly influential and politically well-connected. Especially in the later years of the Fascist regime, he and Francesco Severi (the famous algebraic geometer) were practically in control of much of Italy's mathematical scene. Over the years the INAC became the first workplace for an impressive assembly of mathematicians, including several who were to become among the leading exponents of Italian mathematics and even a few prominent foreign mathematicians. The list includes Renato Caccioppoli, Gianfranco Cimmino, Giuseppe Scorza Dragoni, Carlo Miranda, Tullio Viola, Lamberto Cesari, Sandro Faedo, Giulio Krall, Giuseppe Grioli, Mario Salvadori, Fabio Conforto, Gustav Doetsch, Wolfgang Gröbner, Walter Gautschi, and others (Fig. 2). In addition to this group of researchers, the Institute employed a total of eleven *computers*

[9]"Mi balenò, fin da quei primi anni della riconquistata pace (ahimè, quanto provvisoria!) l'idea della creazione di un Istituto, nel quale matematici, muniti dei più potenti strumenti di calcolo numerico, avessero potuto collaborare con cultori di Scienze sperimentali e con tecnici, per ottenere la concreta risoluzione dei loro problemi di valutazione numerica. Pensavo che la fantasia matematica, a patto che poggi su solide basi analitiche, può essere capace delle più grandi conquiste negli affascinanti problemi che la Scienza della Natura pone al nostro raziocinio, ma se non si voleva che tutto fosse finito, come dice LEONARDO "in parole" era indispensabile fornire il matematico di una potente organizzazione di mezzi per addivenire alla valutazione numerica delle grandezze considerate nei problemi in istudio. Da qui l'impiego delle macchine calcolatrici, anche da parte del matematico, da qui la concezione di laboratori anche per il matematico, che non poteva più essere raffigurato come l'astratto isolato pensatore a cui basta, per il suo lavoro, soltanto carta e matita. Il matematico doveva uscire dal chiuso della sua stanza da studio e scendere tra la folla di coloro che cercano di svelare i misteri della Natura e di conquistarne i nascosti tesori."

Fig. 2 Ciampino (near Rome), 2 May 1955. Luncheon in honor of Mauro Picone on the occasion of his 70th birthday. The list of guests included all the employees of the Istituto per le Applicazioni del Calcolo and several Italian and foreign professors. From right to left one recognizes Carlo Miranda, Mauro Picone, Alessandro Faedo, Tullio Viola (Archivio Storico IAC)

and *draftsmen*. These were highly skilled men and women, many with university degrees, who carried out all the necessary numerical calculations using the computing equipment available at the time, including various electro-mechanical and graphical devices (Fig. 3).[10]

Besides fundamental research in mathematical analysis, differential and integral equations, functional analysis and numerical analysis, the INAC staff was also engaged in a wide variety of applied research projects. These took the form of consulting agreements and research contracts with government agencies (both Italian and foreign), public utility companies, branches of the military, and a number of private companies ranging from major shipyards to small engineering firms. In addition, there were frequent collaborations with university researchers in various scientific and technical fields. One such collaboration with Enrico Fermi resulted in a detailed study by Miranda [60] of the Fermi–Thomas equation of atomic physics. See [1, 63] for descriptions of the manifold activities carried out at the INAC during the 1930s.

Among the topics treated by INAC researchers we find, in addition to "pure" mathematics, problems in classical mechanics (including celestial mechanics), fluid dynamics, structural analysis, elasticity theory (especially the study of beams), atomic physics, electromagnetism, aeronautics, hydraulics, astronomy, and so forth. One of the strong points of INAC researchers was their penchant for developing

[10]It is worth mentioning that in the aftermath of World War II, the research group working at the INAC included mathematicians of the caliber of Gaetano Fichera and Ennio De Giorgi.

Fig. 3 "Computers" at work at the Istituto Nazionale per le Applicazioni del Calcolo (Archivio Storico IAC)

and applying sophisticated techniques of mathematical analysis to solve problems stemming from concrete and urgent applications. Among the preferred tools we find: variational methods, including variants of the Ritz method (these are discussed in greater detail below); fixed point theorems in function spaces; the reformulation of boundary value problems in terms of systems of integral equations; and techniques from asymptotic analysis. Although most of the papers produced by INAC researchers were of a rather theoretical nature, the motivation often came from practical questions that had been submitted to INAC by one or another of its many "customers."[11] Moreover, Picone was keenly interested in the actual numerical implementation of the proposed methods, and extensive numerical calculations were carried out by Institute staff. For an excellent account of the history of the INAC, including descriptions of the computing technology available to INAC researchers, we refer the reader to Nastasi's extensive study [63].

Much of the work done at INAC embodied Picone's philosophy, according to which the mathematical analysis of a problem should not be confined to the study of the existence, uniqueness, and regularity of the solution, but should also supply tools for the (approximate) numerical solution of the problem together with rigorous bounds, in the appropriate norm, of the error incurred. In the next section we describe in some detail how Picone himself put into practice his vision of mathematics.

[11]Because of the strong theoretical flavor of the papers coming out of it, not all applied mathematicians were favorably disposed towards the INAC. See [79] for some of the opinions circulating among contemporary German mathematicians, especially pages 91, 102–103, and 318.

4 Picone's work on numerical PDEs

While Picone proposed a variety of methods that could be used, at least in principle, to compute numerical solutions to differential and integral equations, there are a few to which he attributed particular importance, and which were the topic of several papers by him and his collaborators. These include:

1. Methods based on the Laplace transform;
2. The *variational method* and the *method of weighted least powers*;
3. The *method of Fischer–Riesz integral equations*.

In addition, Picone developed methods for approximating eigenvalues of differential operators (see [35]). In the remainder of this section we take a closer look at Picone's main contributions to the numerical analysis of PDEs, limiting ourselves to items 1 and 2 above.

4.1 Laplace transform methods

Picone first mentions using the *Laplace transform method* for solving heat diffusion problems in [68]; later, in [71] he provided an extension of his technique to a wave propagation problem in three-dimensional space. However, it is only in his *Appunti di Analisi Superiore* [72] that the method will be the subject of a systematic treatment in the broader context of what he called *Metodi delle Trasformate* (*Transform Methods*).

Under suitable assumptions, Picone in [68] proves that every solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \tag{4.1}$$

satisfies on the interval $\left(a', a''\right)$ the following integral equation, for all $s > 0$ and $a \in \left(a', a''\right)$:

$$\int_0^\infty e^{-st} u(x,t)\,dt = \alpha(s)\cosh(\sqrt{s}x) + \beta(s)\sinh(\sqrt{s}x) - \frac{1}{\sqrt{s}}\int_a^x u(\xi,0)\sinh\left[\sqrt{s}(x-\xi)\right]d\xi, \tag{4.2}$$

where $\alpha(s)$ and $\beta(s)$ are functions of s only. Given two linear functionals $L_1[u(x,t)]$ and $L_2[u(x,t)]$ and three functions $f(x)$, $f_1(t)$, $f_2(t)$, under very mild conditions, formula (4.2) provides a method for obtaining a numerical solution of equation (4.1) satisfying the conditions

$$\begin{cases} u(x,0) = f(x), & a' < x < a'', \\ L_1[u(x,t)] = f_1(t), \quad L_2[u(x,t)] = f_2(t), & y \geqslant 0. \end{cases} \tag{4.3}$$

The functionals L_1, L_2 correspond to boundary conditions; e.g., $L_1[u(x,t)] = \lim_{x\to a'} u(x,t)$ and $L_2[u(x,t)] = \lim_{x\to a''} u(x,t)$. Letting

$$L_i\left[\cosh(\sqrt{s}x)\right] = p_{i1}(s), \qquad L_i\left[\sinh(\sqrt{s}x)\right] = p_{i2}(s)$$

and

$$L_i\left[\frac{1}{\sqrt{s}}\int_a^x f(\xi)\sinh\left[\sqrt{s}(x-\xi)\right]d\xi\right] = q_i(s), \quad i = 1, 2,$$

conditions (4.3) lead to the following system of linear equations in the unknown functions $\alpha(s)$ and $\beta(s)$:

$$\begin{cases} p_{11}(s)\alpha(s) + p_{12}(s)\beta(s) = q_1(s) + \displaystyle\int_0^\infty e^{-st} f_1(t)\,dt \\ p_{21}(s)\alpha(s) + p_{22}(s)\beta(s) = q_2(s) + \displaystyle\int_0^\infty e^{-st} f_2(t)\,dt. \end{cases} \tag{4.4}$$

Provided $p(s) = p_{11}(s)p_{22}(s) - p_{12}(s)p_{21}(s) \neq 0$ for all s, this system can be solved (uniquely) for the unknown functions α and β. Thus, the right-hand side of (4.2), denoted now by $F(x,s)$, is known. Denoting by $F^{(n)}(x,s)$ the n-th derivative of $F(x,s)$, it follows from (4.2) that

$$\int_0^\infty e^{-t}t^n u(x,t)\,dy = (-1)^n F^{(n)}(x,1)\,.$$

Therefore the Legendre polynomial series expansion of the solution $u(x,t)$ is obtained, leading to a numerical procedure for approximating $u(x,t)$ on any finite interval on the t-axis. Picone states, without elaborating, that he had the opportunity to experiment with this method "in an interesting application" within the activities of the INAC. In this regard, we note that none of the papers by Picone and Faedo examined here contain any results of actual numerical calculations. At best, these were usually relegated to internal reports or to papers in "applied" journals, often authored by junior collaborators.

