AN ITERATIVE METHOD FOR GENERALIZED SADDLE POINT PROBLEMS∗

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Abstract. We study a stationary iterative scheme for the solution of block-structured linear systems of saddle point type. The scheme consists of an alternating iteration based on the symmetric/skew-symmetric splitting of the coefficient matrix. We prove convergence of the method for a large class of problems, and we consider its use as a preconditioner for Krylov subspace methods. The performance of the method is illustrated by numerical experiments with matrices from various application areas.

Key words. saddle point problems, matrix splittings, iterative methods, preconditioning

AMS subject classifications. Primary 65F10, 65N22, 65F50. Secondary 15A06.

1. Introduction. We consider the solution of systems of linear equations with the following block 2-by-2 structure:

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix} = \begin{bmatrix}
f \\
g
\end{bmatrix}
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{m \times m}, f \in \mathbb{R}^n, g \in \mathbb{R}^m\), and \(m \leq n\). We further assume that matrices \(A, B\) and \(C\) are large and sparse. Systems of the form (1.1) arise in a variety of scientific and engineering applications, including computational fluid dynamics [1, 20, 22, 24, 27, 44], mixed finite element approximation of elliptic partial differential equations [12, 48, 59], optimization [5, 25, 26, 32, 38, 43], optimal control [9, 35], weighted and equality constrained least squares estimation [10], structural analysis [54], electrical networks [54], inversion of geophysical data [34], computer graphics [42], and others.

In many situations of interest \(A\) and \(C\) are symmetric positive semidefinite (frequently \(C = O\)) and \(\text{rank}(B) = m\). Furthermore, matrices \(A\) and \(B\) have no nontrivial common null vector: in symbols, \(\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}\). Under these assumptions, problem (1.1) has a unique solution. A slight generalization of this result is given in Lemma 1.1 below.

When \(A = A^T\) and \(C = O\), (1.1) is equivalent to the equality constrained quadratic minimization problem

\[
\begin{cases}
\text{Minimize} & J(u) = \frac{1}{2} \langle Au, u \rangle - \langle f, u \rangle \\
\text{subject to} & Bu = g.
\end{cases}
\]

Introducing the vector of Lagrange multipliers \(p \in \mathbb{R}^m\), the optimality conditions are

\[Au + B^Tp = f, \quad Bu = g\]
which is precisely problem (1.1) with $C = O$. The solution $(u, p)$ of (1.1) is the unique stationary (saddle) point of the Lagrangian

$$\mathcal{L}(u, p) = \frac{1}{2} \langle Au, u \rangle - \langle f, u \rangle + \langle p, g - Bu \rangle.$$  

Conversely, if $(u, p)$ is the solution of (1.1), then $u$ is the unique solution of (1.2) and $p$ is the corresponding Lagrange multiplier vector.

Problems with $C \neq O$ arise in the augmented Lagrangian method, where a regularization (stabilization) term is added to the Lagrangian (1.3); see, e.g., [24]. Mixed finite element methods are stabilized in this way [12].

Another case of interest is when the $A$ block is nonsymmetric, with positive definite symmetric part $\frac{1}{2}(A + A^T)$. In other words, $A$ is positive real. This situation arises when the steady-state Navier–Stokes equations are linearized by a Picard iteration, leading to the Oseen equations; see [20, 22]. Now the $A$ block corresponds to some discretization of a convection-diffusion operator. In this case, the variational characterization (1.2) of the solution is no longer valid.

A number of solution methods have been proposed in the literature. Besides specialized sparse direct solvers [16, 17] we mention, among others, Uzawa-type schemes [11, 21, 24, 27, 61], block and approximate Schur complement preconditioners [4, 15, 20, 22, 40, 45, 46, 48, 51], splitting methods [18, 30, 31, 49, 55], indefinite preconditioning [23, 35, 38, 43, 48], iterative projection methods [5], iterative null space methods [1, 32, 53] and preconditioning methods based on approximate factorization of the coefficient matrix [25, 50]. Several of these algorithms are based on some form of reduction to a smaller system, for example by projecting the problem onto the null space of $B$, while others work with the original (augmented) matrix in (1.1). The method studied in this paper falls in the second category.

When $A$ is symmetric positive (semi-)definite, the coefficient matrix

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

is symmetric indefinite, and indefinite solvers can be used to solve problem (1.1). Alternatively, one can solve instead of (1.1) the equivalent nonsymmetric system

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}.$$  

Let

$$A = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}.$$  

The nonsymmetric formulation is especially natural when $A$ is nonsymmetric, but positive real (as in the Oseen problem). Whether $A$ is symmetric or not, the nonsymmetric matrix $A$ in (1.5) has certain desirable properties, which are summarized in the following result.

**Lemma 1.1.** Let $A \in \mathbb{R}^{(n+m) \times (n+m)}$ be given by (1.5). Assume $H = \frac{1}{2}(A + A^T)$ is positive semidefinite, $B$ has full rank, $C = C^T$ is positive semidefinite, and $\mathcal{N}(B) \cap \mathcal{N}(H) = \{0\}$. Let $\sigma(A)$ denote the spectrum of $A$. Then

(i) $A$ is nonsingular.

(ii) $A$ is semipositive real: $\langle Av, v \rangle = v^T Av \geq 0$ for all $v \in \mathbb{R}^{n+m}$.
(iii) $A$ is positive semistable, that is, the eigenvalues of $A$ have nonnegative real part:
\[ \Re(\lambda) \geq 0 \] for all $\lambda \in \sigma(A)$.

(iv) If in addition $H = \frac{1}{2}(A + A^T)$ is positive definite, then $A$ is positive stable:
\[ \Re(\lambda) > 0 \] for all $\lambda \in \sigma(A)$.

Proof. To prove (i), let $x = \begin{bmatrix} u \\ p \end{bmatrix}$ be such that $Ax = 0$. Then
\[ \begin{aligned}
Au + B^T p &= 0 \\
-Bu + Cp &= 0.
\end{aligned} \tag{1.6} \]

Now, from $Ax = 0$ we get $x^T Ax = u^T Au + p^T Cp = 0$ and therefore it must be $u^T Au = 0$ and $p^T Cp = 0$, since both of these quantities are nonnegative. But $u^T Au = u^T Hu = 0$, which implies $u \in \mathcal{N}(H)$ since $H$ is symmetric positive semidefinite (see [36], page 400). Similarly, $p^T Cp = 0$ with $C$ symmetric positive semidefinite implies $Cp = 0$ and therefore (using the second of (1.6)) $Bu = 0$. Therefore $u = 0$ since $u \in \mathcal{N}(H) \cap \mathcal{N}(B) = \{0\}$. But if $u = 0$ then from the first of (1.6) we obtain $B^T p = 0$ and therefore $p = 0$ since $B$ has full column rank. Therefore the only solution to $Ax = 0$ is the trivial solution, and $A$ is nonsingular.

To prove (ii) we observe that for any $v \in \mathbb{R}^{n+m}$ we have $v^T A v = v^T \mathcal{H} v$, where
\[ \mathcal{H} = \frac{1}{2} (A + A^T) = \begin{bmatrix} H & O \\ O & C \end{bmatrix} \]
is the symmetric part of $A$. Clearly $\mathcal{H}$ is positive semidefinite, hence $v^T A v \geq 0$.

To prove (iii), let $(\lambda, v)$ be an eigenpair of $A$, with $||v||_2 = 1$. Then $v^* A v = \lambda$ and $(v^* A v)^* = v^* A^T v = \overline{\lambda}$. Therefore \[ \frac{1}{2} v^* (A + A^T) v = \frac{\lambda + \overline{\lambda}}{2} = \Re(\lambda). \] To conclude the proof, observe that
\[ v^* (A + A^T) v = \Re(v)^T (A + A^T) \Re(v) + \Im(v)^T (A + A^T) \Im(v), \]
a real nonnegative quantity.

To prove (iv), assume $(\lambda, v)$ is an eigenpair of $A$ with $v = \begin{bmatrix} u \\ p \end{bmatrix}$, then
\[ \Re(\lambda) = u^* Hu + p^* Cp = \Re(u)^T H \Re(u) + \Im(u)^T H \Im(u) + \Re(p)^T C \Re(p) + \Im(p)^T C \Im(p). \]

This quantity is nonnegative, and it can be zero only if $u = 0$ (since $H$ is assumed to be positive definite) and $Cp = 0$. But if $u = 0$ then from the first of (1.6) we get $B^T p = 0$, hence $p = 0$ since $B$ has full column rank. Hence $v = 0$, a contradiction.

Thus, by changing the sign of the last $m$ equations in (1.1) we may lose symmetry (when $A$ is symmetric), but we gain positive (semi-)definiteness. This can be advantageous when using certain Krylov subspace methods, like restarted GMRES; see [19, 52].

Hereafter, we will refer to problems of the type (1.1) or, equivalently, (1.4) as to generalised saddle point problems, in that we allow $C$ to be nonzero and $A$ to be either symmetric positive semidefinite, or nonsymmetric with positive semidefinite symmetric part.

