SUPPLEMENTARY MATERIALS TO THE PAPER:
ON THE LIMITING BEHAVIOR OF PARAMETER-DEPENDENT
NETWORK CENTRALITY MEASURES

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Abstract. This document contains details of numerical experiments performed to illustrate the theoretical results presented in our accompanying paper.

1. Limiting behavior of PageRank for small $\alpha$. In this section we want to illustrate the behavior of the PageRank vector in the limit of small values of the parameter $\alpha$. We take the following example from [8, pp. 32–33]. Consider the simple digraph $G$ with $n = 6$ nodes described in Fig. 1.1.

![Fig. 1.1: A directed network with six nodes.](image)

The adjacency matrix for this network is

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$ 

The corresponding matrix $H$ is obtained by transposing $A$ and normalizing each

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nonzero column of $A^T$ by the sum of its entries:

$$H = \begin{pmatrix} 0 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & 1/3 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}. $$

Next, we modify the second column of $H$ in order to have a column-stochastic matrix:

$$S = \begin{pmatrix} 0 & 1/6 & 1/3 & 0 & 0 & 0 \\ 1/2 & 1/6 & 1/3 & 0 & 0 & 0 \\ 1/2 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 & 1/2 & 1 \\ 0 & 1/6 & 1/3 & 1/2 & 0 & 0 \\ 0 & 1/6 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}. $$

Note that $S$ is reducible. Finally, we form the matrix

$$P = \alpha S + \frac{(1 - \alpha)}{6} 11^T, \quad \text{where } \alpha \in (0, 1).$$

This matrix is strictly positive for any $\alpha \in (0, 1)$, hence for each such $\alpha$ there is a unique dominant eigenvector, the corresponding PageRank vector.

Now we compute the PageRank vector $p = p(\alpha)$ for different values of $\alpha$, and compare the corresponding rankings of the nodes of $G$. We begin with $\alpha = 0.9$, the value used in [8, p. 39]. Rounded to five digits, the corresponding PageRank vector is

$$p(0.9) = (0.03721, 0.05396, 0.04151, 0.37510, 0.20600, 0.28620)^T.$$

Therefore, the nodes of $G$ are ranked by their importance as $(4 \ 6 \ 5 \ 2 \ 3 \ 1)$.

Next we compute the PageRank vector for $\alpha = 0.1$:

$$p(0.1) = (0.15812, 0.16603, 0.16067, 0.17812, 0.16703, 0.17002)^T.$$

Therefore, the nodes of $G$ are ranked by their importance as $(4 \ 6 \ 5 \ 2 \ 3 \ 1)$, exactly the same ranking as before. The scores are now closer to one another (since they are all approaching the uniform probability $1/6$), but not so close as to make the ranking impossible, or different than in the case of $\alpha = 0.9$.

For $\alpha = 0.01$ we find

$$p(0.01) = (0.16583, 0.16666, 0.16610, 0.16778, 0.16667, 0.16695)^T.$$

Again, we find that the nodes of $G$ are ranked as $(4 \ 6 \ 5 \ 2 \ 3 \ 1)$, exactly as before.

Finally, for $\alpha = 0.001$ we find, rounding this time the results to seven digits:

$$p(0.001) = (0.1665833, 0.166666, 0.1666111, 0.1667778, 0.1666667, 0.1666945)^T.$$

It is worth noting that our matrices and vectors are the transposes of those found in [8] since we write our probability distribution vectors as column vectors rather than row ones.
As before, the ranking of the nodes is unchanged.

Clearly, as $\alpha$ gets smaller it becomes more difficult to rank the nodes, since the corresponding PageRank values get closer and closer together, and more accuracy is required. For this reason, it is better to avoid tiny values of $\alpha$. This is especially true for large graphs, where most of the individual entries of the PageRank vector are very small. But the important point here is that even for very small nonzero values of $\alpha$ the underlying graph structure continues to influence the rankings of the nodes. Taking values of $\alpha$ close to 1 is probably not necessary in practice, especially recalling that $\alpha$ values near 1 result in slow convergence of the PageRank iteration.