In [71] Picone considers the solution of the wave equation ("propagation problem")

$$p\,\frac{\partial^2 u}{\partial t^2} + q\,\Delta u = F(x,y,z,t), \quad (x,y,z)\in D, \quad t \geq 0 \tag{4.5}$$

subject to homogeneous Dirichlet conditions on ∂D and to initial conditions $u(x,y,z,0) = f(x,y,z)$, $u_t(x,y,z,0) = g(x,y,z)$ in D. Here, for simplicity, D is assumed to be the parallelepiped defined by $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. Picone begins by observing that when p is a (positive) function of space only and q a function of time only, the solution of (4.5) can be expanded as

$$u(x,y,z,t) \sim \sum_{n=1}^{\infty} \lambda_n u_n(t)\varphi_n(x,y,z)\,, \tag{4.6}$$

where the φ_n and λ_n are the solutions of the generalized eigenvalue problem

$$\Delta\varphi + \lambda\,p\,\varphi = 0 \quad \text{in } D, \quad \varphi = 0 \quad \text{on } S\,, \tag{4.7}$$

where $S := \partial D$ and where the eigenfunctions φ_n are assumed to be orthogonal with respect to the weight function p and normalized so that

$$\lambda_n \int_0^a \int_0^b \int_0^c p\,\varphi_n^2\,dx\,dy\,dz = 1\,.$$

The functions $u_n(t)$ are obtained by solving the following infinite system of linear second-order ODEs:

$$\frac{d^2 u_n}{dt^2} - \lambda_n\, q(t)\, u_n = F_n(t)\,, \quad (n = 0, 1, \dots)\,,$$

where

$$F_n(t) = \int_0^a \int_0^b \int_0^c \varphi_n(x, y, z)\, F(x, y, z, t)\, dx\, dy\, dz\,,$$

subject for all n to the initial conditions

$$\begin{cases} u_n(0) = \int_0^a\int_0^b\int_0^c p(x, y, z)\,\varphi_n(x, y, z)\, f(x, y, z)\, dx\, dy\, dz\,, \\ \left[\frac{du_n}{dt}\right]_{t=0} = \int_0^a\int_0^b\int_0^c p(x, y, z)\,\varphi_n(x, y, z)\, g(x, y, z)\, dx\, dy\, dz\,. \end{cases} \tag{4.8}$$

Picone calls this method of integration "classical" and states that it "goes back to Bernoulli". He refers to the generic u_n as the *transform* of the solution u. He then goes on to state that this method provides a uniqueness proof and "quite often, broad conditions" for the existence of the solution to problem (4.5). When p and q are both constant, Picone observes that the eigenvalues and eigenfunctions are explicitly known and the method reduces to the well known solution by Fourier series. However, Picone emphasizes that in the general case finding the eigensolutions of (4.7) is an impossible task.

In order to overcome this fundamental difficulty, Picone considers (for the case where q is constant) the Laplace transform of u:

$$u_\tau(x, y, z) = \int_0^\infty e^{-\tau t} u(x, y, z, t)\, dt\,.$$

He then observes that, under suitable conditions on u and F and for a certain set I of values of the parameter τ, the transform u_τ of the solution u of (4.5) satisfies the Dirichlet problem

$$\Delta u_\tau + \tau^2 \frac{p}{q} u_\tau = \frac{F_\tau}{q} + \frac{p}{q}(g + \tau f) \quad \text{in } D, \quad u_\tau = 0 \ \text{ on } S\,, \tag{4.9}$$

where F_τ denotes the Laplace transform of F. Picone then states that once u_τ has been obtained "for a certain system of values of τ", one can recover u by inversion of the Laplace transform. One caveat, as Picone mentions, is that this method requires the well-posedness ("compatibilità") of the boundary value problem (4.9), which is guaranteed for example when p and q are of opposite sign. Assuming that the set I contains the positive integers and using the completeness of the system $\{e^{-kt} \mid k = 1, 2, \dots\}$ on the interval $(0, \infty)$, Picone asserts the uniqueness of the solution u of (4.5) as a consequence of the uniqueness of the solution u_τ of (4.9). Finally, Picone shows that u can be expressed by a Legendre polynomial series expansion.

Picone is well aware of the difficulties inherent in this approach to the solution of propagation problems, and states that the obtained series expansion “rarely lends itself, as experience has shown, to the effective numerical computation of the solution” [71, page 114].

Because of the limited range of applicability of these “classical methods”, Picone recommends for more general problems the use of the so-called *variational method*.

4.2 The method of weighted least powers and the variational method

Picone was an expert on the calculus of variations and it is therefore not surprising that he was keenly interested in developing variational approaches to the solution of PDEs. Already in [65, 66] Picone had outlined a solution method for the Laplace equation $\Delta u = 0$ on a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) subject to the condition $u = f$ on $\partial\Omega$, with f assumed to be square-integrable on $\partial\Omega$. Assuming the existence of the solution, Picone develops an approximation scheme based on a least squares approach, together with a proof of the fact that the resulting approximations converge in the L_2 norm (“in media”) to the desired solution; moreover, the convergence is uniform on compact subsets of Ω. In [66, page 359] he mentions that the same approach had been developed a few years earlier by the prominent French physicist Marcel Brillouin, a professor at the Collège de France [11], although only in some special cases and without any convergence proofs. Picone thanks Henri Villat, the editor of the *Journal de Mathématiques Pures et Appliquées*, for pointing out this fact to him.[12]

4.2.1 The method of weighted least powers

In [67] Picone returns on the topic with renewed enthusiasm. This is arguably Picone’s first substantial paper on the numerical solution of differential and integral equations, and that is the reason why we consider the year 1928 as the conventional beginning of our story. In this memoir, Picone presents a number of new results and a comparison between the method of Ritz and the *method of weighted least powers*, a novel approach that he proposes as “a natural and, perhaps, not fruitless” [67, page 227] generalization of the classical method of least squares.

Picone states the problem in very general terms:

General problem Given $m+1$ functionals $F[y]$, $F_1[y], \ldots, F_m[y]$, dependent on the unknown function y only, find the solution y of the following system:

$$F[y] = 0, \quad F_1[y] = 0, \; \ldots, \; F_m[y] = 0. \tag{4.10}$$

[12]It is perhaps worth mentioning that the short note [66] appears to have replaced a much longer follow-up memoir that Picone had announced in [65], also to be published in *Journal de Mathématiques Pures et Appliquées*. It is likely that Picone decided not to pursue the topic further (beyond the short note [66]) after being informed by Villat, then the newly appointed editor-in-chief of the *Journal*, of Brillouin’s priority.

In most cases, the first equation corresponds to the functional equation and the remaining ones to the side conditions (e.g., boundary conditions) that the solution y must satisfy. This problem requires addressing the following questions:

1. Determine whether the system (4.10) admits solutions;
2. If so, determine how many;
3. Give an approximate solution method which can be implemented in practice [67, page 226], together with an upper bound on the approximation error.

The method of weighted least powers consists of the following. Assume $\{\varphi_n\}$ is a complete system of functions defined on the domain on which the unknown function y is to be determined. Note that the basis functions $\{\varphi_n\}$ are not required to satisfy any condition other than completeness, in particular they need not satisfy any prescribed boundary condition. Consider the functional (sum of weighted powers)

$$J[y] = \int_{\Omega} |F[y]|^p \, d\Phi + \sum_{k=1}^{m} \int_{\Omega_k} |F_k[y]|^{p_k} \, d\Phi_k, \tag{4.11}$$

where

- the integrals are Stieltjes integrals;
- the sets Ω and Ω_k, $k = 1, \dots, m$ are those on which the equations $F_k[y] = 0$, $k = 1, \dots, m$ must be satisfied;
- the functions Φ and Φ_k, $k = 1, \dots, m$, are nonnegative additive set functions (measures) corresponding to the sets Ω and Ω_k, $k = 1, \dots, m$, respectively;
- The exponents p_k, $k = 1, \dots, m$ are given positive numbers.[13]

Letting

$$y_n = \sum_{i=0}^{n} a_i^{(n)} \varphi_i,$$

it is possible to determine the coefficients $a_i^{(n)}$ so as to ensure

$$\lim_{n\to\infty} J[y_n] = J[y].$$

Thus, if y is a solution of the problem, it is necessarily

$$\lim_{n\to\infty} J[y_n] = 0.$$

For any n, consider the function $J_n(\alpha_0, \alpha_1, \dots, \alpha_n) = J[\alpha_0\varphi_0 + \alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n]$ and assume that each such function attains an absolute minimum. Let $A_0^{(n)}, A_1^{(n)}, \dots, A_n^{(n)}$ denote the values of the variables $\alpha_0, \alpha_1, \dots, \alpha_n$ for which J_n attains such minimum value. Then, letting

$$Y_n = A_0^{(n)}\varphi_0 + A_1^{(n)}\varphi_1 + \cdots + A_n^{(n)}\varphi_n\,, \tag{4.12}$$

one has $0 \leqslant J[Y_n] \leqslant J[y_n]$, from which Picone obtains the following result.

[13] The condition $p_k > 1$ is not explicitly stated here, but is implicitly used in the paper.