In this paper we propose a new approach for solving generalised saddle point problems based on an alternating symmetric/skew-symmetric iteration [2] applied to (1.4). This approach is very general in that it does not require the submatrix $A$ to be nonsingular or symmetric; hence, it is applicable to a broad class of problems. The method is described in section 2, and some of its convergence properties are studied in section 3. The alternating symmetric/skew-symmetric scheme is a stationary
iteration that can be accelerated by Krylov subspace methods; in other terms, the
alternating symmetric/skew-symmetric splitting can be used as a preconditioner for
Krylov subspace methods. This is considered in section 4. Numerical experiments are
presented in section 5. Finally, in section 6 we draw our conclusions.

2. The alternating iteration. In [2], the following method was proposed for
solving linear systems $Ax = b$ with a positive real coefficient matrix $A \in \mathbb{R}^{n \times n}$. Consider the symmetric/skew-symmetric splitting

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^T) \quad \text{and} \quad S = \frac{1}{2}(A - A^T)$$

are the symmetric and skew-symmetric part of $A$, respectively.

Given an initial guess $x^0$ and $\alpha > 0$, the symmetric/skew-symmetric splitting
iteration computes a sequence $\{x^k\}$ as follows:

$$\begin{align*}
(H + \alpha I)x^{k+\frac{1}{2}} &= (\alpha I - S)x^k + b, \\
(S + \alpha I)x^{k+1} &= (\alpha I - H)x^{k+\frac{1}{2}} + b
\end{align*}$$

(where $k = 0, 1, \ldots$). This scheme is reminiscent of the classical ADI (Alternating
Direction Implicit) method; see, e.g., [58]. It alternates between the symmetric and
the skew-symmetric part of $A$, with an additional shift added to guarantee convergence
(and also to make sure $S$ is invertible). From the point of view of preconditioning
adopted here, the idea is to obtain a robust solver by combining preconditioning
with $H + \alpha I$, which is effective for matrices with dominant symmetric part, with
preconditioning with $S + \alpha I$, which is effective for matrices with dominant skew-
symmetric part. Preconditioning with the symmetric part goes back to [14, 60];
preconditioning with the (shifted) skew-symmetric part is more recent, see [29, 41].

Next, we show how to apply the iteration to problem (1.4). Begin by writing the
linear system as

$$A x = b$$

where

$$A = \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}, \quad x = \begin{bmatrix} u \\ p \end{bmatrix}, \quad b = \begin{bmatrix} f \\ -g \end{bmatrix}.$$ 

Now recall that $A = \mathcal{H} + \mathcal{S}$, where

$$\mathcal{H} = \frac{1}{2}(A + A^T), \quad \mathcal{S} = \frac{1}{2}(A - A^T)$$

are the symmetric and skew-symmetric part of $A$, respectively. Also, write $A = H + S$,
then we have

$$\mathcal{H} = \begin{bmatrix} H & O \\ O & C \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} S & B^T \\ -B & O \end{bmatrix}.$$ 

Let $\alpha > 0$ be a parameter. Consider the following two splittings of $A$:

$$A = (\mathcal{H} + \alpha \mathcal{I}) - (\alpha \mathcal{I} - \mathcal{S})$$
and

\[ A = (H + \alpha I) - (\alpha I - H). \]

Here \( I \) denotes the \((n + m)\)-by-\((n + m)\) identity matrix. We explicitly observe that

\[ \begin{bmatrix} H + \alpha I_n & O \\ O & C + \alpha I_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S + \alpha I_n \quad B^T \\ -B \quad \alpha I_m \end{bmatrix} \]

are both nonsingular matrices.

The algorithm is obtained by alternating between these two splittings. Given an initial guess \( x^0 = \begin{bmatrix} u^0 \\ p^0 \end{bmatrix} \), the symmetric/skew-symmetric iteration computes a sequence \( \{x^k\} \) as follows:

\[
\begin{cases}
(H + \alpha I)\bar{x}^{k+} = (\alpha I - S)\bar{x}^k + b, \\
(S + \alpha I)\bar{x}^{k+1} = (\alpha I - H)\bar{x}^{k+} + b.
\end{cases}
\]

A few remarks are in order. Algorithm (2.4) is simply the alternating iteration of [2] applied to problem (1.4). However, the convergence analysis of [2] assumes that the coefficient matrix is positive real (that is, \( H \) must be positive definite). In the context of (generalized) saddle point problems, this is true only when submatrices \( H \) and \( C \) in (1.4) are both SPD, which is almost never the case in practice. To handle the general case, a different analysis is required. We provide this analysis in the next section.

The algorithm has the following intuitive justification when \( A = A^T \) and \( C = O \). The iteration (1.4) decouples the unconstrained problem

\[ \text{Minimize} \quad J(u) = \frac{1}{2} \langle Au, u \rangle - \langle f, u \rangle, \]

which is determined by the matrix \( A \), from the constraints (determined by \( B \)). In many cases, each of the two individual subproblems (quadratic minimization and constraint satisfaction) is relatively easy to solve: the coupled problem, however, is often rather challenging. By alternating between the two subproblems, both features of the original problem are captured.

At each iteration of (2.4), it is required to solve two sparse linear systems with coefficient matrices \( H + \alpha I \) and \( S + \alpha I \). Note that in many applications \( H, S \) and \( C \) are block-diagonal, thus providing additional decoupling into smaller subproblems, which can be advantageous in a parallel implementation.

Clearly, the choice of the solution methods used to perform the two half-steps in (2.4) is highly problem-dependent. The alternating algorithm (2.4) is just a general scheme that can incorporate whatever solvers are appropriate for a given problem.

The first half-step of algorithm (2.4) necessitates the solution of two (uncoupled) linear systems of the form

\[
\begin{cases}
(H + \alpha I_n)\bar{u}^{k+} = \alpha u^k - Su^k + f - B^T p^k, \\
(C + \alpha I_m)\bar{p}^{k+} = \alpha p^k + g + Bu^k.
\end{cases}
\]

Both systems in (2.5) are SPD, and any sparse solver for SPD systems can be applied. This could be a sparse Cholesky factorization, or a preconditioned conjugate gradient (PCG) scheme, or some specialized solver. Note that the addition of a positive term \( \alpha \) to the main diagonal of \( H \) (and \( C \)) improves the condition number. This, in turn,
tends to improve the rate of convergence of iterative methods applied to (2.5). More precisely, if $H$ is normalized so that its largest eigenvalue is equal to 1, then for the spectral condition number of $H + \alpha I$ we have

$$
\kappa(H + \alpha I) = \frac{1 + \alpha}{\lambda_{\min}(H) + \alpha} \leq 1 + \frac{1}{\alpha},
$$

independent of the size of the problem. Note that even a fairly small value of $\alpha$, such as $\alpha = 0.1$, yields a small condition number ($\kappa(H + \alpha I) \leq 11$). Unless $\alpha$ is very small, rapid convergence of the CG method applied to (2.5) can be expected, independent of the number $n$ of unknowns.

In the case of mixed finite element formulations of Stokes and elliptic problems, even for arbitrarily small $\alpha$ there are $h$-independent methods to perform the solves in (2.5). In the case of Stokes, the first solve in (2.5) reduces to solving two or three (depending on whether the problem is two- or three-dimensional) Poisson-type problems, which can be done in $O(n)$ operations using multigrid. For mixed formulations of second order elliptic problems, $S = O$ and the matrix $H = A$ in (2.5) is just a mass matrix, which has condition number $O(1)$ independent of $h$, again leading to an $O(n)$ solution. Also the matrix $C$, when not zero, is typically a mass matrix requiring $O(m)$ work.

The second half-step of algorithm (2.4) requires the solution of two coupled linear systems of the form

$$
\begin{align*}
(\alpha I_n + S)u^{k+1} + B^T p^{k+1} &= (\alpha I_n - H)u^{k+1} + f \equiv f^k, \\
-Bu^{k+1} + \alpha p^{k+1} &= (\alpha I_m - C)p^{k+1} + g \equiv g^k.
\end{align*}
$$

This system can be solved in several ways. Of course, a sparse LU factorization could be used if the problem is not too large. An alternative approach is to eliminate $u^{k+1}$ from the second equation using the first one (Schur complement reduction), leading to a smaller (order $m$) linear system of the form

$$
B(I_n + \alpha S)^{-1} B^T \alpha^2 I_m) p^{k+1} = B(I_n + \alpha S)^{-1} f^k + \alpha g^k.
$$

Once the solution $p^{k+1}$ to (2.7) has been computed, the vector $u^{k+1}$ is given by $u^{k+1} = (\alpha I_n + S)^{-1}(f^k - B^T p^{k+1})$. When $S = O$, system (2.7) simplifies to

$$
(BB^T + \alpha^2 I_m) p^{k+1} = Bf^k + \alpha g^k,
$$

and $u^{k+1} = \frac{1}{\lambda}(f^k - B^T p^{k+1})$. If $BB^T$ is sufficiently sparse, system (2.8) could be formed explicitly and solved by a sparse Cholesky factorization. Otherwise, a PCG iteration with a simple preconditioner not requiring access to all the entries of the coefficient matrix $BB^T + \alpha^2 I_m$ could be used.

