Consider the function
\[ f(x) = \frac{1 - \cos x}{x}. \]

(a) Explain why evaluating \( f(x) \) may give inaccurate results when \( x \approx 0 \).

(b) Rewrite \( f(x) \) so as to avoid the problem in (a). (Hint: Multiply and divide \( f(x) \) by the same function \( g(x) \), for an appropriate choice of \( g(x) \).)

Solution:

(a) When \( x \) is close to zero, the value of \( \cos x \) is very close to 1. Subtracting two numbers that have very nearly the same value may lead to catastrophic cancellation on the computer, leading to results without any correct significant digits.

(b) For example,
\[ \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} = \frac{1 - \cos^2 x}{x(1 + \cos x)} = \frac{\sin^2 x}{x(1 + \cos x)}, \]
and this expression does not contain any “dangerous” operations.