Theorem Denoted by $A_i^{(n)}$ the values of the α_i for which J_n attains its minimum and with the functions Y_n defined as in (4.12), a necessary condition for the system (4.10) to admit a solution is that the following holds:

$$\lim_{n\to\infty} J[Y_n] = 0; \tag{4.13}$$

in any event, $J[Y_n]$ has a limit to which it converges monotonically without increasing.

Picone applies his method to three classical problems: a Fredholm integral equation of the second kind, the Dirichlet problem for a self-adjoint second-order elliptic PDE (in two space variables), and a two-point boundary value problem for a linear second-order ODE. Picone notes how the second and third problem can both be reduced to the first, but does not make use of this fact and all three problems are dealt with directly. It is interesting to note that Picone does not make any explicit assumptions at the outset on the function spaces where the solution y is being sought. In his analysis, he freely switches from the L_2 (or L_p) norm to the L_∞ norm by making ad hoc (and at times tacit) assumptions on the fly. For the first two cases, Picone limits himself to the classical method of least squares: that is, the Stieltjes integrals in (4.11) reduce to standard (Lebesgue) integrals, and the exponents p_k are assumed to be equal to 2. For the third problem, however, he uses the more general method of weighted least powers, with $p > 1$ and variable weights. He shows in the first two cases that, provided $\lambda = 1$ is not an eigenvalue of the corresponding operator, the sequence $\{y_n\}$ converges in the mean (i.e., in L_2) to the unique solution, y; in the third case he is able to prove a stronger result, namely, the pointwise and uniform convergence of the sequences $\{y_n\}$ and $\{y_n'\}$ to y and y', respectively. Moreover, for this third problem Picone discusses in detail the question of the rate of convergence of the approximants $\{y_n\}$, establishing sharp bounds for the approximation errors under suitable assumptions on the data and on the basis functions $\{\varphi_n\}$. In particular, Picone observes that for problems with analytic data, using the Fourier basis results in convergence of the error to zero (in the L_∞ norm) faster than any power of $1/n$. This is likely to be one of the earliest instances of what is now known as a *spectral method* together with a rigorous error analysis.

Concerning references to earlier work, Picone limits himself to a few references in the footnotes. Besides the more applied works [11] and [81] (the second paper includes numerous tables of numerical results), Picone mentions Krylov's works [51] and [52], although in a somewhat critical manner. For instance, Picone notes that Krylov requires the basis functions $\{\varphi_n\}$ to satisfy exactly some prescribed boundary conditions. This, Picone states, is not necessary and in certain cases may lead to difficulties, both theoretical and practical. We also note here that Picone does not appear to have known of Galerkin's work.

As we shall see, the latter is a recurring theme in Picone's work on boundary value problems. Rather than approximants that satisfy exactly the boundary conditions (and only inexactly the differential equation), Picone[14] will often point out the practical advantages of the complementary approach whereby the approximants are not required to satisfy exactly the boundary conditions except, of course, in the limit as $n \to \infty$.[15] This approach will later become commonplace; it is discussed, for example, in the influential book by Collatz [14]. As we know today, however, issues related to boundary conditions are often rather delicate and require careful handling.

In reality, in [67] Picone does not discuss in any detail the actual numerical implementation of the methods he discusses. In particular, he says nothing about the difficult problem of numerically determining the coefficients $A_0^{(n)}, A_1^{(n)}, \ldots, A_n^{(n)}$ when powers other than $p = 2$ are used in the method of weighted least powers. Only in the conclusions he expresses the need for numerical experimentation with the proposed methods in order to assess their effectiveness for solving real problems of mathematical physics, and states that he hopes to carry out such experiments in the near future with the help of his students at the University of Naples.

The last part of the paper (before the concluding remarks) is dedicated to the analysis and critical assessment of Ritz's method for solving two-point boundary problems for linear second-order self-adjoint ODEs. In his critique of Ritz's method Picone first observes that it imposes certain restrictions on the eigenvalues of the underlying differential operator that do not apply to the method of weighted least powers (in modern terms, this amounts to the usual coercivity condition). Next, Picone studies the convergence (pointwise and uniform) of the approximate solutions and their derivatives under very general conditions, and establishes an upper bound for the approximation error. These bounds depend on quantities that are generally difficult to compute. To simplify the analysis, Picone assumes analytic data, zero end point conditions ($y(0) = 0$, $y(\pi) = 0$), and the Fourier basis for the functions φ_n. In this case he is only able to prove the upper bound

$$\|y - Y_n\|_\infty < \frac{C}{\sqrt{n^3}}, \quad n = 1, 2, \ldots,$$

where C is an explicitly computable constant. This estimate is clearly much worse than the result previously obtained for the method of weighted least powers. This misleads Picone in his assessment of Ritz's method; indeed, the founder of INAC believes to have proved in a definitive manner the superiority of the method of least squares and of his own generalization (the method of weighted least powers) over Ritz's method. In the conclusions, Picone poses the rhetorical question:

> Therefore I ask, should one really persist in giving, for the approximate solution of differential equations, such an eminent place to the method of Ritz, forgetting the old method of least powers?
> I dare not to believe it.

[14]Picone was not the first to advocate for this approach. The idea was already present in Brillouin's mentioned work on the method of least squares [11].

[15]The almost contemporary method of Trefftz [86, 87] is also based on a similar principle.

I like to think that this work may succeed in convincing that the method of least powers has an infinitely broader scope, thus justifying my opinion.[16]

With the benefit of hindsight, we know that Picone's enthusiasm was largely unjustified (for one thing, his estimate is far from sharp). In due time, his own pupil Sandro Faedo will reach diametrically opposite conclusions, showing that the two methods are essentially identical (in terms of convergence rates) except that Ritz's method has "great practical efficacy" [27, page 80].

Incidentally, Picone's words just cited are indirect proof of the fact that Ritz's method may have enjoyed greater success during the 1920s than recognized in [37]. It is also interesting to observe that in the papers of Picone and in those of Faedo, the name of Rayleigh is never used in connection with Ritz's.

4.2.2 The variational method

Numerical investigations carried out at INAC in the years up to 1934 led to a systematic study of computational methods for solving wave propagation problems (see Picone's account in [69]). One of the main findings was that the methods developed up to that time, such as those based on the use of Laplace transforms, performed satisfactorily and reliably only in those propagation problems characterized by the property that the solution, after a transient phase in which it oscillates, monotonically approaches a well-defined, finite steady-state as $t \to \infty$. In those cases where the solution never ceases to oscillate and a steady-state (stationary) regime does not exist, such methods, while still convergent in theory, were found to be practically useless. As Picone observes, those are precisely the cases that most interest the engineers, e.g., in problems of aircraft design.

Such state of affairs induced Picone to conduct further researches, leading to the note [70] in which he introduced the so-called *metodo variazionale* (variational method) for the solution of a very broad class of propagation problems. This method will be the subject of several of Picone's papers in subsequent years. Starting from the observation that Ritz's method produces approximants that satisfy exactly the boundary conditions but only approximately the minimum condition (and thus the differential equation), he develops a "reciprocal" approach, called by him the variational method [70, 71], which results in approximants that exactly satisfy a minimum condition but only approximately match the problem data. As already mentioned, the idea that the approximants should not be restricted to functions satisfying the boundary conditions exactly was already present in Picone's earlier work; indeed, it was already implicit in the method of least squares

[16] "Io domando perciò, è dunque proprio il caso di persistere a dare, per il calcolo approssimato delle soluzioni delle equazioni differenziali, un posto così eminente al metodo di Ritz, dimenticando il vecchio metodo delle minime potenze?

Io oso non crederlo.

Mi lusingo che questo lavoro possa valere a convincere che il metodo delle minime potenze ha una portata infinitamente maggiore, rimanendo così giustificata la mia opinione."

applied by Brillouin in some special cases, albeit without any convergence proofs (as Picone pointed out in [66]).

As usual, Picone begins by posing the problem in very broad terms, considering the most general time-dependent second-order equation on a compact, simply connected region in three-dimensional space.