Also note that up to a scaling factor, the coefficient matrix of the coupled system in (2.6) is a normal matrix of the form “identity-plus-symmetric.” There exist optimal three-term recurrence methods that can be applied to systems of this kind; see [14, 60] and, more generally, [37]. The CGW (Concus–Golub–Widelund) method of [14] and [60] is of particular interest when the system comes from mixed finite element formulations of Stokes and elliptic problems (where $S = O$). Indeed, it follows from Thms. 4.2 and 4.3 in [56] that the rate of convergence of the CGW iteration applied to (2.6) depends only on the largest singular value of $\alpha^{-1} B$; when $B$ represents a discretization of the divergence operator, as in the case of Stokes and elliptic problems, $B$ can be scaled so that $\sigma_{\text{max}}(B) = O(1)$ as $h \to 0$, and
therefore the rate of convergence is independent of $h$ (and improves as $\alpha$ increases, as one would expect). This means that the CGW iteration converges in a number of steps independent of $h$, and since the cost of each iteration is dominated by matrix-vector products with $B$ and $B^T$, the total cost of solving the second half-step (2.6) is $\mathcal{O}(n + m)$. This remains true even for the case of the Oseen equations, where $S \neq O$. Now the rate of convergence depends on the spectral radius of

$$
\alpha^{-1} \begin{bmatrix}
S & B^T \\
-B & O
\end{bmatrix},
$$

where $S$ represent the convective terms.

Other iterative methods for the solution of shifted skew-symmetric systems can be found, e.g., in [47] and [29].

We have therefore argued that in many important cases, each iteration of the alternating scheme (2.4) can be executed in $\mathcal{O}(n + m)$ work, which is of the same order as the cost of a matrix-vector product with the coefficient matrix $A$. Of course, the constant that multiplies $n + m$ can be quite large, but it can be made arbitrarily small by using a large value of the convergence parameter $\alpha$. Unfortunately, as can be expected, a large value of $\alpha$ tends to result in slow convergence of the overall alternating iteration (2.4). Ideally, one would like to choose $\alpha$ so as to minimize the total amount of work required to solve (1.4). However, estimating such an optimal value of $\alpha$ appears to be difficult in general; see the next section.

Until now we have assumed that the linear systems in (2.4) are solved exactly. In [2], the use of inexact solves was investigated in the context of solving positive real systems. This is important in order to reduce the cost of the alternating method, especially for three-dimensional PDE problems. In [2], the authors consider inexact solves corresponding to performing a few steps of an inner iteration, and derived sufficient conditions on the accuracy of the inner iterations that guarantee the convergence of the outer iteration. Also, it was shown in [2] that by tuning the convergence tolerance of the inner iterations in a suitable manner, it is possible to recover the asymptotic rate of convergence of the exact alternating iteration. The conclusion is that in practical implementations, inexact solves considerably increase the competitiveness of the algorithm. Here we observe that when the alternating method is used as a preconditioner for a Krylov method, then inexact solves are a natural choice, and there is no theoretical restriction on the accuracy of the inner solves. In the section on numerical experiments, we will investigate the use of inexact inner solves corresponding to incomplete factorizations of the coefficient matrices in (2.4).

Also note that in (2.4), the scalar matrix $\alpha I$ could be replaced by a matrix of the form $\alpha \mathcal{F}$, where $\mathcal{F}$ is SPD. This idea, in the context of ADI methods, goes back to Wachspress and Habeter [57]; see also [58], p. 242. It is straightforward to see that this is equivalent to applying the alternating iteration (2.4) to the symmetrically preconditioned system

$$
(2.9) \quad \hat{A}\hat{x} = \hat{b}, \quad \hat{A} := \mathcal{F}^{-1/2} A \mathcal{F}^{-1/2}, \quad \hat{x} = \mathcal{F}^{1/2} x, \quad \hat{b} = \mathcal{F}^{-1/2} b.
$$

In this paper we limit ourselves to the case where $\mathcal{F}$ is the $(n + m) \times (n + m)$ diagonal matrix having the $i$th diagonal entry equal to the $i$th diagonal entry of $A$ if this is nonzero, and 1 otherwise. As we show in the section on numerical experiments, this simple diagonal preconditioning considerably improves the overall performance of the method in many cases.
In the next section we turn to the study of the convergence of the general scheme (2.4), assuming that the solves in (2.5) and (2.6) are performed exactly (rather than approximately, as in an inexact inner-outer setting).

3. Convergence analysis. To analyze the convergence of (2.4), we eliminate the intermediate vector \( x^{k+1} \) and write the iteration in fixed point form as

\[
x^{k+1} = T_\alpha x^k + c
\]

where

\[
T_\alpha := (S + \alpha I)^{-1} (\alpha I - H) (H + \alpha I)^{-1} (\alpha I - S)
\]

is the iteration matrix of the method, and

\[
c := (S + \alpha I)^{-1} [I + (\alpha I - H)(H + \alpha I)^{-1}] b.
\]

The fixed point iteration (3.1) converges for arbitrary initial guesses \( x^0 \) and right-hand sides \( b \) to the solution \( x = A^{-1} b \) if and only if \( \rho(T_\alpha) < 1 \), where \( \rho(T_\alpha) \) denotes the spectral radius of \( T_\alpha \).

It is easy to show that the spectral radius of \( T_\alpha \) cannot exceed 1. Indeed, note that \( T_\alpha \) is similar to

\[
\hat{\hat{T}}_\alpha := (\alpha I + S)T_\alpha (\alpha I + S)^{-1} = (\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1}.
\]

Now, the matrix \( (\alpha I - S)(\alpha I + S)^{-1} \) is the Cayley transform of the skew-symmetric matrix \( S \), and is therefore orthogonal [36]. It follows that

\[
\rho(T_\alpha) = \rho(\hat{\hat{T}}_\alpha) \leq ||\hat{\hat{T}}_\alpha||_2 = ||(\alpha I - H)(\alpha I + H)^{-1}||_2.
\]

Also, \( (\alpha I - H)(\alpha I + H)^{-1} \) is symmetric, therefore

\[
||(\alpha I - H)(\alpha I + H)^{-1}||_2 = \max_{\lambda \in \sigma(H)} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \leq 1,
\]

where \( \sigma(H) \) denotes the spectrum of the symmetric matrix \( H \). Putting together (3.3) and (3.4) we get

\[
\rho(T_\alpha) \leq \max_{\lambda \in \sigma(H)} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \leq 1.
\]

If \( A \) is positive real (i.e., \( H \) is SPD), then the second inequality in (3.5) is strict, and convergence follows. This is the proof that was given in [2] for the positive real case. However, in the context of generalized saddle point problems (1.4), the matrix \( H \) is only positive semidefinite and generally singular. In this case,

\[
\max_{\lambda \in \sigma(H)} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| = 1
\]

and we cannot conclude that the alternating iteration is convergent. For matrices whose symmetric part is positive semidefinite and singular, the alternating iteration is not convergent in general. The simplest example is given by a skew-symmetric matrix: in this case the iteration matrix \( T_\alpha \) is orthogonal, and therefore \( \rho(T_\alpha) = 1 \).
However, as the following theorem shows, the alternating iteration converges for a broad class of generalized saddle point problems.

**Theorem 3.1.** Consider problem (1.4) and assume that $A$ is positive real, $C$ is symmetric positive semidefinite, and $B$ has full rank. Then the iteration (3.1) is unconditionally convergent; that is, $\rho(T_\alpha) < 1$ for all $\alpha > 0$.

**Proof.** As already observed, the iteration matrix $T_\alpha$ is similar to

$$
\hat{T}_\alpha = (\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1} = \mathcal{R} \mathcal{U},
$$

where

$$
\mathcal{R} := (\alpha I - H)(\alpha I + H)^{-1}
$$
is symmetric and

$$
\mathcal{U} := (\alpha I - S)(\alpha I + S)^{-1}
$$orthogonal. Now, $\mathcal{R}$ is orthogonally similar to the $(n+m)$-by-$(n+m)$ diagonal matrix

$$
D = \begin{bmatrix}
\frac{\alpha - \mu_1}{\alpha + \mu_1} & & & \\
& \frac{\alpha - \mu_2}{\alpha + \mu_2} & & \\
& & \ddots & \\
& & & \frac{\alpha - \mu_m}{\alpha + \mu_m}
\end{bmatrix}
$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the (positive) eigenvalues of $H$ and $\nu_1, \nu_2, \ldots, \nu_m$ are the (nonnegative) eigenvalues of $C$. That is, there is an orthogonal matrix $V$ of order $n + m$ such that

$$
V^T \mathcal{R} V = D = \begin{bmatrix}
D_1 & O \\
O & D_2
\end{bmatrix},
$$

where $D_1 = \text{diag}\left(\frac{\alpha - \mu_1}{\alpha + \mu_1}, \frac{\alpha - \mu_2}{\alpha + \mu_2}, \ldots, \frac{\alpha - \mu_n}{\alpha + \mu_n}\right)$ and $D_2 = \text{diag}\left(\frac{\alpha - \nu_1}{\alpha + \nu_1}, \frac{\alpha - \nu_2}{\alpha + \nu_2}, \ldots, \frac{\alpha - \nu_m}{\alpha + \nu_m}\right)$.

