Math 315, Quiz 2

NOTE: You must show all details of your work to receive credit.

1. Consider the matrix

\[
A = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

Find the LU factorization of \( A \), where \( L \) is unit lower triangular and \( U \) is upper triangular. Is pivoting necessary for this problem? Do all computations in exact arithmetic.

Solution:

Clearly, no pivoting is necessary at step 1 since \( 2 > | -1 | = 1 \). Let

\[
L_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Then

\[
L_1A = \begin{bmatrix}
2 & -1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

Again, no row interchanges are needed since \( \frac{3}{2} > | -1 | = 1 \). Let

\[
L_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \frac{2}{3} & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Then

\[
L_2L_1A = \begin{bmatrix}
2 & -1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & 0 \\
0 & 0 & \frac{4}{3} & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]
As before, no row interchanges are needed since $\frac{4}{3} > -1 = 1$. Let

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}.$$ 

Then

$$L_3L_2L_1A = L^{-1}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = U.$$ 

Finally,

$$L = L_1^{-1}L_2^{-1}L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}.$$ 

(Note that $A$ is diagonally dominant, which explains why pivoting is not needed.)
2. Use the $A = LU$ factorization of $A$ to solve $Ax = b$, where $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution:

First we solve $Ly = b$ by forward substitution, obtaining

$$y = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{7} \\ \frac{5}{4} \end{bmatrix}.$$ 

Finally we solve $Ux = y$ by back-substitution, obtaining

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$