Problem Suppose that an elastic body in three-dimensional space can be mapped onto the unit cube C $(0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1)$ in (x, y, z)-space, and that it starts vibrating at the instant t_0. Denoting with $u_1(x, y, z, t)$, $u_2(x, y, z, t)$, $u_3(x, y, z, t)$ the components of the displacement at time t of the generic point $(x, y, z) \in C$, the phenomenon is described by the following equations:

- three equations in the interior of C, for $h = 1, 2, 3$:

$$\sum_{k=1}^{3}\left(p_{hk}^{11}\frac{\partial^2 u_k}{\partial x^2} + p_{hk}^{22}\frac{\partial^2 u_k}{\partial y^2} + p_{hk}^{33}\frac{\partial^2 u_k}{\partial z^2} + p_{hk}^{44}\frac{\partial^2 u_k}{\partial t^2} + p_{hk}^{23}\frac{\partial^2 u_k}{\partial y\partial z} + p_{hk}^{31}\frac{\partial^2 u_k}{\partial z\partial x}\right.$$
$$\left. + p_{hk}^{12}\frac{\partial^2 u_k}{\partial x\partial y} + p_{hk}^{1}\frac{\partial u_k}{\partial x} + p_{hk}^{2}\frac{\partial u_k}{\partial y} + p_{hk}^{3}\frac{\partial u_k}{\partial z} + p_{hk}^{4}\frac{\partial u_k}{\partial t} + p_{hk}u_k\right)$$
$$= p_h(x, y, z, t); \tag{4.14}$$

- eighteen equations on the faces of C, for $h = 1, 2, 3$:

$$\begin{cases} \sum_{k=1}^{3}\left[a_{hk}^{01}\frac{\partial u_k}{\partial x} + a_{hk}^{02}\frac{\partial u_k}{\partial y} + a_{hk}^{03}\frac{\partial u_k}{\partial z} + a_{hk}^{04}\frac{\partial u_k}{\partial t} + a_{hk}^{0}u_k\right]_{x=0} = a_h^0(y, z, t) \\ \sum_{k=1}^{3}\left[a_{hk}^{11}\frac{\partial u_k}{\partial x} + a_{hk}^{12}\frac{\partial u_k}{\partial y} + a_{hk}^{13}\frac{\partial u_k}{\partial z} + a_{hk}^{14}\frac{\partial u_k}{\partial t} + a_{hk}^{1}u_k\right]_{x=1} = a_h^1(y, z, t) \\ \sum_{k=1}^{3}\left[b_{hk}^{01}\frac{\partial u_k}{\partial x} + b_{hk}^{02}\frac{\partial u_k}{\partial y} + b_{hk}^{03}\frac{\partial u_k}{\partial z} + b_{hk}^{04}\frac{\partial u_k}{\partial t} + b_{hk}^{0}u_k\right]_{y=0} = b_h^0(z, x, t) \\ \sum_{k=1}^{3}\left[b_{hk}^{11}\frac{\partial u_k}{\partial x} + b_{hk}^{12}\frac{\partial u_k}{\partial y} + b_{hk}^{13}\frac{\partial u_k}{\partial z} + b_{hk}^{14}\frac{\partial u_k}{\partial t} + b_{hk}^{1}u_k\right]_{y=1} = b_h^1(z, x, t) \\ \sum_{k=1}^{3}\left[c_{hk}^{01}\frac{\partial u_k}{\partial x} + c_{hk}^{02}\frac{\partial u_k}{\partial y} + c_{hk}^{03}\frac{\partial u_k}{\partial z} + c_{hk}^{04}\frac{\partial u_k}{\partial t} + c_{hk}^{0}u_k\right]_{z=0} = c_h^0(x, y, t) \\ \sum_{k=1}^{3}\left[c_{hk}^{11}\frac{\partial u_k}{\partial x} + c_{hk}^{12}\frac{\partial u_k}{\partial y} + c_{hk}^{13}\frac{\partial u_k}{\partial z} + c_{hk}^{14}\frac{\partial u_k}{\partial t} + c_{hk}^{1}u_k\right]_{z=1} = c_h^1(x, y, t). \end{cases} \tag{4.15}$$

In these equations the coefficients and the right-hand sides are given continuous functions which depend, in general, on both time and position.

Equations (4.14) and (4.15) must be supplemented with the initial conditions, for $t_0 = 0$ and $k = 1, 2, 3$:

$$\begin{cases} u_k(x, y, z, 0) = u_k^{(0)}(x, y, z), \\ \left[\dfrac{\partial u_k}{\partial t}(x, y, z, t)\right]_{t=0} = u_k^{(1)}(x, y, z), \end{cases} \tag{4.16}$$

under the assumption that the given functions $u_k^{(0)}$ and $u_k^{(1)}$ are finite and continuous together with their first and second partial derivatives.

4.2.3 Description of the variational method

Picone's variational method is somewhat involved. Our presentation closely follows Picone's original text and notations from [70].

Let $(0, T)$ be the time interval of interest. For a given function $f(x_1, x_2, \dots, x_r)$, Picone defines the *first total derivative of* f as the partial derivative of order r given by $\frac{\partial^r f(x_1,x_2,\dots,x_r)}{\partial x_1 \partial x_2 \dots \partial x_r}$, and the *total derivative of* f *of order* p as $\frac{\partial^{pr} f(x_1,x_2,\dots,x_r)}{\partial x_1^p \partial x_2^p \dots \partial x_r^p}$.

Given $f(x, y, z)$ defined on C, Picone introduces the following notations:

- f^{000}, f^{010}, f^{001}, f^{011}, f^{100}, f^{110}, f^{101}, f^{111} denote the values that f attains on the vertices of C;
- $f_{yz}^{00}(x)$, $f_{yz}^{01}(x)$, $f_{yz}^{10}(x)$, $f_{yz}^{11}(x)$ denote the second derivatives of the four functions of x obtained by restriction of f to the edges of C described by the equations $y = 0$ and $z = 0$, $y = 0$ and $z = 1$, $y = 1$ and $z = 0$, $y = 1$ and $z = 1$ (the derivatives $f_{zx}^{00}(y)$, $f_{zx}^{01}(y)$, $f_{zx}^{10}(y)$, $f_{zx}^{11}(y)$, $f_{xy}^{00}(z)$, $f_{xy}^{01}(z)$, $f_{xy}^{10}(z)$, $f_{xy}^{11}(z)$ are defined analogously);
- $f_x^0(y, z)$, $f_x^1(y, z)$ denote the total second derivatives of the functions of y and z obtained by restriction of f on the faces of C described by the equations $x = 0$, $x = 1$ (the derivatives $f_y^0(z, x)$, $f_y^1(z, x)$, $f_z^0(x, y)$, $f_z^1(x, y)$ are defined analogously);
- $f''(x, y, z)$ denotes the second total derivative of $f(x, y, z)$ with respect to x, y, z.

Next, Picone defines the class of functions where the solutions are sought, distinguishing between the case of a finite time interval and the case of an infinite one. In the case of a finite interval of integration, he stipulates that the solutions (displacements) u_1, u_2 and u_3 must be twice continuously differentiable with respect to x, y, z and t; moreover, their second partial derivatives with respect to time, $\frac{\partial^2 u_i}{\partial t^2}$, are required to have continuous second total derivatives on the edges, faces, and interior of the unit cube C.

Letting

$$\frac{\partial^2 u}{\partial t^2} = U(x, y, z, t)\,,$$

and making use of the notations previously described, Picone obtains the expression

$$
\begin{aligned}
U = & \ (1-x)(1-y)(1-z)U^{000}(t) + (1-x)y(1-z)U^{010}(t) \\
& + (1-x)(1-y)zU^{001}(t) + (1-x)yzU^{011}(t) + x(1-y)(1-z)U^{100}(t) \\
& + xy(1-z)U^{110}(t) + x(1-y)zU^{101}(t) + xyzU^{111}(t) \\
& + \sum_{(x,y,z)} \Bigg\{ (1-y)(1-z)\int_0^1 G(x,\xi)U^{00}_{yz}(\xi,t)\,d\xi \\
& \qquad + y(1-z)\int_0^1 G(x,\xi)U^{10}_{yz}(\xi,t)\,d\xi \\
& \qquad + (1-y)z\int_0^1 G(x,\xi)U^{01}_{yz}(\xi,t)\,d\xi + yz\int_0^1 G(x,\xi)U^{11}_{yz}(\xi,t)\,d\xi \Bigg\} \\
& + \sum_{(x,y,z)} \Bigg\{ (1-x)\int_0^1\int_0^1 G(y,\eta)G(z,\zeta)U^{0}_{x}(\eta,\zeta,t)\,d\eta\,d\zeta \\
& \qquad + x\int_0^1\int_0^1 G(y,\eta)G(z,\zeta)U^{1}_{x}(\eta,\zeta,t)\,d\eta\,d\zeta \Bigg\} \\
& + \int_0^1\int_0^1\int_0^1 G(x,\xi)G(y,\eta)G(z,\zeta)U''(\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta,
\end{aligned}
\tag{4.17}
$$

where $G(x,\xi)$ denotes the *Burkhardt's function*:

$$
G(x,\xi) = \begin{cases} \xi(x-1) & \text{if } \ \xi \leqslant x \\ x(\xi-1) & \text{if } \ \xi \geqslant x \end{cases}
$$

and the summation indices are obtained from those indicated under the summation symbols by means of the cyclic permutations (x, y, z), (ξ, η, ζ).