Note that

$$
\frac{\alpha - \mu_i}{\alpha + \mu_i} < 1 \text{ for } 1 \leq i \leq n \quad \text{and} \quad \frac{\alpha - \nu_i}{\alpha + \nu_i} \leq 1 \text{ for } 1 \leq i \leq m.
$$

It follows that $\mathcal{R} \mathcal{U}$ is orthogonally similar to

$$
V^T \mathcal{R} V V^T \mathcal{U} V = \mathcal{D} \mathcal{Q}
$$

where $Q := V^T \mathcal{U} V$, being a product of orthogonal matrices, is orthogonal. Hence, the iteration matrix $T_\alpha$ is similar to $\mathcal{D} \mathcal{Q}$, and therefore

$$
\rho(T_\alpha) = \rho(\mathcal{D} \mathcal{Q}) = \rho(\mathcal{Q} \mathcal{D}).
$$

We claim that $\rho(\mathcal{Q} \mathcal{D}) < 1$, for all $\alpha > 0$. To show this, partition $\mathcal{Q}$ conformally to $D$:

$$
\mathcal{Q} = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}.
$$
Then
\[
QD = \begin{bmatrix} Q_{11}D_1 & Q_{12}D_2 \\ Q_{21}D_1 & Q_{22}D_2 \end{bmatrix}.
\]

Now, let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( QD \) and let \( x \in \mathbb{C}^{n+m} \) be a corresponding eigenvector with \( \|x\|_2 = 1 \). We assume \( \lambda \neq 0 \), or else there is nothing to prove. We want to show that \( |\lambda| < 1 \). Clearly, \( QDx = \lambda x \) implies \( Dx = \lambda Q^Tx \) and taking norms:
\[
\|Dx\|_2 = |\lambda| \|Q^Tx\|_2 = |\lambda|.
\]
Therefore
\[
|\lambda|^2 = \|Dx\|^2_2 = \sum_{i=1}^{n+m} \left( \frac{\alpha - \mu_i}{\alpha + \mu_i} \right)^2 x_i^* x_i + \sum_{i=n+1}^{n+m} \left( \frac{\alpha - \nu_i}{\alpha + \nu_i} \right)^2 x_i^* x_i \leq \|x\|^2_2 = 1.
\]

Hence, as we already knew, the spectral radius of \( T_\alpha \) cannot exceed unity.

To prove that \( |\lambda| < 1 \) (strictly), we show that there exists at least one \( i \) (1 \( \leq i \leq n \)) such that \( x_i \neq 0 \). Using the assumption that \( B \) has full rank, we will show that \( x_i = 0 \) for all \( 1 \leq i \leq n \) implies \( x = 0 \), a contradiction. Indeed, if the eigenvector \( x \) is of the form
\[
x = \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}
\]
(where \( \hat{x} \in \mathbb{C}^m \)), the identity \( QDx = \lambda x \) becomes
\[
(3.7) \quad QDx = \begin{bmatrix} Q_{11}D_1 & Q_{12}D_2 \\ Q_{21}D_1 & Q_{22}D_2 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix} = \begin{bmatrix} Q_{12}D_2\hat{x} \\ Q_{22}D_2\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \hat{x} \end{bmatrix}
\]
so that, in particular, it must be \( Q_{12}D_2\hat{x} = 0 \). We will prove shortly that \( Q_{12} \) has full column rank; hence, it must be \( D_2\hat{x} = 0 \). But by (3.7) we have \( \lambda \hat{x} = Q_{22}D_2\hat{x} = 0 \) and since \( \lambda \neq 0 \) by assumption, it must be \( \hat{x} = 0 \) (a contradiction, since \( x \neq 0 \)).

To conclude the proof we need to show that \( Q_{12} \in \mathbb{R}^{n \times m} \) has full column rank. Recall that \( Q = \mathcal{V}^T U \mathcal{V} \) with
\[
\mathcal{V} = \begin{bmatrix} V_{11} & O \\ O & V_{22} \end{bmatrix},
\]
where \( V_{11} \in \mathbb{R}^n \) is the orthogonal matrix that diagonalizes \( (\alpha I_n - H)(\alpha I_n + H)^{-1} \) and \( V_{22} \in \mathbb{R}^m \) is the orthogonal matrix that diagonalizes \( (\alpha I_m - C)(\alpha I_m + C)^{-1} \). Recall that the orthogonal matrix \( U \) is given by
\[
(\alpha I - S)(\alpha I + S)^{-1} = \begin{bmatrix} \alpha I_n - S & -B^T \\ B & \alpha I_m \end{bmatrix} \begin{bmatrix} \alpha I_n + S & B^T \\ -B & \alpha I_m \end{bmatrix}^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.
\]

An explicit calculation reveals that
\[
U_{12} = -2B^T[\alpha I_m + B(\alpha I_n + S)^{-1}B^T]^{-1}.
\]
The matrix \( \alpha I_m + B(\alpha I_n + S)^{-1}B^T \) is nonsingular: we prove this by showing that \( P := B(\alpha I_n + S)^{-1}B^T \) is positive real and therefore its eigenvalues have positive real part. Indeed, let \((\mu, v)\) be an eigenpair of \( P \), with \( \mu \in \mathbb{C} \) and \( v \in \mathbb{C}^m \), \( \|v\|_2 = 1 \). Then
\[
Pv = B(\alpha I_n + S)^{-1}B^Tv = \mu v.
and therefore
\[ v^* P v = v^* B (\alpha I_n + S)^{-1} B^T v = \mu. \]

Setting \( w = B^T v \) we have
\[ \mu = w^* (\alpha I_n + S)^{-1} w \quad \text{and} \quad \bar{\mu} = w^* (\alpha I_n - S)^{-1} w \]

since \( S^* = S^T = -S \). Thus,
\[ \Re(\mu) = \frac{1}{2} (\mu + \bar{\mu}) = \frac{1}{2} w^* [(\alpha I_n + S)^{-1} + (\alpha I_n - S)^{-1}] w. \]

Now observe that
\[ (\alpha I_n + S)^{-1} + (\alpha I_n - S)^{-1} = (\alpha I_n - S)^{-1} [(\alpha I_n - S) + (\alpha I_n + S)] (\alpha I_n + S)^{-1}, \]

and since\( [(\alpha I_n + S)^{-1}]^T = (\alpha I_n - S)^{-1} \), the matrix \((\alpha I_n + S)^{-1} + (\alpha I_n - S)^{-1}\) is congruent to
\[ (\alpha I_n - S) + (\alpha I_n + S) = 2\alpha I_n \]

and therefore positive definite, since congruence transformations preserve the inertia of a matrix. This shows that \( \Re(\mu) > 0 \) for any eigenvalue \( \mu \) of \( P \); therefore \( \alpha I_n + P \) is nonsingular for all \( \alpha > 0 \).

Furthermore,
\[ Q = V^T U V = \begin{bmatrix} V^T_{11} U_{11} V_{11} & V^T_{12} U_{12} V_{22} \\ V^T_{21} U_{21} V_{11} & V^T_{22} U_{22} V_{22} \end{bmatrix} \]

and therefore
\[ Q_{12} = V^T_{11} U_{12} V_{22} = -2V^T_{11} B^T [\alpha I_n + B (\alpha I_n + S)^{-1} B^T]^{-1} V_{22}, \]

showing that \( Q_{12} \) has full column rank since since \( V^T_{11} \) and \( V_{22} \) are orthogonal and \( B^T \)

has full column rank. This completes the proof. \( \square \)

**Remark 3.1.** There exists a unique invertible matrix \( M \in \mathbb{R}^{n+m} \) such that
\[ T_\alpha = I - M^{-1} A. \]

By Theorem 2.1 in [7], such a representation of \( T_\alpha \) is possible if and only if \( A \) and \( I - T_\alpha \) have the same null space:
\[ \mathcal{N}(A) = \mathcal{N}(I - T_\alpha). \]

Since we know from Theorem 3.1 that \( \rho(T_\alpha) < 1 \) for all \( \alpha > 0 \), we can conclude that \( I - T_\alpha \) is nonsingular, as is \( A \) by Lemma 1.1. Following [7] (proof of Theorem 3.2; see also (3.2) above), the matrix \( M^{-1} \) is given by
\[ M^{-1} = (S + \alpha I)^{-1} [I + (\alpha I - H)(H + \alpha I)^{-1}] \]

and a straightforward manipulation yields
\[ M \equiv M_\alpha = \frac{1}{2\alpha} (H + \alpha I)(S + \alpha I). \]
This expression for the preconditioning matrix will be useful when we consider GMRES acceleration of the alternating iteration (2.4) in the next section.

In some applications the submatrix $A$ in (1.1) is only semipositive real (singular). As already mentioned, system (1.1) (or, equivalently, (1.4)) is nonsingular provided that $B^T$ has full column rank and \( \mathcal{N}(H) \cap \mathcal{N}(B) = \{0\} \), where $H$ denotes the symmetric part of $A$. In this case the alternating iteration (2.4) is still well-defined, but it may happen that $\rho(T_\alpha) = 1$ for all values of $\alpha > 0$. A simple example with $n = 2$, $m = 1$ is given by

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C = [0].
\]

Nevertheless, the following result holds.

**Theorem 3.2.** Suppose the symmetric part $H$ of $A$ and $C = C^T$ are positive semidefinite, $B$ has full rank, and $\mathcal{N}(H) \cap \mathcal{N}(B) = \{0\}$. Then, $\rho(T_\alpha) \leq 1$ for all $\alpha > 0$; moreover, $1$ is not an eigenvalue of $T_\alpha$, and the matrix $(1 - \beta)I + \beta T_\alpha$ has spectral radius less than $1$ for all $\beta \in (0, 1)$ and all $\alpha > 0$.