As discussed in the paper (Theorem 6.1), the rankings given by PageRank approach those obtained using the vector $H_1$ (equivalently, $S_1$) in the limit as $\alpha \to 0^+$. This vector is given by

$$H_1 = \left( \begin{array}{cccccc} 1/3 & 5/6 & 1/2 & 3/2 & 5/6 & 1 \end{array} \right)^T.$$

The corresponding ranking is again $\{4, 6, 5, 2, 3, 1\}$, with nodes 5 and 2 tied in third place. This is in complete agreement with our analysis. Moreover, it suggests that an inexpensive alternative to computing the PageRank vector could be simply taking the row sums of $H$. This of course amounts to ranking the nodes of the digraph using a kind of weighted in-degree. This ranking scheme is much more crude than PageRank, as we can see from the fact that it assigns the same score to nodes 2 and 5, whereas PageRank clearly gives higher importance to node 5 when $\alpha = 0.9$. We make no claims about the usefulness of this ranking scheme for real directed networks, but given its low cost it may be worthy of further study.

2. Numerical experiments on undirected networks. In this section we present the results of numerical experiments aimed at illustrating the limiting behavior of walk-based, parameterized centrality measures using various undirected networks. We focus our attention on exponential-type and resolvent-type centrality measures, and study their relation to degree and eigenvector centrality.

The rankings produced by the various centrality measures are compared using the intersection distance method (for more information, see [6] and [1, 4]). Given two ranked lists $x$ and $y$, the top-$k$ intersection distance is computed by:

$$isim^k(x, y) := \frac{1}{k} \sum_{i=1}^{k} \frac{|x_i \Delta y_i|}{2i}$$

where $\Delta$ is the symmetric difference operator between the two sets and $x_k$ and $y_k$ are the top $k$ items in $x$ and $y$, respectively. The top-$k$ intersection distance gives the average of the normalized symmetric differences for the lists of the top $i$ items for all $i \leq k$. If the ordering of the top $k$ nodes is the same for the two ranking schemes, $isim_k(x, y) = 0$. If the top $k$ are disjoint, then $isim_k(x, y) = 1$. Unless otherwise specified, we compare the intersection distance for the full set of ranked nodes.

The networks come from a range of sources, although most can be found in the University of Florida Sparse Matrix Collection [5]. The first is the Zachary Karate Club network, which is a classic example in network analysis [9]. The Intravenous Drug User and the Yeast PPI networks were provided by Prof. Ernesto Estrada and are not present in the University of Florida Collection. The three Erdős networks correspond to various subnetworks of the Erdős collaboration network and can be found in the Pajek group of the UF Collection. The ca-GrQc and ca-HepTh networks
Table 2.1: Basic data for the networks used in the experiments.

<table>
<thead>
<tr>
<th>Graph</th>
<th>n</th>
<th>nnz</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zachary Karate Club</td>
<td>34</td>
<td>156</td>
<td>6.726</td>
<td>4.977</td>
</tr>
<tr>
<td>Drug User</td>
<td>616</td>
<td>4024</td>
<td>18.010</td>
<td>14.234</td>
</tr>
<tr>
<td>Yeast PPI</td>
<td>2224</td>
<td>13218</td>
<td>19.486</td>
<td>16.134</td>
</tr>
<tr>
<td>Pajek/Erdos971</td>
<td>472</td>
<td>2628</td>
<td>16.710</td>
<td>10.199</td>
</tr>
<tr>
<td>Pajek/Erdos972</td>
<td>5488</td>
<td>14170</td>
<td>14.448</td>
<td>11.886</td>
</tr>
<tr>
<td>Pajek/Erdos982</td>
<td>5822</td>
<td>14750</td>
<td>14.819</td>
<td>12.005</td>
</tr>
<tr>
<td>Pajek/Erdos992</td>
<td>6100</td>
<td>15030</td>
<td>15.131</td>
<td>12.092</td>
</tr>
<tr>
<td>SNAP/ca-GrQc</td>
<td>5242</td>
<td>28980</td>
<td>45.617</td>
<td>38.122</td>
</tr>
<tr>
<td>SNAP/ca-HepTh</td>
<td>9877</td>
<td>51971</td>
<td>31.035</td>
<td>23.004</td>
</tr>
<tr>
<td>SNAP/as-735</td>
<td>7716</td>
<td>26467</td>
<td>46.893</td>
<td>27.823</td>
</tr>
<tr>
<td>Gleich/Minnesota</td>
<td>2642</td>
<td>6606</td>
<td>3.2324</td>
<td>3.2319</td>
</tr>
</tbody>
</table>

are collaboration networks corresponding to the General Relativity and High Energy Physics Theory subsections of the arXiv and can be found in the SNAP group of the UF Collection. The as-735 network can also be found in the SNAP group and represents the communication network of a group of Autonomous Systems on the Internet. This communication was measured over the course of 735 days, between November 8, 1997 and January 2, 2000. The final network is the network of Minnesota roads and can be found in the Gleich group of the UF Collection. Basic data on these networks, including the order $n$, number of nonzeros, and the largest two eigenvalues, can be found in Table 2.1. All of the networks, with the exception of the Yeast PPI network, are simple. The Yeast PPI network has several ones on the diagonal, representing the self-interaction of certain proteins. All are undirected.