Now, the initial conditions imply

$$
u = u^{(0)}(x,y,z) + tu^{(1)}(x,y,z) + \int_0^1 (t-\tau)U(x,y,z,\tau)\,d\tau\,,
$$

and owing to (4.17) we obtain:

$$
\begin{aligned}
u &= u^{(0)}(x,y,z) + tu^{(1)}(x,y,z)\\
&\quad + (1-x)(1-y)(1-z)\int_0^t (t-\tau)U^{000}(\tau)\,d\tau + \cdots + xyz\int_0^t (t-\tau)U^{111}(\tau)\,d\tau\\
&\quad + \sum_{(x,y,z)} \Big\{ (1-y)(1-z)\int_0^t\int_0^1 (t-\tau)G(x,\xi)U_{yz}^{00}(\xi,\tau)\,d\tau\,d\xi + \cdots\\
&\qquad\qquad + yz\int_0^t\int_0^1 (t-\tau)G(x,\xi)U_{yz}^{11}(\xi,\tau)\,d\tau\,d\xi \Big\}\\
&\quad + \sum_{(x,y,z)} \Big\{ (1-x)\int_0^t\int_0^1\int_0^1 (t-\tau)G(y,\eta)G(z,\zeta)U_x^{0}(\eta,\zeta,\tau)\,d\tau\,d\eta\,d\zeta\\
&\qquad\qquad + x\int_0^t\int_0^1\int_0^1 (t-\tau)G(y,\eta)G(z,\zeta)U_x^{1}(\eta,\zeta,\tau)\,d\tau\,d\eta\,d\zeta \Big\}\\
&\quad + \int_0^t\int_0^1\int_0^1\int_0^1 (t-\tau)G(x,\xi)G(y,\eta)G(z,\zeta)U''(\xi,\eta,\zeta,\tau)\,d\tau\,d\xi\,d\eta\,d\zeta.
\end{aligned}
\tag{4.18}
$$

This last expression contains the unknown functions:

$$\Phi_1(t) = \int_0^t (t-\tau)U^{000}(\tau)\,d\tau, \ldots, \Phi_8(t) = \int_0^t (t-\tau)U^{111}(\tau)\,d\tau\,,$$

where

$$
\begin{cases}
\Phi_1(0) = \Phi_2(0) = \ldots = \Phi_8(0) = 0\\
\Phi_1'(0) = \Phi_2'(0) = \ldots = \Phi_8'(0) = 0,
\end{cases}
\tag{4.19}
$$

and the unknown functions:

$$U_{yz}^{00}(x,t),\, U_{yz}^{10}(x,t),\, U_{yz}^{01}(x,t),\, U_{yz}^{11}(x,t),\, U_x^{0}(y,z,t),\, U''(x,y,z,t),$$

which depend on the time t and on one, two and three space variables, respectively.

Denoting with $F(x_1, x_2, \ldots, x_r, t)$ any such function in which $x_1, x_2, \ldots, x_r$ represent space variables, with $1 \le r \le 3$ and C_r the domain $0 \leqslant x_i \leqslant 1, i = 1, 2, \ldots, r$ and with $\{X_k(x_1, x_2, \ldots, x_r) \,|\, k = 1, 2, \ldots\}$ a complete orthonormal system of real-valued functions defined on C_r, Picone separates variables and writes

$$F(x_1, x_2, \ldots, x_r, t) = \sum_{k=1}^{n(F)} X_k(x_1, x_2, \ldots, x_r)\psi_k(t)\,,$$

where $n(F)$ is a fixed positive integer which depends on F but is otherwise arbitrary.

Substituting this expression into (4.18) results in expansions of the form

$$f(x_1, x_2, \ldots, x_r, t) = \sum_{k=1}^{n(F)} Y_k(x_1, x_2, \ldots, x_r)\varphi_k(t)\,,$$

where

$$Y_k(x_1, x_2, \ldots, x_r) = \int_{C_r} G(x_1, \xi_1) \ldots G(x_r, \xi_r) X_k(\xi_1, \xi_2, \ldots, \xi_r)\, d\xi_1\, d\xi_2 \ldots d\xi_r$$

and $$\varphi_k(t) = \int_0^t (t-\tau)\psi_k(\tau)\, d\tau\, .$$

Denoting the functions $\varphi_k(t)$ with $\Phi_9(t), \Phi_{10}(t), \ldots, \Phi_\nu(t)$, where ν depends on the chosen value of $n(F)$, we have

$$\begin{cases} \Phi_9(0) = \Phi_{10}(0) = \ldots = \Phi_\nu(0) = 0, \\ \Phi_9'(0) = \Phi_{10}'(0) = \ldots = \Phi_\nu'(0) = 0. \end{cases} \tag{4.20}$$

One thus obtains for the functions u_k expressions of the form:

$$u_k = u_k^{(0)} + t u_k^{(1)} + \sum_{i=1}^{\nu_k} A_{ki}(x, y, z)\Phi_{ki}(t), \tag{4.21}$$

where the functions $A_{ki}(x, y, z)$ are known and the $\Phi_{ki}(t)$ are unknown functions which vanish, together with their first derivative, for $t = 0$. To determine the $\Phi_{ki}(t)$ from (4.21) one first selects positive, continuous *weight functions* $q^h(x, y, z, t)$, $q_x^{0h}(y, z, t)$, $q_x^{1h}(y, z, t), \ldots$ Consider now the functional

$$\Omega_T\,[u_1, u_2, u_3] = \int_0^T \sum_{h=1}^{3} \Bigg\{ \int_0^1 \int_0^1 \int_0^1 q^h \cdot (E^h[u_1, u_2, u_3])^2\, dx\, dy\, dz$$
$$+ \sum_{(x,y,z)} \int_0^1 \int_0^1 \left[q_x^{0h} \cdot (E_x^{0h}[u_1, u_2, u_3])^2 + q_x^{1h} \cdot (E_x^{1h}[u_1, u_2, u_3])^2 \right] dy\, dz \Bigg\}\, dt, \tag{4.22}$$

where $E^h[u_1, u_2, u_3]$, $E_x^{0h}[u_1, u_2, u_3]$, $E_x^{1h}[u_1, u_2, u_3], \ldots, E_z^{1h}[u_1, u_2, u_3]$ denote the left-hand sides of equations (4.14) and (4.15), respectively.

This functional vanishes if and only if the functions u_1, u_2, u_3 satisfy the equations (4.14) and (4.15). Furthermore, under suitable assumptions Ω_T admits one and only one system of functions

$$\Phi_{11}^*(t), \ldots, \Phi_{1\nu_1}^*(t), \Phi_{21}^*(t), \ldots, \Phi_{2\nu_2}^*(t), \Phi_{31}^*(t), \ldots, \Phi_{3\nu_3}^*(t),$$

which minimize it on the class of all functions that are absolutely continuous on $(0, T)$ together with their first derivatives and verifying the conditions (4.19) and (4.20).

Finally, the approximation obtained with the variational method is as follows:

$$u_k^* = u_k^{(0)} + t u_k^{(1)} + \sum_{i=1}^{\nu_k} A_{ki}(x, y, z)\Phi_{ki}^*(t), \quad k = 1, 2, 3\,,$$

and the value of $\Omega_T\left[u_1^*, u_2^*, u_3^*\right]$ measures the corresponding residual in the L_2 norm.

As Picone himself points out, one of the advantages of the variational method is that it reduces the problem of solving a PDE in four independent variables x, y, z, t

to that of determining the functions $\Phi^*_{ki}(t)$, which depend only on time. Furthermore, Picone writes,

> From the practical standpoint, as experience has also shown, such method of approximation is preferable to that based on the Laplace transform which requires, besides the solution of a system of PDEs, albeit fixed and not dependent on time, the inversion of the said Laplace transform, an operation which is often impossible to accomplish numerically.[17]

But the greatest achievement that Picone claims for his method is without any doubt the fact that it can drastically reduce the computational effort:

> The method [...] has been applied to a problem of heat propagation [...] previously treated with the Laplace transform leading to the same results, which are thus fully validated, with just one week of work, whereas the previous method had taken several months.[18]

The variational method was applied successfully by INAC researchers, including B. Barile, W. Gröbner and C. Minelli, to the solution of a wide variety of technical and scientific applications [5, 43, 59].

In [70], Picone also proves that the L_2-norm of the approximation error tends to zero under the two following assumptions:

1. The system (4.14), (4.15) and (4.16) admits a solution with twice continuously differentiable components $u_1(x, y, z, t)$, $u_2(x, y, z, t)$, $u_3(x, y, z, t)$;
2. The derivatives $\frac{\partial^2 u_k}{\partial t^2 \partial x^2 \partial y^2 \partial z^2}$, $k = 1, 2, 3$, exist and are continuous on $C \times [0, T]$.

In [26], Faedo will succeed in proving convergence in the L_2-norm using only assumption 1. We now turn to a brief overview of Faedo's career and to his contributions to the quantitative analysis of PDEs.

5 Faedo and his work

Alessandro Faedo – better known as Sandro – was born in Chiampo (near Vicenza, in the North-East of Italy) on November 18, 1913 and died in Pisa on June 16, 2001 (Fig. 4).

Like Picone, he studied in Pisa at the Scuola Normale Superiore, where he was awarded the *Laurea* degree in Mathematics under the supervision of Leonida

[17] "Dal punto di vista pratico, come anche l'esperienza ha dimostrato tale metodo di approssimazione è preferibile a quello attraverso la trasformata di Laplace della soluzione, per il quale, oltre all'integrazione di un sistema alle derivate parziali, sia pure fissato ed indipendente dal tempo, si richiede la inversione della detta trasformata di Laplace operazione questa che riesce spesso numericamente impraticabile." Thus, although he does not use this terminology, Picone appears to be aware of the ill-posedness of the problem of inverting the Laplace transform.

[18] "Il metodo [...] è stato applicato ad un problema di propagazione del calore [...] precedentemente trattato col metodo della trasformata di Laplace [permettendo] di conseguire i risultati già ottenuti, confermandoli pienamente, con una sola settimana di lavoro, laddove il primitivo metodo aveva richiesto qualche mese."

