Supplementary Information

1. Analytic Results

We prove a result about the existence of graphs for which the global balance tends to zero as the number of nodes tends to infinite. Here, \( C_n \) and \( K_n \) stand for the cycle and complete graphs, respectively. The cycle is the graph in which all the nodes are connected to two other nodes. The complete graph is the graph in which every pair of nodes is connected by an edge.

**Theorem 1.** Let \( G_n \) be the graph whose adjacency matrix is given by:

\[
A(G_n) = 2A(C_n) - A(K_n).
\]

Then, \( K(G_n) \to 0 \) as \( n \to \infty \).

**Proof.** First we start by proving that the matrices \( A(C_n) \) and \( A(K_n) \) commute. Let \( E = \mathbf{1} \cdot \mathbf{1}^T \) where \( \mathbf{1} \) is an all-ones vector. Obviously, \( A(K_n) = E - I \), where \( I \) is the corresponding identity matrix. Then, \( A(K_n) \cdot A(C_n) = E \cdot A(C_n) - A(C_n) \) and \( A(C_n) \cdot A(K_n) = A(C_n) \cdot (E - A(C_n)) \). The two matrices commute if \( A(K_n) \cdot A(C_n) = A(C_n) \cdot A(K_n) \), which implies that \( E \cdot A(C_n) = A(C_n) \cdot E \). It can be easily checked that

\[
E \cdot A(C_n) = \begin{bmatrix} k_1 & 1 & & \cdots & 1 \\ k_2 & 1 & & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ k_n & & & & 1 \end{bmatrix},
\]

and

\[
A(C_n) \cdot E = \begin{bmatrix} k_1 \mathbf{1}^T \\ k_2 \mathbf{1}^T \\ \vdots \\ k_n \mathbf{1}^T \end{bmatrix},
\]

where \( k_i \) is the degree of the node \( i \). Then, if the graph is regular, \( k_1 = k_2 = \cdots = k_n = r \) and

\[
A(C_n) \cdot E = E \cdot A(C_n) = r[1 \quad \cdots \quad 1],
\]

which proves that the adjacency matrix of a complete graph
and that of any regular graph commute. Because the cycle is a regular graph, the first part of the proof is complete.

Because of the commutativity between the adjacency matrices of the cycle and complete graph we can start by writing the ratio

$$\frac{Z(G_n)}{Z([G_n])} = \frac{\text{tr}[\exp(2A(C_n)) \cdot \exp(-A(K_n))]}{\text{tr}[\exp(2A(C_n)) \cdot \exp(A(K_n))]}. \quad (S4)$$

Using the eigenvalues and eigenvectors of the adjacency matrices of cycles and complete graphs we have

$$\langle \exp(A(C_n)) \rangle_{pp} = \frac{1}{n} \sum_{j=0}^{n/2} e^{2\cos\left(\frac{j\pi}{n}\right)}, \quad (S5)$$

$$\langle \exp(A(C_n)) \rangle_{pq} = \frac{1}{n} \sum_{j=0}^{n/2} e^{2\cos\left(\frac{j\pi}{n}\right)} \cos\left(\frac{2\pi(p-q)}{n}\right), \quad (S6)$$

$$\langle \exp(A(K_n)) \rangle_{pp} = \frac{e^{n-1} + n - 1}{ne}, \quad (S7)$$

$$\langle \exp(A(K_n)) \rangle_{pq} = \frac{e^{n-1} - 1}{ne}, \quad (S8)$$

$$\langle \exp(-A(K_n)) \rangle_{pp} = \frac{1}{ne^{n-1}} + \frac{(n-1)e}{n}, \quad (S9)$$

$$\langle \exp(-A(K_n)) \rangle_{pq} = \frac{1}{ne^{n-1}} - \frac{e}{n}. \quad (S10)$$

For $j = 1,2,\cdots,n$ the angles $j\pi/(n+1)$ uniformly cover the interval $[0,\pi]$, thus enabling the usage of the following integral approximation:

$$\langle \exp(A(C_n)) \rangle_{pp} \approx \frac{1}{\pi} \int_{0}^{\pi} e^{2\cos \theta} d\theta = I_0(2), \quad (S11)$$
\[
\left(\exp(A(C_n))\right)_{pq} \approx \frac{1}{\pi} \int_{0}^{\pi} e^{2 \cos \theta} \cos(\theta(p-q)) = I_{d(p,q)}(2),
\]

(S12)

where \( I_{\alpha}(x) \) is the Bessel function of the first kind and \( \alpha = d(p,q) \) is the shortest path distance between the nodes \( p \) and \( q \) in the network.