**Proof.** The fact that the spectral radius of $T_\alpha$ cannot exceed unity has already been established; see (3.5) or (3.6). Also, $1 \notin \sigma(T_\alpha)$ since $A$ is nonsingular. Take now $\beta \in (0, 1)$ and observe that the eigenvalues of $(1 - \beta)I + \beta T_\alpha$ are of the form $1 - \beta + \beta \lambda$, where $\lambda \in \sigma(T_\alpha)$. It is easy to see that since $|\lambda| \leq 1$ and $\lambda \neq 1$, all the quantities $1 - \beta + \beta \lambda$ have magnitude strictly less than $1$.

The previous result implies that for any choice of the initial guess $\hat{x}^0 = x^0$, the sequence $\{\hat{x}^k\}$ defined by

\[
\hat{x}^{k+1} = (1 - \beta)\hat{x}^k + \beta \hat{x}^{k+1} = (1 - \beta)\hat{x}^k + \beta (T_\alpha \hat{x}^k + c)
\]

$(k = 0, 1, \ldots)$ converges to the unique solution of problem (1.4) for all $\beta \in (0, 1)$ and all $\alpha > 0$. We mention that this technique is routinely used in the solution of singular linear systems and especially in the computation of the stationary distribution vector of Markov chains; see, e.g., [39] or [7]. The presence of the parameter $\beta$, unfortunately, adds another complication to the method. Numerical experiments suggest that a value of $\beta$ slightly less than $1$, like $\beta = 0.99$, should be used. When Krylov subspace acceleration is used, however, there is no need to use this technique (that is, one can use $\beta = 1$).

Under the assumptions of Theorem 3.1, the asymptotic rate of convergence of the alternating iteration is governed by the spectral radius of $T_\alpha$, so it makes sense to try to choose $\alpha$ so as to make $\rho(T_\alpha)$ as small as possible. In general, this is a difficult problem. From the analysis in [2] we know that if $\mathcal{H}$ is SPD, then

\[
\rho(T_\alpha) \leq \max_{\lambda \in \sigma(\mathcal{H})} \frac{\alpha - \lambda}{\alpha + \lambda} \leq \max_{\lambda \in [\gamma_{\min}, \gamma_{\max}]} \frac{\alpha - \lambda}{\alpha + \lambda}
\]

where $\gamma_{\min}$ and $\gamma_{\max}$ are the smallest and largest eigenvalues of $\mathcal{H}$, respectively. The upper bound (3.10) is minimized when

\[
\alpha = \alpha_* = \sqrt{\gamma_{\min} \gamma_{\max}}
\]

which yields

\[
\rho(T_{\alpha_*}) = \frac{\sqrt{\gamma_{\max}} - \sqrt{\gamma_{\min}}}{\sqrt{\gamma_{\max}} + \sqrt{\gamma_{\min}}} = \frac{\kappa(\mathcal{H}) - 1}{\kappa(\mathcal{H}) + 1}
\]
where \( \kappa(\mathcal{H}) = \frac{\max_{i,j}}{\min_{i,j}} \) is the condition number of \( \mathcal{H} \). Hence, the better conditioned is \( \mathcal{H} \) (that is, \( H \) and \( C \)), the faster the asymptotic convergence of the alternating method when the “optimal” value of the parameter \( \alpha = \alpha_* \) is used. However, \( \alpha_* \) is not easy to compute or estimate, and it can be quite different from the value of \( \alpha \) that actually minimizes the spectral radius of \( \mathcal{T}_\alpha \). Also, since the iteration matrix is nonnormal, it is not clear that the asymptotic convergence factor is a meaningful measure of the actual number of iterations required to satisfy a prescribed tolerance. Furthermore, in our setting \( \mathcal{H} \) is generally singular, since \( C \) is almost invariably singular (as is \( A \) in some cases). Therefore, the above analysis does not apply. Of course, the parameter \( \alpha \) can be estimated by trial-and-error, an acceptable approach if one has to solve a sequence of linear systems \( Ax = b \) with the same coefficient matrix \( A \) and different right-hand-sides \( b \).

Very recently, an expression for the optimal \( \alpha \) has been obtained in [3] under the assumption that \( A \) is symmetric (more generally, Hermitian) positive definite, \( B \) has full rank, and \( C = O \). This requires knowledge of the smallest and largest singular values of a matrix product involving \( A^{-2} \), \( B \), and a preconditioning matrix \( K \). Explicit formulas expressing the optimal value of \( \alpha \) as a function of the mesh size \( h \) have been given for some model PDE problems in [6]. However, choosing \( \alpha \) so as to minimize the spectral radius of the iteration matrix is not necessarily the best choice when the algorithm is used as a preconditioner for a Krylov subspace method. Remarkably, it can be shown that for certain important problems the alternating iteration results in an \( h \)-independent preconditioner for GMRES when \( \alpha \) is chosen sufficiently small, corresponding to a spectral radius very close to 1; see [6] and the numerical experiments in section 5 below.

In any case, minimizing the spectral radius or even the number of GMRES iterations does not imply optimal performance in terms of CPU time. Indeed, the efficient implementation of the method almost invariably require that the two linear systems (2.5) and (2.6) be solved inexactly. Clearly, the choice of \( \alpha \) will influence the cost of performing the two solves. Indeed, “large” values of \( \alpha \) will make the iterative solution of (2.5) and (2.6) easy; on the other hand, it is clear from (3.6) that the nonzero eigenvalues of \( \mathcal{T}_\alpha \) approach 1 as \( \alpha \to \infty \) (and also as \( \alpha \to 0 \)), and convergence of the alternating iteration slows down. Hence, there is a trade-off involved. If we define the “optimal” value of \( \alpha \) as the one that minimizes the total amount of work needed to compute an approximate solution, this will not necessarily be the same as the \( \alpha \) that minimizes the number of (outer) iterations. Overall, the analytic determination of such an optimal value for \( \alpha \) appears to be daunting.

4. Krylov subspace acceleration. Even with the optimal choice of \( \alpha \), the rate of convergence of the stationary iteration (2.4) can be quite slow (\( \rho(\mathcal{T}_\alpha) \approx 1 \)). In this case we can use GMRES (or any other nonsymmetric Krylov subspace method) to accelerate the convergence of the iteration. In other words, we can use the alternating method as a preconditioner for GMRES or its restarted version, GMRES\((m)\).

As we argued in Remark (3.1), the linear system \( Ax = b \) is equivalent to (i.e., has the same solution as) the linear system

\[
(I - \mathcal{T}_\alpha) x = M_\alpha^{-1} Ax = c,
\]

where \( c = M_\alpha^{-1} b \) is given in (3.2). This equivalent (left-preconditioned) system can be solved with GMRES. Hence, the matrix \( M_\alpha \) can be seen as a preconditioner for GMRES. Equivalently, we can say that GMRES is used to accelerate the convergence of the alternating iteration applied to \( Ax = b \).
Note that as a preconditioner we can use
\[ \mathcal{M}_\alpha = (\mathcal{H} + \alpha I)(S + \alpha I) \]
instead of the expression given in (3.9), since the factor \( \frac{1}{2\alpha} \) has no effect on the preconditioner.

The GMRES method can also be applied to the right-preconditioned system
\[ \mathcal{A}\mathcal{M}_\alpha^{-1}\mathbf{y} = \mathbf{b} \]
where \( \mathbf{y} = \mathcal{M}\mathbf{x} \). Note that \( \mathcal{M}_\alpha^{-1}\mathcal{A} \) and \( \mathcal{A}\mathcal{M}_\alpha^{-1} \) are similar and therefore have the same eigenvalues. However, as is well known, the convergence behavior of GMRES can be different depending on whether left or right preconditioning is being used.

Suppose now that \( \mathcal{A} \) is positive real, \( \mathcal{C} \) is positive semidefinite (possibly \( \mathcal{C} = 0 \)) and \( \mathcal{B} \) has full rank. Since \( \mathcal{M}_\alpha^{-1}\mathcal{A} = \mathcal{I} - \mathcal{T}_\alpha \), it immediately follows from Theorem 3.1 that for all \( \alpha > 0 \) the eigenvalues of the preconditioned matrix \( \mathcal{M}_\alpha^{-1}\mathcal{A} \) (or of \( \mathcal{A}\mathcal{M}_\alpha^{-1} \)) are entirely contained in \( B(1,1) \), the open disk of radius 1 centered at \( (1,0) \). In particular, the preconditioned matrix is positive stable. The smaller the spectral radius of \( \mathcal{T}_\alpha \), the more clustered the eigenvalues of the preconditioned matrix (around 1). If the symmetric part of \( \mathcal{A} \) is only positive semidefinite, then from Theorem 3.2 we have that
\[ \sigma(\mathcal{A}\mathcal{M}_\alpha^{-1}) \subset \overline{B(1,1)} \setminus \{0\}. \]

These results are reassuring, since it is known that GMRES\((m)\) applied to a \( n \times n \) linear system \( \mathcal{A}\mathbf{x} = \mathbf{b} \) may fail to converge for all values of the restart \( m \) less than \( n \) if \( \mathcal{A} \) is indefinite (in the sense that \( \mathcal{A} \) has eigenvalues with both negative and positive real part).