Fig. 4 Sandro Faedo around 35 years of age (Archivio Storico IAC)

Tonelli in 1936. After his graduation, Faedo moved to Rome where he became at first an assistant of the famous geometer Federigo Enriques, then of Gaetano Scorza and finally of Tonelli, who in the years 1939–1942 held appointments in both Pisa and Rome. In the late 1930s and during the 1940s Faedo also worked as a consultant for INAC, establishing a close relationship with Picone (who later will not hesitate to claim Faedo as one of his pupils). He obtained the *libera docenza* (venia docendi) in Mathematical Analysis in 1941, and won the competitive examination for the chair of Mathematical Analysis at the University of Pisa in 1946 succeeding Tonelli himself, who had prematurely died that same year.

The presence of Faedo in Pisa marked a turning point for Pisan mathematics. With his great energy and his skills as an organizer, innovator and promoter he rebuilt a great mathematical school and revitalized the Mathematics Institute in Pisa bringing it to levels of absolute excellence. From 1958 to 1972 he was Rector of the University of Pisa. Early in his tenure he was able to secure the appointment as professors of an amazing constellation of mathematical talents including Aldo Andreotti, Jacopo Barsotti, Enrico Bombieri (who was to receive the Fields medal in 1974), Ennio De Giorgi, Guido Stampacchia, Gianfranco Capriz, Giovanni Prodi and many other scholars.

From the mid-1960s onward, Faedo will essentially cease his research activities to dedicate all his time and energy to his work as a manager and organizer. His political skills and organizational talents did not benefit mathematics only. He was among the first in Italy to recognize the importance of computers in the development of modern science and society and he promoted the creation of the *Centro*

Studi Calcolatrici Elettroniche (CSCE). After an initial suggestion in 1954 by Enrico Fermi to the then Rector of the University of Pisa, Enrico Avanzi, Faedo was charged with forming a small team which included among others the renowned physicist Marcello Conversi and the electrical engineer Ugo Tiberio. The team had as its objective the realization (which was completed in 1961) of the *Calcolatrice Elettronica Pisana* (CEP), the first electronic computer entirely designed and built in Italy.

In the early 1960s Faedo undertook a long journey in the United States, with the goal of learning as much as possible about state-of-the-art computer technology. As a result of this trip Faedo was able to obtain for the University of Pisa an IBM 7090, which at that time was the best and most powerful computer in production. In 1964 Faedo founded the National University Computing Center (CNUCE), originally part of the Italian National Research Council (CNR). On the strength of this record and despite the skepticism of many colleagues, in 1969 Faedo convinced the Ministry of Education to set up at the University of Pisa the first degree-granting program in Computer Science in Italy.

Thanks to his authoritativeness, Sandro Faedo became President of the CNR from 1972 to 1976. Among the most notable results of his office we mention the successful deployment of the *Sirius* satellite program, with the goal of testing the use of very short waves in telecommunications, and the start of the first national research projects funded by the CNR.

In 1976, at the end of his tenure as CNR president, Faedo began an intense parliamentary activity. For two terms, from 1976 to 1983, he was elected Senator of the Italian Republic and as chairman of the Senate Education Committee he participated in the drafting of legislature aimed at reforming the Italian university system.

In the course of his long career, Faedo was elected to membership in many scientific academies and received numerous honors, including those of *Cavaliere di Gran Croce dell'Ordine al Merito della Repubblica Italiana* and of *Officier de la Légion d'Honneur*.

Faedo's scientific activity was primarily in the fields of analysis and numerical analysis, under the influence of Leonida Tonelli and Mauro Picone, respectively. Among the analysis topics treated by Faedo we mention the Laplace transform for functions of several variables, problems in measure theory, the existence theory for abstract linear equations in Banach spaces, and even some foundational aspects (like a study of Zermelo's principle in infinite-dimensional function spaces).

His main contributions, however, concern the calculus of variations, the theory of linear ordinary differential equations and the theory of partial differential equations [58]. In the calculus of variations Faedo contributed a number of papers devoted to extensions of the classical theory, including several papers on minimum conditions for integrals over unbounded intervals and for integrals of the Fubini–Tonelli type. Concerning the theory of ordinary differential equations, Faedo worked primarily on the stability theory and asymptotic properties of solutions of systems of linear ODEs. Faedo's main contributions to PDEs are closely connected with his work in numerical analysis and will be considered in some detail in the next section.

To conclude this rapid overview of Faedo's scientific contributions, and as further proof of his breadth of interest and eclecticism, we mention his studies on ocean tides and dykes (see [32] and [33]). We also note that as part of his work as a consultant for INAC, Faedo carried out extensive numerical calculations on a variety of applied problems.

6 Faedo's work on numerical PDEs

Faedo's most important mathematical contributions, destined to having a lasting impact, are in the field of partial differential equations. His papers in this area focus on what he called the *existential and quantitative analysis* of PDEs, in particular in the elliptic and hyperbolic cases. Clearly influenced by Picone, Faedo always strives to provide constructive existence proofs which can be translated into practical numerical solution procedures.

The period from 1941 to 1953 is among the most productive in Faedo's scientific career. In his papers, Faedo focused his attention on specific aspects of Ritz's method [27, 29, 30] and gave significant contributions [23–26] to the theoretical foundations of Picone's *metodo variazionale*. In view of Faedo's later contribution [28], these last papers appear primarily as preparing the ground for Faedo's own original work, which was motivated to a large extent by a desire the overcome some of the limitations of Picone's methods.

Faedo's single most important paper, published in 1949 in the *Annali della Scuola Normale Superiore di Pisa*, contains the description and analysis of a method for solving time-dependent PDEs [28] (see also [31]). Later this method, which Faedo called *method of moments*, was to become universally known as the *Faedo–Galerkin method*. Faedo's work on this method is discussed next.

6.1 The method of moments (Faedo–Galerkin method)

Faedo's paper [28] is by far his most important and the one destined to have the greatest impact. Already from the paper's title one can clearly discern the influence of Picone on Faedo, with its emphasis on both the existential and the quantitative aspects of the solution of partial differential equations. This paper, however, provides a classical example of the pupil surpassing the master.

Faedo begins his paper by stating that the method of Ritz and Picone's method of weighted least powers have considerable importance in the study of equilibrium problems (i.e., *static* problems). Both methods are based on the minimization of an integral of the calculus of variations; however, while the integral in Ritz's method is always regular,[19] regardless of the number of independent variables involved, in the case of the method of weighted least powers the integral is regular only if it depends on a function of a single variable. This implies that for the method of Picone it has not been possible to obtain convergence results (pointwise or at least in the mean) as

[19] For the definition of regular variational problem see, for instance, [61].

general as those available for the method of Ritz. Moreover, Faedo points out that in [27] he had carried out a comparison of the two methods, showing the relationship between them and establishing for Ritz's methods error bounds similar to those given by Picone for his own method.

Faedo continues by briefly recalling the idea behind Ritz's method, also in order to introduce the terminology and notation for the rest of the paper. Faedo considers a linear, second-order elliptic equation $L[u] = f$ in a domain C subject to the homogeneous Dirichlet boundary condition $u(P) = 0$ for each point P on the boundary FC of C. Chosen a suitable complete system of functions $\{\varphi_i(P)\}$ defined on C and such that $\varphi_i(P) = 0$ if $P \in FC$, one looks for approximate solutions of the form

$$u_n(P) = \sum_{i=1}^{n} c_{i,n}\varphi_i(P)\,,$$

where the coefficients $c_{i,n}$ are obtained by solving the system of linear algebraic equations

$$\int_C [L[u_n] - f]\,\varphi_i(P)\,dC = 0, \quad i = 1, \ldots, n\,.$$

The function u_n is the n-th approximation to the solution u, and under rather general conditions the sequence $\{u_n\}$ converges to the solution u of the boundary value problem as $n \to \infty$. It is interesting to note that Faedo formulates the method of Ritz as a *method of moments* rather than as a minimum problem. Indeed, Faedo's formulation is precisely the same used for the more general *Galerkin's method* which, as is well known, is equivalent to Ritz's method when applied to a self-adjoint problem. In Faedo's papers (as in Picone's), however, there is never any mention of Galerkin's method.

Faedo recalls that in the case of time-dependent propagation (or evolution) problems, Picone had earlier introduced his *variational method*, according to which one constructs a multiple integral and a sequence of functions approximating the minimizer of the integral, in the tradition of Hilbert's and Tonelli's *direct methods* of the calculus of variations. Unfortunately, Faedo observes, such integral is *always* non-regular for the hyperbolic equations governing wave propagation phenomena, thus making it very difficult to find convergence criteria for Picone's method.

In order to circumvent this fundamental difficulty, Faedo proposes what is essentially a separation of variables approach. In a propagation problem, the unknown function u depends both on space (that is, on the point $P \in C$) and on time, t. Given a second-order hyperbolic equation $L[u] = f$ subject to the boundary and initial conditions $u = 0$ on FC and $u(P, 0) = g(P)$, $u_t(P, 0) = g_1(P)$ and chosen a complete orthonormal system of basis functions $\{\varphi_i(P)\}$ which depend only on space, Faedo seeks an approximate solution of the form

$$u_n(P, t) = \sum_{i=1}^{n} c_{i,n}(t)\varphi_i(P)\,,$$

where the coefficients $c_{i,n}(t)$ – now functions of t – can be found by solving the following system of ODEs:

$$\int_C [L[u_n] - f]\,\varphi_i(P)dC = 0, \quad i = 1, \ldots, n$$

subject to the initial conditions

$$c_{i,n}(0) = \int_C g(P)\varphi_i(P)dC, \quad c'_{i,n}(0) = \int_C g_1(P)\varphi_i(P)dC, \quad i = 1, \ldots, n\,.$$

In modern terms, Faedo's method is an example of *semi-discretization*, since the problem is discretized with respect to space but not with respect to time. Faedo calls it *method of moments*, and states that a similar method had been previously introduced, under the name of *generalized harmonic analysis*, by Kryloff and Bogoliouboff [53], but only for a rather special case, under restrictive assumptions, and without any proof of convergence.

