1. Consider the function \( f(x) = 1 + 10x - x^2 \) and let \( S_{1,n}(x) \) be its piecewise linear interpolant at a set of \( n + 1 \) equidistant nodes \( 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1 \) on the interval \([0, 1]\). Estimate the number of nodes needed to guarantee that the quadrature error satisfies
\[
|I(f) - R(f)| := \left| \int_0^1 f(x)dx - \int_0^1 S_{1,n}(x)dx \right| < 10^{-4}.
\]
Be sure to explain how you got your estimate. **Hint:** Use the fact that a piecewise linear interpolant \( S_{1,n}(x) \) of a twice continuously differentiable function \( f(x) \) on an interval \([a, b]\) satisfies the error bound
\[
\max_{a \leq x \leq b} |S_{1,n}(x) - f(x)| \leq \frac{Mh^2}{8}, \quad \text{where} \quad M := \max_{a \leq x \leq b} |f''(x)|.
\]

**Solution:**

First note that \( f'(x) = 10 - 2x \) and \( f''(x) = -2 \), hence \( M = \max_{0 \leq x \leq 1} |f''(x)| = 2 \). We have
\[
|I(f) - R(f)| = \left| \int_0^1 f(x)dx - \int_0^1 S_{1,n}(x)dx \right| = \left| \int_0^1 [f(x) - S_{1,n}(x)]dx \right| \leq \int_0^1 |f(x) - S_{1,n}(x)|dx \leq \int_0^1 \frac{Mh^2}{8}dx = \frac{Mh^2}{8} = \frac{h^2}{4}.
\]

Therefore, the bound on the error will be satisfied if \( \frac{h^2}{4} < 10^{-4} \), that is, if \( \frac{h}{2} < 10^{-2} \), or (taking reciprocals) \( \frac{2}{h} > 100 \). Recalling that \( h = \frac{1}{n-1} \), the condition becomes \( n > 51 \).