More precisely, assume \( \mathcal{A} \) is diagonalizable, with \( \mathcal{A} = \mathcal{X}\mathcal{D}\mathcal{X}^{-1} \). Then it is well known (Theorem 4 in [52]) that the residual norm provided at the \( k \)th step of GMRES satisfies
\[ \|\mathbf{r}^k\|_2 \leq \kappa(\mathcal{X})\varepsilon^{(k)}\|\mathbf{r}^0\|_2, \tag{4.2} \]
where
\[ \varepsilon^{(k)} := \min_{p \in P_k} \max_{\lambda_i \in \sigma(\mathcal{A})} |p(\lambda_i)|. \]

Here, \( P_k \) denotes the set of all polynomials \( p(\lambda) \) of degree not greater than \( k \) such that \( p(0) = 1 \). If all the eigenvalues \( \lambda_i \) of \( \mathcal{A} \) are contained in a circle centered at the point \( (1,0) \) and having radius \( \rho \), then a special case of Theorem 5 in [52] implies that \( \varepsilon^{(k)} \leq \rho^k \). For a normal matrix \( \kappa(\mathcal{X}) = 1 \), and therefore the bound (4.2) shows that a clustered spectrum (around 1) will lead to rapid convergence of GMRES. The same can be expected for a “nearly normal” matrix (i.e., one for which \( \kappa(\mathcal{X}) \) is not too large). These results assume that the full GMRES algorithm is being used. Another well-known result, due to Elman (see [19] and [52], p. 866), states that when \( \mathcal{A} \) is positive real with symmetric part \( \mathcal{H} \), the following error bound holds:
\[ \|\mathbf{r}^k\|_2 \leq (1 - \gamma/\delta)^{k+1}\|\mathbf{r}_0\|_2, \tag{4.3} \]
where \( \gamma = (\lambda_{\min}(\mathcal{H}))^2 \), \( \delta = \lambda_{\max}(\mathcal{A}^T\mathcal{A}) \). This proves the convergence of restarted GMRES, GMRES\((m)\), for all \( m \) when \( \mathcal{A} \) is positive real.
If a matrix is positive real, then it is positive stable; the converse, however, is not true. A counterexample is given by a matrix of the form

\[
A = \begin{bmatrix}
1 & 0 \\
a & 1
\end{bmatrix}
\]

where \(a\) is any real number with \(|a| \geq 2\). We already know from Theorem 3.1 and Theorem 3.2 that \(M_\alpha^{-1} A\) (and therefore \(AM_\alpha^{-1}\)) is positive stable for all \(\alpha\). The question then arises whether \(M_\alpha^{-1} A\) (or \(AM_\alpha^{-1}\)) is positive real, for in this case the convergence of GMRES\((m)\) would be guaranteed for all restarts \(m\). Unfortunately, numerical experiments show that this is not true in general. However, when \(A\) is SPD and \(C = O\) we can prove that the preconditioned matrix is positive real provided that \(\alpha\) is sufficiently large.

**Theorem 4.1.** Assume \(A\) is SPD, \(C = O\) and \(B\) has full rank. Then there exists \(\alpha^* > 0\) such that \(M_\alpha^{-1} A\) is positive real for all \(\alpha > \alpha^*\). An analogous result holds for the right-preconditioned matrix, \(AM_\alpha^{-1}\).

*Proof.* For brevity, we prove the theorem only for the left-preconditioned matrix; the proof for the right-preconditioned one is similar. Up to a positive scalar multiple, the symmetric part of the preconditioned matrix \(M_\alpha^{-1} A\) is given by

\[
B = (S + \alpha I)^{-1} (H + \alpha I)^{-1} A + A^T (H + \alpha I)^{-1} (\alpha I - S)^{-1}
\]

(where we have used the fact that \(S^T = -S\)). This matrix is congruent to

\[
(S + \alpha I)B(S + \alpha I)^T = (H + \alpha I)^{-1} A(\alpha I - S) + (S + \alpha I)A^T (H + \alpha I)^{-1}
\]

which, in turn, is congruent to the inverse-free matrix

\[
Z = A(\alpha I - S)(H + \alpha I) + (H + \alpha I)(S + \alpha I)A^T.
\]

A direct calculation shows that

\[
Z = \begin{bmatrix}
Z_\alpha & -2\alpha AB^T \\
-2\alpha BA & 2\alpha BB^T
\end{bmatrix}
\]

where

\[
Z_\alpha := 2\alpha A^2 + 2\alpha B^T B + 2\alpha^2 A + B^T BA + AB^T B.
\]

We want to show that this matrix is SPD for sufficiently large \(\alpha\). To this end, we observe that \(Z\) can be split as

\[
Z = 2 \begin{bmatrix}
\alpha A^2 & -\alpha AB^T \\
-\alpha BA & \alpha BB^T
\end{bmatrix} + \begin{bmatrix}
M_\alpha & O \\
O & O
\end{bmatrix}
\]

where

\[
M_\alpha := 2\alpha^2 A + 2\alpha B^T B + B^T BA + AB^T B.
\]

The first matrix on the right-hand side of (4.4) is symmetric positive semidefinite, since

\[
\begin{bmatrix}
\alpha A^2 & -\alpha AB^T \\
-\alpha BA & \alpha BB^T
\end{bmatrix} = \begin{bmatrix}
\alpha A & O \\
-\alpha B & I_m
\end{bmatrix} \begin{bmatrix}
\alpha^{-1} I_m & O \\
O & O
\end{bmatrix} \begin{bmatrix}
\alpha A & -\alpha B^T \\
O & I_m
\end{bmatrix}.
\]
Next, we observe that
\[ M_\alpha = \alpha(2B^TB + 2\alpha A) + (B^TBA + AB^TB) \]
is similar to a matrix of the form \( \alpha I_n + W \), where \( W = W^T \) is generally indefinite. This matrix can be made SPD by taking \( \alpha \) sufficiently large. Specifically, \( M_\alpha \) is SPD for all \( \alpha > \alpha^* \), where
\begin{equation}
\alpha^* = -\lambda_{\text{min}}(B^TBA + AB^TB)
\end{equation}
(note that \( B^TBA + AB^TB \) is generally indefinite). Hence, for \( \alpha > \alpha^* \) the matrix \( Z \) is the sum of two symmetric positive semidefinite matrices; therefore, it is itself symmetric positive semidefinite. Finally, it must be nonsingular for all \( \alpha > \alpha^* \) (and therefore positive definite). Indeed, it is clear from (4.4) that when \( M_\alpha \) is positive definite, any null vector of \( Z \) must be of the form
\[ x = \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix} \quad \text{where} \quad \hat{x} \in \mathbb{R}^m. \]
But then
\[ Zx = \begin{bmatrix} \alpha A^2 & -\alpha AB^T \\ -\alpha BA & \alpha BB^T \end{bmatrix} \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix} = \begin{bmatrix} -2\alpha AB^T\hat{x} \\ 2\alpha BB^T\hat{x} \end{bmatrix} \]
which cannot be zero unless \( \hat{x} = 0 \), since \( B^T \) has full column rank and \( A \) is nonsingular. Hence \( Z \) has no nontrivial null vectors for \( \alpha > \alpha^* \). This shows that the symmetric part of the preconditioned matrix is SPD for all \( \alpha > \alpha^* \), since it is congruent to a matrix which is SPD for all such values of \( \alpha \).

It is worth mentioning that in all cases that we were able to check numerically, we found the symmetric part of the preconditioned operator to be positive definite already for rather small values of \( \alpha \).

We conclude this section by noting that use of the alternating preconditioner within GMRES requires solving a linear system of the form
\[ (\mathcal{H} + \alpha I)(S + \alpha I)z = r \]
at each iteration. This is done by first solving
\begin{equation}
(\mathcal{H} + \alpha I)v = r
\end{equation}
for \( v \), followed by
\begin{equation}
(S + \alpha I)z = v.
\end{equation}
The solution of these two systems has already been discussed in section 2. In the next section we experiment with both exact and inexact solves.

5. Numerical experiments. In this section we present a sample of numerical experiments conducted in order to assess the effectiveness of the alternating algorithm (2.4) both as a stationary iterative scheme and as a preconditioner for GMRES. All experiments were performed in Matlab. Our codes have not been optimized for highest efficiency and therefore we do not report timings or flop counts. For the same reason, we abstain from making comparisons with other preconditioners for saddle
point problems. Nevertheless, we think that the results of the experiments presented here provide evidence of the fact that our approach is worth further consideration. See also the experiments reported in [3].

We target matrices from different applications areas, but mostly from PDE problems. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e., when \(|b - Ax^k|_2 \leq 10^{-9}|b|_2\)).

5.1. Second order equations in first order system form. Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a bounded open set. Here we consider the numerical solution of boundary value problems for the following second order elliptic PDE:

\[
- \nabla \cdot (K \nabla p) = g \quad \text{in} \quad \Omega
\]

where \( K = K(r) \) is a strictly positive function or tensor for \( r \in \Omega \) and \( g(r) \) is a given forcing term. The equation (5.1) is complemented by appropriate boundary conditions.

The PDE (5.1) is equivalent to the following system of two first order PDEs:

\[
\begin{aligned}
K^{-1} u - \nabla p &= 0 \\
- \nabla \cdot u &= g.
\end{aligned}
\]

Discretization of these equations leads to large sparse linear systems in saddle point form (1.4).