Concerning his approach, Faedo writes:

> The method of moments is therefore to that of Ritz as the variational method is to that of weighted least powers. There is, however, no variational problem corresponding to the method of moments, unlike the case of Ritz's method [...] The method lends itself not only to the quantitative analysis of propagation problems, but is also an effective tool for obtaining existence theorems and for establishing qualitative properties, also of an asymptotic nature, of the solution.[20]

The last statement is especially remarkable in view of later developments in the qualitative theory of systems of ordinary differential equations.

Having thus introduced the basic idea, Faedo begins the detailed exposition of his method by formulating the following one-dimensional propagation problem:

Problem On the half-strip $S = \{(x, t) : 0 \leq x \leq c\,, t \geq 0\}$ in the xt-plane, find $u(x, t)$ continuous in S (together with its first and second partial derivatives) satisfying the initial conditions $u(x, 0) = g(x)$, $u_t(x, 0) = g_1(x)$, $0 \leq x \leq c$, the boundary conditions $u(0, t) = f_1(t)$, $u(c, t) = f_2(t)$, $t \geq 0$ and the partial differential equation

$$L[u] = -u_{xx} + a_1 u_{xt} + a_2 u_{tt} + a_3 u_x + a_4 u_t + a_5 u = f \tag{6.1}$$

in S, where the coefficients a_i and the right-hand side f are functions of x and t defined on S.

[20]"Il metodo dei momenti sta quindi a quello di Ritz come il metodo variazionale a quello delle minime potenze ponderate. Però non ha più senso un problema di Calcolo delle Variazioni che traduca il metodo dei momenti come invece accade per quello di Ritz [...] Il metodo si presta non solo all'analisi quantitativa dei problemi di propagazione ma è anche un efficace strumento per ottenere teoremi di esistenza e per stabilire proprietà qualitative, anche di carattere asintotico, della soluzione."

For problem (6.1), Faedo establishes the following existence, uniqueness and convergence theorems.

Uniqueness theorem If the coefficients a_i in equation (6.1) are continuous together with $\frac{\partial a_1}{\partial x}$, $\frac{\partial a_2}{\partial t}$ and if $a_2 > 0$ in S, then (6.1) admits at most one solution.

Concerning this result, Faedo emphasizes how it is independent of a similar theorem given by Picone in [72], since the coefficients are now assumed to be functions of both x and t. He then observes:

> The method of Ritz can be regarded as a direct method of the Calculus of Variations [...] and, as such, it not only provides a procedure for approximate computations, but it may also be used as a tool for obtaining existence theorems for boundary value problems for self-adjoint equations of elliptic type. It is therefore natural to ask if one can also obtain existence theorems for propagation problems with the method of moments.[21]

However, since Faedo's method of moments lacks the variational interpretation which underlies the method of Ritz, Faedo needs to use different tools in order to obtain the desired existence theorem for equation (6.1). Thus, instead of the notion of semicontinuity, due to Tonelli and its school, which is appropriate for treating problems of maxima and minima, Faedo turns to the classical compactness criteria of Ascoli and Arzelà. It is precisely a clever application of such techniques, combined with the classical Cauchy–Schwarz and Bessel inequalities, that allows Faedo to obtain the following result.

Existence theorem If the coefficients a_i and the right-hand side f in (6.1) are continuous in S together with their first and second partial derivatives with respect to t; if $\frac{\partial a_1}{\partial x}$ exists and is continuous in S, while the other coefficients a_i $(i = 2, \ldots, 5)$ and the right-hand side f have continuous first partial derivatives with respect to x for $t = 0$, and if $a_2 > 0$ in S; if $f_1(t)$ and $f_2(t)$ are continuous for $t \geq 0$ together with their derivatives up to fourth order; if $g(x)$ and $g_1(x)$ are continuous for $0 \leq x \leq c$ together with their derivatives up to third and second order, respectively; and if the following compatibility conditions are satisfied at the point $x = 0$, $t = 0$:

$$f_1 = g\,,\ f_1' = g_1\,,\ -g'' + a_1 g_1' + a_2 f_1'' + a_3 g' + a_4 f_1' + a_5 f_1 = f,$$

and at the point $x = c$, $t = 0$:

$$f_2 = g\,,\ f_2' = g_1\,,\ -g'' + a_1 g_1' + a_2 f_2'' + a_3 g' + a_4 f_2' + a_5 f_2 = f,$$

[21]"Il metodo di Ritz si può considerare come un metodo diretto di Calcolo delle Variazioni [...] e, come tale, non solo dà un procedimento per il calcolo approssimato, ma può servire quale strumento per ottenere teoremi di esistenza per problemi al contorno per equazioni auto-aggiunte di tipo ellittico.
Viene quindi naturale di chiedersi se col metodo dei momenti si possono pure ottenere teoremi di esistenza per i problemi di propagazione."

then equation (6.1) has a solution $u(x,t)$, which is unique by virtue of the Uniqueness Theorem. Furthermore, the derivatives $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x \partial t}$ are *absolutely continuous in the sense of Tonelli* (see [85]) on any rectangle $S_T = \{(x,t) : 0 \le x \le c, 0 \le t \le T\}$ and therefore the derivatives $\frac{\partial^3 u}{\partial t^3}$, $\frac{\partial^3 u}{\partial t^2 \partial x}$, $\frac{\partial^3 u}{\partial x^2 \partial t}$ exist almost everywhere in S and are integrable on every S_T. If, moreover, the coefficients and the right-hand side of equation (6.1) are differentiable with respect to x, it follows that the derivative $\frac{\partial^3 u}{\partial x^3}$ exists almost everywhere in S.

In section III of [28], Faedo proves a fundamental theorem which shows how his method allows to approximate not only the solution u of the propagation problem described by (6.1), but also its derivatives, which can be of considerable importance in applications, for instance in those problems where the velocity or the current are of interest. This theorem amounts to a convergence result for the method of moments with a particular choice of the basis functions $\{\varphi_i\}$:

Convergence theorem I Under the same assumptions as in the EXISTENCE THEOREM, choosing the system of functions $\{\varphi_i(x) = \sin \frac{\pi i x}{c}, i = 1, 2, \ldots\}$, the method of moments yields a sequence $u_n(x,t)$ which converges uniformly together with the derivatives $\frac{\partial u_n}{\partial x}$, $\frac{\partial u_n}{\partial t}$, $\frac{\partial^2 u_n}{\partial t^2}$, $\frac{\partial^2 u_n}{\partial x \partial t}$ on every S_T to the solution $u(x,t)$ and to the corresponding derivatives. Moreover, the sequence $\frac{\partial^2 u_n}{\partial x^2}$ converges in the mean to $\frac{\partial^2 u}{\partial x^2}$ for all $t \ge 0$ and $0 \le x \le c$.

In an additional theorem (Th. IV [28, page 4]), Faedo also provides a priori bounds for the solution $u(x,t)$ and its derivatives and emphasizes how such upper bounds may be useful in the applications in order to obtain some indications on the order of magnitude of the quantities entering the phenomena under study.

Finally, as already mentioned, the following theorem allows Faedo to conclude that in the context of propagation problems, the method of moments is in the same relationship to the method of Ritz as the variational method is to the method of weighted least powers, but with the added benefit that for the method of moments it is possible to state general convergence criteria:

Convergence theorem II Under the same hypotheses as in the UNIQUENESS THEOREM and assuming that the propagation problem (6.1) has the solution $u(x,t)$, if the system of functions $\{\varphi_i(x)\}$ is admissible for the variational method[22] and satisfies $\varphi_i(0) = \varphi_i(c)$, $i = 1, 2, \ldots$, the sequence $u_n(x,t)$ converges uniformly to $u(x,t)$ for $n \to \infty$ on every S_T, where u_n is the n-th approximation computed by the method of moments.

As pointed out by Faedo, one of the advantages of the method of moments over the variational method is the fact that it makes it easier to obtain approximate

[22] The admissibility conditions for the variational method had been previously specified by Faedo in [23].

solutions to propagation problems. Indeed, in order to obtain the n-th approximation in the method of moments it is necessary to solve an initial value problem for a second-order system of n ordinary differential equations in normal form; in contrast, the variational method necessitates the solution of a *fourth-order* system of n differential equations with two end-point conditions on the time interval; in the case of an infinite time interval, asymptotic conditions need to be prescribed. Furthermore, when the problem is posed on a domain of the form S_T the n-th approximation in the method of moments is independent of T, whereas the approximation obtained with the variational method depends explicitly on T. In fairness, Faedo observes that the variational method presents the advantage of being less restrictive concerning the choice of basis functions φ_i, which do not have to satisfy any prescribed boundary conditions. Furthermore, Faedo recognizes that his earlier work on the theoretical foundations of the variational method turned out to be very useful in his analysis of the method of moments.