Then, using the fact that

\[
\sum_{j=1}^{\infty} I_j(2) = \frac{1}{2}(e^2 - I_0(2)),
\]

(S13)

we finally obtain

\[
\lim_{n \to \infty} K(G_n) = \lim_{n \to \infty} -\frac{\left[ \frac{1}{ne^{n-1}} + \frac{(n-1)e}{n} \right] (nI_0(2)) + \left( \frac{1}{ne^{n-1}} - \frac{e}{n} \right) \sum_{j=1}^{\infty} I_j(2)}{\left[ \frac{e^{n-1}}{n} + \frac{(n-1)e}{ne} \right] (nI_0(2)) + \left( \frac{e^{n-1}-1}{ne} \right) \sum_{j=1}^{\infty} I_j(2)}
\]

(S14)

\[
= \lim_{n \to \infty} -\frac{\left[ \frac{1}{e^{n-1}} + \frac{(n-1)e}{e} \right] (I_0(2)) + \left( \frac{1}{ne^{n-1}} - \frac{e}{n} \right) \left( \frac{e^2 - I_0(2)}{2} \right)}{\left[ \frac{e^{n-1}}{e} + \frac{(n-1)e}{e} \right] (I_0(2)) + \left( \frac{e^{n-1}-1}{ne} \right) \left( \frac{e^2 - I_0(2)}{2} \right)} = 0.
\]
2. Numerical Results

2.1 All-negative undirected sub-networks

Table S1. Balance indices in the all-negative sub-networks of the undirected versions of the three online social networks studied.

<table>
<thead>
<tr>
<th>Network</th>
<th>$K$</th>
<th>$K$ (rnd)</th>
<th>$K_3$</th>
<th>$K_3$ (rnd)</th>
<th>$\bar{C}^a$</th>
<th>$\bar{C}^a$ (rnd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Epinions</td>
<td>$4.10 \cdot 10^{-11}$</td>
<td>$\sim 10^{-4}$</td>
<td>0.652</td>
<td>0.681</td>
<td>0.012</td>
<td>0.022</td>
</tr>
<tr>
<td>Slashdot</td>
<td>$1.38 \cdot 10^{-6}$</td>
<td>0.025</td>
<td>0.758</td>
<td>0.851</td>
<td>0.005</td>
<td>0.010</td>
</tr>
<tr>
<td>WikiElections</td>
<td>$3.95 \cdot 10^{-5}$</td>
<td>$\sim 10^{-6}$</td>
<td>0.569</td>
<td>0.890</td>
<td>0.028</td>
<td>0.031</td>
</tr>
</tbody>
</table>

$^a$Average Watts-Strogatz clustering coefficient reported by Leskovec et al.\textsuperscript{9}

2.2 Fit of van’t Hoff Equations for the Online Social Networks
Fig. S1. Nonlinear change of the balance ($\ln K$) with the weight of the links (inverse temperature, $\beta$) in the online social network Epinions. The circles represent the values from the simulation and the solid line represents the fit using the equation 

$$\ln K = -\frac{\Delta H^o}{R} \beta + a \ln \beta^{-1} + b \beta^{-1} + c \beta^{-2} + \cdots + e \beta^{-5} + C,$$

where $\frac{\Delta H^o}{R} = -0.4716$, $a = -0.279$, $b = -0.02308$, $c = 0.002422$, $d = -4.625 \cdot 10^{-5}$, $e = 2.535 \cdot 10^{-7}$, and $C = 0.2223$. The squared correlation coefficient and root of the mean standard error are, respectively: $R^2 = 0.9922$ and $RMSE = 0.008625$. 
Fig. S2. Linear change of the balance ($\ln K$) with the weight of the links ($\beta$) in the online social networks WikiElections and Slashdot. The circles and squares represent the values from the simulation and the solid lines represent the fits using the $\ln K = a\beta + b$, where $a = -\frac{\Delta H}{R}$ and with the parameters given in Table S2.
Table S2. Fitting parameters for the van’t Hoff plots of the online social networks of Slashdot and Wikielections.

<table>
<thead>
<tr>
<th>Network</th>
<th>a</th>
<th>b</th>
<th>$R^2$</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slashdot</td>
<td>-2.676</td>
<td>0.001023</td>
<td>1.0000</td>
<td>0.001611</td>
</tr>
<tr>
<td>WikiElections</td>
<td>-10.8</td>
<td>0.1161</td>
<td>0.9999</td>
<td>0.06957</td>
</tr>
</tbody>
</table>

$R^2$ is the squared correlation coefficient and RMSE is the root of the mean standard error.