We begin with the simplest possible case, namely, Poisson’s equation on the unit square:

\[
- \Delta p = - \nabla \cdot (\nabla p) = g \quad \text{in} \quad \Omega = [0, 1] \times [0, 1].
\]

This corresponds to taking \( K \equiv 1 \) in (5.1). We discretize form (5.2) of the problem using finite differences with a forward difference for the gradient and a backward difference for the divergence. Using an \( N \times N \) uniform grid with mesh size \( h = \frac{1}{N+1} \) results in a linear system of type (1.4) with \( n = 2N^2 \) and \( m = N^2 \), for a total system size of \( 3N^2 \) equations in as many unknowns.

As shown in [6], for this model problem a Fourier analysis at the continuous (differential operator) level can be applied to completely analyze the spectrum of the iteration operator \( T_\alpha \). This allows us to find the optimal value \( \alpha^* \) of the parameter as a function of \( h \), showing that the spectral radius for the stationary iteration (2.4) behaves as \( 1 - c\sqrt{h} \) as \( h \to 0 \). The optimal value \( \alpha^* \) itself behaves as \( h^{-\frac{1}{2}} \) as \( h \to 0 \). More interestingly, the spectral analysis in [6] indicates that when GMRES acceleration is used, a better choice is to use a small value of \( \alpha \), for it can be shown that as \( \alpha \to 0 \) the spectrum of the preconditioned matrix clusters around 2-3 values, resulting in convergence within 2-3 iterations (independent of \( h \)).

This behavior is illustrated in Table 5.1. We take the forcing term to be the function \( g(x, y) = \sin \pi x \sin \pi y \) and we impose Neumann boundary conditions for \( x = 0, x = 1 \) and homogeneous Dirichlet boundary conditions for \( y = 0, y = 1 \). The numerical results are in agreement with the theoretical analysis. In particular, note that convergence is attained in two steps (independent of \( h \)) when the iteration is optimized for GMRES acceleration. Here we used \( \alpha = 0.001 \), but the behavior of the preconditioned iteration is not very sensitive to the choice of \( \alpha \), at least for \( \alpha \) small.

In Figure 5.1 we display the eigenvalues of the preconditioned matrix \( M_\alpha^{-1} A \) in the case of \( h = \frac{1}{10} \) for two values of \( \alpha \). On the left we used the value \( \alpha = \alpha^* \) that
minimizes the spectral radius, which is given by \( \rho(T_{\alpha}) = 0.8062 \). On the right we used \( \alpha = 0.01 \), showing the clustering near 0 and 2 predicted by the theory developed in [6]. Now the spectral radius of the iteration matrix is very close to 1. The cluster near 0 contains \( m = 81 \) eigenvalues, the one near 2 the remaining \( n = 162 \). Notice the scaling of the vertical axis in Figure 5.1(b).

Next we consider a somewhat harder problem, namely, the anisotropic equation

\[-100 p_{xx} - p_{yy} = g \quad \text{in} \quad \Omega = [0,1] \times [0,1].\]

Since this problem has constant coefficients, the technique used in [6] for Poisson’s equation can be used to optimize the method. The results in Table 5.2 show that the anisotropy in the coefficients drastically decreases the rate of convergence. However, in this case there is an easy fix: as the results reported in Table 5.3 show, it is enough to apply the scaling (2.9) to restore the effectiveness of the solver. We note that a similar scaling has been used in [13] in a somewhat different context.

Although \( h \)-independent convergence is a desirable property, in practice we would like to minimize the total amount of work to obtain the solution rather than to minimize the number of iterations. In these simple model problems the linear system (4.5) is diagonal and trivial to solve, but (4.6) is nearly as difficult to solve as the original problem. Indeed, if Schur complement reduction is applied, we see that this amounts to solving a problem of the form (2.8), which can be seen as a discrete

<table>
<thead>
<tr>
<th>( h )</th>
<th>Iterative</th>
<th>No Prec.</th>
<th>Preconditioned</th>
<th>Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>66</td>
<td>54</td>
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<td>103</td>
<td>140</td>
<td>19</td>
<td>2</td>
</tr>
<tr>
<td>1/50</td>
<td>146</td>
<td>286</td>
<td>25</td>
<td>2</td>
</tr>
<tr>
<td>1/100</td>
<td>207</td>
<td>574</td>
<td>34</td>
<td>2</td>
</tr>
</tbody>
</table>
version of the shifted problem \((-\nabla \cdot KV + \alpha^2) p = f\). For large \(\alpha\) this problem is much easier to solve than (5.1); however, for small values of \(\alpha\) (the ones that give \(h\)-independent convergence rates with GMRES acceleration) this is just as hard to solve as the original equation. Hence, in practice it may be necessary to strike a compromise between rate of convergence of the (outer) iteration and computational effort needed at each step. If an incomplete factorization (or an inner iteration) is used to perform inexact inner solves, then a value of \(\alpha\) that is not too small should be preferred.

Some results for the two-dimensional anisotropic problem are presented in Table 5.3 (under the heading “Inexact”). The solve (4.6) is approximated by a no-fill incomplete LU factorization ILU(0) of the matrix \(S + \alpha I\). This operation is extremely cheap compared to the exact solve. We use a fixed value \(\alpha = 1\) of the parameter. In spite of the simplicity of the approximation used, we can see that the inexact preconditioned GMRES iteration converges much faster than either the stationary iteration with exact solves, or the GMRES iteration with just diagonal preconditioning. Better results can be expected by replacing ILU(0) with a more accurate incomplete factorization. On the other hand, using a no-fill incomplete factorization to approximately solve (4.6) for very small values of \(\alpha\) (like \(\alpha = 0.001\)) leads to poor results, since in this case \(S + \alpha I\) is far from being diagonally dominant.

Finally, we consider a more difficult problem with large jumps in the coefficients \(K\). The system is discretized using a discontinuous Galerkin finite element scheme. This radiation diffusion problem arises in a nuclear engineering application and was supplied to us by James Warsa of Los Alamos National Laboratory. For more details, see [59] and the references therein. For this problem \(n = 2592, m = 864, n + m = 3456\) and \(A\) contains 93,612 nonzero entries. Here \(C \neq 0\) (and indeed it is SPD).

The results for this problem are presented in Table 5.4, where the entries in the first row correspond to GMRES with diagonal preconditioning (2.9). We give results for full GMRES and for restarted GMRES with restart every 20 steps. Here we cannot apply Fourier analysis to optimize the choice of \(\alpha\) as we did in the constant coefficient cases. Therefore, we experiment with different values of \(\alpha\). While the
fastest convergence rate for the stationary iterative methods correspond to $\alpha = 0.25$, a somewhat bigger $\alpha$ works best if the method is used as a preconditioner for GMRES. In any case the method is not overly sensitive to the choice of $\alpha$ when GMRES acceleration is used.

We stress here again the importance of the diagonal scaling (2.9), which results in a reduction by a factor of two in the number of iterations for this problem.

### 5.2. Stokes and Oseen problems

In this section we present a few results for discretizations of Stokes and Oseen problems. Recall that the Stokes system is

\[
\begin{align*}
-\Delta u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{align*}
\]

in $\Omega \subset \mathbb{R}^d$, together with suitable boundary conditions. Here $u$ denotes the velocity vector field and $p$ the pressure scalar field. Discretization of (5.4) using stabilized finite elements leads to saddle point problems of the type (1.4) with a symmetric positive definite $A$ and a symmetric positive semidefinite $C$.

The Oseen equations are obtained when the steady-state Navier–Stokes equations are linearized by Picard iteration:

\[
\begin{align*}
-\nu \Delta u + (\nabla \cdot \mathbf{v}) u + \nabla p &= f \\
\n\nabla \cdot u &= 0.
\end{align*}
\]

Here the vector field $\mathbf{v}$ is the approximation of $u$ from the previous Picard iteration. The parameter $\nu > 0$ represents viscosity. Various approximation schemes can be used to discretize the Oseen system (5.5) leading to a generalized saddle point system of type (1.4). Now the $A$ block corresponds to a discretization of the convection-diffusion operator $L[u] := -\nu \Delta u + (\nabla \cdot \mathbf{v}) u$. It is nonsymmetric but the symmetric part is positive (semi-)definite.

We generated several test problems using the IFISS software package, kindly provided by Howard Elman and David Silvester. We used this package to generate discretizations of leaky lid driven cavity problems for both the Stokes and Oseen equations. The discretization used is stabilized Q1-P0 finite elements. In all cases the default value of the stabilization parameter ($\beta = 0.25$) was used. It should be mentioned that the matrices generated by this package are actually singular, since $B$
Table 5.5  
Results for Stokes problem.