It should be observed that Faedo does not discuss the actual solution of the initial value problem for the second-order system of ODEs satisfied by the coefficients $c_i(t)$. Either he considered this an easy problem, or he regarded his method primarily as a theoretical tool for obtaining (constructively) existence and uniqueness results for hyperbolic problems. Regardless, the numerical solution of such systems was a well-studied problem that did not present insurmountable difficulties at the time, at least in principle.

In closing this brief account of the method which will become known as the *Faedo–Galerkin method*, it is interesting to note that there is little in Faedo's paper about the actual rate of convergence of the approximants.

7 Related contributions

Of the many students and collaborators of Picone, Faedo was the one who gave the most important contributions to the quantitative analysis of PDEs.[23]

Faedo's work did not go unnoticed. Several authors, both in Italy and abroad, extended and improved his method. Before giving a brief overview of these developments, we must mention an important paper by Eberhard Hopf [50], written in 1950 and published in 1951, in which the author, independently of Faedo, introduced the same method of moments for obtaining existence results for the time-dependent, incompressible Navier–Stokes equations of fluid mechanics. As is well known, these equations are nonlinear.[24] Faedo had considered nonlinear problems before, but only in the context of Picone's variational method [26, Section 5].

[23] Among other notable pupils of Picone who contributed, either directly or indirectly, to Numerical Analysis we mention Gaetano Fichera, Gianfranco Cimmino, Lamberto Cesari and Carlo Miranda. On Cimmino's numerical analysis papers, see [6].

[24] Years later, Jacques–Louis Lions will write [57, page 216] that the method of Faedo–Galerkin is most useful in the treatment of nonlinear problems.

The next significant development was the adaptation by J. W. Green of Faedo's method to parabolic problems [42]. This paper (published in 1953 but completed by April 1952, about three years after the publication of Faedo's paper) contains existence and uniqueness theorems as well as a priori bounds on the solution, convergence results (in the mean), and estimates of the *asymptotic* rate of convergence of the method.[25] The paper's abstract states,

> The aim of this paper is to adapt to certain parabolic partial differential equations an expansion method of solution developed by S. Faedo for hyperbolic equations. In order to make possible a moderately compact presentation, the equations treated are not the most general to which the method is applicable, but are the simplest nontrivial relatives of the heat equation [...] The method of solution not only shows the existence of a solution, but describes a definite procedure for approximating it. Some remarks are made on the possibility of estimating the error.

Green calls Faedo's method *the moment method* or, in a footnote, *Galerkin's method*; for the latter he provides a reference to the 1946 English edition of Sokolnikoff's *Mathematical Theory of Elasticity* [80].

In Italy, Faedo's work was continued by A. Chiffi and U. Barbuti [3, 4, 13]. In particular, Barbuti considered the application of the method to semilinear hyperbolic problems, mostly as a technique for obtaining existence results.

The principal promoter of Faedo's work, however, turned out to be the great French mathematician Jacques–Louis Lions (1928–2001). Lions' 1956 paper [56] contains the first occurrence we were able to find of the phrase *Faedo–Galerkin method*, in the context of parabolic PDEs formulated in distributional terms.[26] In his 1961 book [57], in a section titled *Méthodes d'approximation par des projections* (Approximation methods by projections) he attributes the method to Galerkin for the elliptic case, to Faedo for the case of second-order hyperbolic equations, and to Green for the case of second-order parabolic equations. He then goes on to state that the method was later systematically used by I. M. Višik, O. A. Ladyzhenskaya, and by Lions himself, and that E. Hopf had independently introduced the method for nonlinear equations.

There is no doubt that Lions' fame and universally recognized authoritativeness greatly contributed to the popularity of the Faedo–Galerkin method, especially among French mathematicians, and also to Faedo's getting credit for his contribution. The Soviet school also made large use of the Faedo–Galerkin method, although some authors in the Russian literature will always refer to the method as *Galerkin's method*, even when treating time-dependent PDEs. A notable exception is Olga Ladyzhenskaya; see, e.g., her classic monograph on the boundary value problems of mathematical physics [54] or her lecture notes on semigroups and evolution equations [55].

[25] As pointed out by Green, however, he was unable to estimate how long it would take for the asymptotic regime to kick in.

[26] "On va maintenant démontrer l'existence de u en utilisant la méthode de Faedo–Galerkine." See [56, page 131].

Nowadays, descriptions of the Faedo–Galerkin method are routinely found in the literature; see, for example, the encyclopaedic work [18] or the treatment given in S. Salsa's textbook [77]. In combination with modern numerical methods like finite elements, the Faedo–Galerkin approach remains a powerful technique also for current research, for instance on the Navier–Stokes equations [44, 45], as well as on the numerical solution of PDEs posed on evolving surfaces [20].

8 The contributions of Picone and Faedo in historical perspective

When attempting to assess the respective contributions of Picone and Faedo to the numerical analysis of PDEs in the light of later developments, it might appear that only Faedo succeeded in making a contribution of lasting importance. While Faedo's name can still be found in contemporary monographs, textbooks and research papers, Picone's has largely fallen into oblivion, and seldom appears in numerical analysis books or papers published after the 1960s.

Nevertheless, it is important to recognize the great indirect contribution made by Picone with his enormous effort, scientific as well as organizational, to establish numerical analysis as an independent branch of modern analysis on the same level of dignity and importance (if not higher) than other, more traditional branches. Picone's true legacy lies in the founding of a major mathematical school, with ramifications not only in Italy but also abroad, and in the creation of the INAC, one of the first institutes in the world entirely devoted to numerical analysis; Picone's vision of applied and numerical mathematics as a fundamental subject for both mathematicians and applied scientists was very much ahead of its time and was to be fully vindicated by later developments.

As we have shown in this paper, Faedo's work on the quantitative analysis of PDEs was directly inspired (perhaps even ordered!) by Picone himself. Many of Faedo's papers are devoted to extending, analyzing and improving (even correcting, on occasions) Picone's work on the method of Ritz, on the method of weighted least powers and on the variational method. Even Faedo's most important creation, the Faedo–Galerkin method, was a direct outgrowth of his work on wave propagation problems, which he studied upon Picone's suggestion.

While Faedo had the advantage of standing on the shoulders of giants like Tonelli and Picone, Picone himself had to find his own place in the mathematical landscape entirely on his own and without the benefit of any previous example or role model from which to draw inspiration; indeed, for many years he had to constantly fight against the opposition of the mathematical establishment, with only the support of very few allies. Thus, as authoritatively argued by Gaetano Fichera in [36], one could say that Picone's contribution was more important, lasting and fundamental than those of most of his pupils, even though many of them were to attain much higher visibility and recognition by the wider mathematical community.

When looking at the papers of Picone and Faedo, written between the mid-1920s until the early 1950s, from the vantage point of today's knowledge, the modern reader is of course struck by how much the subject has changed in the intervening decades. The development of the notions of consistency, stability and convergence of numerical schemes and their relationship, which took place during the 1950s, completely changed the way numerical analysts look at the entire subject. Another seismic change was brought about by the massive infusion of functional-analytic techniques into the field, especially the use of weak formulations and the theory of Sobolev spaces, which are now standard features in numerical analysis. At the time when Picone and Faedo were writing their papers, such techniques were still in the process of being developed (by mathematicians like S. Sobolev himself in the Soviet Union, J. Leray in France, and a few others). In the papers of Picone, Faedo and other contemporary Italian mathematicians on PDEs, the regularity assumptions on the coefficients are usually rather high and only strong solutions are typically considered. The techniques used are the ones traditionally favored by the Italian school of analysis, which tended to be classical; the references to works by non-Italian authors are infrequent and occasionally tinged by nationalistic tones, as was typical of the period.

Only in the course of the 1950s will Italian mathematicians fully absorb the language of modern function spaces, as one can see by studying the papers of Cimmino, Fichera, Stampacchia, Miranda, Magenes and others from that period. With the advent of the finite element method around 1960, numerical analysts and even engineers will begin to recognize the indispensable nature of the language and techniques of functional analysis, eventually leading to the modern theory of numerical methods for PDEs. The Faedo–Galerkin method itself will find its most general and rigorous formulation only after the shift to modern functional-analytic methods, not unlike numerous other approaches for solving PDEs.

9 Conclusions

The second quarter of the 20th century saw important advances in the numerical solution of boundary and initial value problems for partial differential equations. The best known among the works that laid the foundation for much of the developments that were to follow the advent of digital computers in the late 1940s are the seminal paper by Courant, Friedrichs and Lewy [16] on finite difference schemes, and the early outline of the finite element method given by Courant in [15]. This important work was carried out by Richard Courant and his coworkers first in Germany, and subsequently in the United States. A rather different, largely independent line of research was meanwhile being pursued in Italy around the charismatic figure of Mauro Picone. This work was influenced by earlier techniques, like Ritz's method, which were rooted in the classical tradition of direct methods in the calculus of variations, going all the way back to Hilbert. While the methods developed by Picone ultimately failed to meet with the success he had hoped for, this research activity was far from fruitless, eventually leading one of his younger collaborators, Sandro Faedo, to the development of the so-called Faedo–Galerkin method for the treatment of evolution problems. This method, which has since become classical, has been widely applied and extended by a number of researchers and has been shown to be an effective tool, both computational and theoretical, in the study of complex problems such as the Navier–Stokes system in incompressible fluid dynamics and many others.

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