<table>
<thead>
<tr>
<th>α</th>
<th>Iterative</th>
<th>GMRES</th>
<th>GMRES(20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td></td>
<td>103</td>
<td>194</td>
</tr>
<tr>
<td>0.01</td>
<td>&gt; 1000</td>
<td>101</td>
<td>205</td>
</tr>
<tr>
<td>0.1</td>
<td>801</td>
<td>53</td>
<td>60</td>
</tr>
<tr>
<td>0.2</td>
<td>135</td>
<td>33</td>
<td>36</td>
</tr>
<tr>
<td>0.3</td>
<td>78</td>
<td>29</td>
<td>30</td>
</tr>
<tr>
<td>0.4</td>
<td>107</td>
<td>29</td>
<td>34</td>
</tr>
<tr>
<td>0.5</td>
<td>134</td>
<td>30</td>
<td>39</td>
</tr>
<tr>
<td>0.6</td>
<td>137</td>
<td>32</td>
<td>47</td>
</tr>
<tr>
<td>0.7</td>
<td>165</td>
<td>35</td>
<td>51</td>
</tr>
<tr>
<td>0.8</td>
<td>222</td>
<td>37</td>
<td>58</td>
</tr>
<tr>
<td>0.9</td>
<td>250</td>
<td>39</td>
<td>63</td>
</tr>
<tr>
<td>1.0</td>
<td>277</td>
<td>42</td>
<td>67</td>
</tr>
</tbody>
</table>

Table 5.6  
Results for Stokes problem with inexact solves.

<table>
<thead>
<tr>
<th>α</th>
<th>GMRES</th>
<th>GMRES(20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>210</td>
<td>699</td>
</tr>
<tr>
<td>0.1</td>
<td>58</td>
<td>62</td>
</tr>
<tr>
<td>0.2</td>
<td>35</td>
<td>36</td>
</tr>
<tr>
<td>0.3</td>
<td>32</td>
<td>30</td>
</tr>
<tr>
<td>0.4</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>0.5</td>
<td>37</td>
<td>41</td>
</tr>
<tr>
<td>0.6</td>
<td>40</td>
<td>48</td>
</tr>
<tr>
<td>0.7</td>
<td>42</td>
<td>54</td>
</tr>
<tr>
<td>0.8</td>
<td>44</td>
<td>59</td>
</tr>
<tr>
<td>0.9</td>
<td>45</td>
<td>64</td>
</tr>
<tr>
<td>1.0</td>
<td>46</td>
<td>68</td>
</tr>
</tbody>
</table>

has rank \( m - 1 \). This does not cause any difficulty to the iterative solvers considered here. In particular, even if \( \lambda = 1 \) is now an eigenvalue of the iteration matrix \( T_\alpha = I - M_\alpha^{-1} A \), the stationary iteration is still convergent, with a rate of convergence governed by \( \gamma(T_\alpha) := \max \{|\lambda|; \lambda \in \sigma(T_\alpha), \lambda \neq 1\} \).

For the Stokes problem we used a 16 \times 16 grid. For the Oseen problem we used two grids, 16 \times 16 and 32 \times 32. The first grid corresponds to \( n = 578 \) and \( m = 256 \), for a total of 834 unknowns. For the second grid \( n = 2178 \) and \( m = 1024 \), for a total of 3202 unknowns. Two values of the viscosity parameter were used for the Oseen problems, \( \nu = 0.01 \) and \( \nu = 0.001 \). We experiment with both full GMRES and GMRES(20). Diagonal scaling (2.9) greatly improves the rate of convergence in all cases, and it is used throughout.

Tables 5.5-5.6 are for the Stokes problem. In Table 5.5 we show results obtained using exact inner solves. Although there is no value of \( \alpha \) that yields convergence in two steps, the alternating iteration is able to significantly improve the convergence of GMRES. Note that the behavior of the preconditioned iteration is not overly sensitive to the choice of \( \alpha \); in contrast, the rate of convergence of the stationary iteration without GMRES acceleration depends strongly on \( \alpha \). In Table 5.6 we replaced the
exact solves in (2.4) with incomplete factorizations. We used drop tolerance-based incomplete Cholesky for the first system in (2.4) and ILU for the second one. In both cases the drop tolerance was set to $tol = 0.05$. The incomplete factors are much sparser than the complete ones; yet, as the results in Table 5.6 show, the rate of convergence suffers almost no deterioration.

Tables 5.7-5.8 contain the experimental results for the Oseen problem on the small ($16 \times 16$) grid with viscosity $\nu = 0.01$. The results for GMRES with diagonal scaling (2.9), reported in the first row, indicate that the Oseen problem is harder than the Stokes problem. Here we see a surprising result: while the stationary iteration converges more slowly than for the Stokes problem, the preconditioned GMRES iteration is now faster. We think this could be due to the fact that the coefficient matrix has a more substantial skew-symmetric part in this case, and preconditioning with the (shifted) skew-symmetric part becomes more effective. The results obtained with exact solves (by incomplete factorization), reported in Table 5.8, show a slight deterioration in convergence rates. This deterioration is more than compensated by
the reduced cost of each preconditioned iteration.

In Tables 5.9-5.10 we report results for the Oseen problem on the $16 \times 16$ grid and a viscosity parameter $\nu = 0.001$. Generally speaking, the Oseen problem becomes harder to solve as the viscosity gets smaller; see the results for diagonally scaled GMRES, and for the stationary iteration. However, the combination of the iteration and GMRES acceleration results in even faster convergence than in the previous case of $\nu = 0.01$. In Figure 5.2 we display the eigenvalues of the preconditioned matrix corresponding to the Oseen problem on the $16 \times 16$ grid. The plot on the left corresponds to a viscosity $\nu = 0.01$ and the one on the right to $\nu = 0.001$; we used the values of $\alpha$ that resulted in the smallest number of preconditioned GMRES iterations ($\alpha = 0.6$ and $\alpha = 0.8$, respectively). Note the stronger clustering of the spectrum for the case with $\nu = 0.001$.

Unfortunately, this apparent robustness with respect to $\nu$ is lost as soon as the exact solves in (2.4) are replaced by inexact solves by incomplete factorization, especially with restarted GMRES; see Table 5.10. The same value of the drop tolerance $col = 0.05$ was used in all cases. Whether it is possible to solve the inner problems inexactly and still preserve robustness with respect to $\nu$ remains an open question.

Finally, in Table 5.11 we present results for the Oseen problem with $\nu = 0.001$ on
the finer grid. The preconditioned GMRES iteration appears to be fairly robust with respect to the mesh size $h$ and the viscosity parameter $\nu$ when exact solves are used.

5.3. A problem with singular $A$. Here we consider a saddle point problem arising in geophysics and supplied to us by Eldad Haber of Emory University; see [28, 33, 34]. In this application the submatrix $A$ turns out to be singular. In the example at hand $n = 1241$, $m = 729$, $n + m = 1970$ and $A$ contains 25,243 nonzeros. The $A$ block is singular, with rank($A$) = 876. In this problem, $C = O$.

We present results for this problem in Tables 5.12-5.13. Diagonal scaling (2.9) drastically improves the convergence of the preconditioned iterations. However, the convergence of the stationary iteration (2.4) without GMRES acceleration remains extremely slow. Likewise for GMRES with no preconditioning or diagonal preconditioning alone. In the first table we report results with exact solves, and in the second table results where the linear solves in (2.4) are replaced by no-fill incomplete factorizations. Again we see a deterioration in convergence rates, but each iteration is now
far cheaper than in the case of exact solves, resulting in reduced solution costs overall.

6. Conclusions. In this paper we have studied the extension of the alternating method of [2] to generalized saddle point problems. Because these linear systems have coefficient matrices with singular symmetric part, they are not positive real. Thus, the convergence analysis carried out in [2] for the positive real case does not apply, and convergence has to be established using different arguments from those used in [2]. Other approaches to studying convergence have been proposed recently in [3] and [6]; see also [8].

We have also investigated the idea of using the alternating iteration as a preconditioner for a nonsymmetric Krylov subspace method, such as GMRES. In some important special cases, we have established properties of the preconditioned matrices that were relevant for restarted GMRES, at least from a qualitative point of view.

Our numerical experiments suggest that the (stationary) alternating iteration may converge too slowly to be of practical importance, especially in the absence of any indication for the choice of the convergence parameter \( \alpha \). However, when the method is used as a preconditioner for GMRES, rapid convergence was observed in many of our test runs, and the combination of GMRES and the alternating iteration appears to be quite robust. In addition, when used as a preconditioner, the method does not appear to be overly sensitive to the choice of the convergence parameter \( \alpha \). As demonstrated already in [6] for some model problems, there are important examples of systems of PDEs where the combination of iteration (2.4) with an appropriate choice of the optimization parameter \( \alpha \) and GMRES acceleration results in an \( h \)-independent solver, or with a weak dependence on \( h \).

Our numerical experiments show that diagonal scaling (2.9) greatly improves the performance of the solver, at virtually no extra cost. We have also performed some experiments with inexact solves. Although the rate of convergence deteriorates somewhat, the cost of each iteration is now greatly reduced, leading to a reduction in overall costs in many cases.
In conclusion, the alternating method first proposed in [2] has many interesting mathematical properties, has a wider range of applicability than initially thought, and provides a framework for constructing powerful preconditioners for Krylov subspace methods.

Future work should focus on developing efficient implementations of the algorithm, with particular attention to the problem of striking a balance between the rate of convergence of the outer (preconditioned) iteration, and the amount of work spent performing the inner (inexact) solves. Here we have presented a few results using incomplete factorizations, but iterative methods may be a better (more flexible) option. For the Oseen equations with small viscosity parameter \( \nu \), it remains an open problem to find inexact inner solves that do not lead to a serious deterioration of the rate of convergence of the outer iteration. While the shifted symmetric part \( (4,5) \) usually poses no great challenge in PDE problems, the solution of the shifted skew-symmetric part \( (4,6) \) appears to be more problematic and warrants further research.

REFERENCES