2.1. Exponential subgraph centrality and total communicability. We examined the effects of changing $\beta$ on the exponential subgraph centrality and total communicability rankings of nodes in a variety of undirected real world networks, as well as their relation to degree and eigenvector centrality. Although the only restriction on $\beta$ is that it must be greater than zero, there is often an implicit upper limit that may be problem-dependent. For the analysis in this section, we impose the following limits: $0.1 \leq \beta \leq 10$. To examine the sensitivity of the exponential subgraph centrality and total communicability rankings, we calculate both sets of scores and rankings for various choices of $\beta$. The values of $\beta$ tested are: 0.1, 0.5, 1, 2, 5, 8 and 10.

The rankings produced by the matrix exponential-based centrality measures for all choices of $\beta$ were compared to those produced by degree centrality and eigenvector centrality, using the intersection distance method described above. Plots of the intersection distances for the rankings produced by various choices of $\beta$ with those produced by degree or eigenvector centrality can be found in Figs. 2.1 and 2.2. The intersection distances for rankings produced by successive choices of $\beta$ can be found in Fig. 2.3.

In Figure 2.1, the rankings produced by exponential subgraph centrality and total communicability are compared to those produced by degree centrality. For small values of $\beta$, both sets of rankings based on the matrix exponential are very close to those produced by degree centrality (low intersection distances). When $\beta =$
0.1, the largest intersection distance between the degree centrality rankings and the exponential subgraph centrality rankings for the networks examined is slightly less than 0.2 (for the Minnesota road network). The largest intersection distance between the total communicability rankings with $\beta = 0.1$ and the degree centrality rankings is 0.3 (for the as-735 network). In general, the (diagonal-based) exponential subgraph centrality rankings tend to be slightly closer to the degree rankings than the (row sum-based) total communicability rankings for low values of $\beta$. As $\beta$ increases, the intersection distances increase, then level off. The rankings of nodes in networks with a very large (relative) spectral gap, such as the karate, Erdos971 and as-735 networks, stabilize extremely quickly, as expected. The one exception to the stabilization is the intersection distances between the degree centrality rankings and exponential subgraph centrality (and total communicability rankings) of nodes in the Minnesota road network. This is also expected, as the tiny ($< 0.001$) spectral gap for the Minnesota road network means that it will take longer for the exponential subgraph centrality (and total communicability) rankings to stabilize as $\beta$ increases. It is worth noting that the Minnesota road network is quite different from the other ones: it is (nearly) planar, has large diameter and a much more regular degree distribution.

The rankings produced by exponential subgraph centrality and total communicability are compared to those produced by eigenvector centrality for various values of $\beta$ in Figure 2.2. When $\beta$ is small, the intersection distances are large but, as $\beta$ increases, the intersection distances quickly decrease. When $\beta = 2$, they are essentially
Fig. 2.2: The intersection distances between eigenvector centrality and the exponential subgraph centrality (blue circles) or total communicability (blue crosses) rankings of the nodes in the networks in Table 2.1. The red lines show the intersection distance between eigenvector centrality and degree centrality.

zero for all but one of the networks examined. Again, the outlier is the Minnesota road network. For this network, the intersection distances between the exponential-based centrality rankings and the eigenvector centrality rankings still decrease as $\beta$ increases, but at a much slower rate than for the other networks. This is also expected, in view of the very small spectral gap. Again, the rankings of the nodes in the karate, Erdos971, and as-735 networks, which have very large relative spectral gaps, stabilize extremely quickly.

In Figure 2.3, the intersection distances between the rankings produced by exponential subgraph centrality and total communicability are compared for successive choices of $\beta$. Overall, these intersection distances are quite low (the highest is 0.25 and occurs for the exponential subgraph centrality rankings of the as-735 network when $\beta$ increases from 0.1 to 0.5). For all the networks examined, the largest intersection distances between successive choices of $\beta$ occur as $\beta$ increases to two. For higher values of $\beta$, the intersection distance drops, which corresponds to the fact that the rankings are converging to those produced by eigenvector centrality. In general, there is less change in the rankings produced by the total communicability scores for successive values of $\beta$ than for the rankings produced by the exponential subgraph centrality scores.

If the intersection distances are restricted to the top 10 nodes, they are even lower. For the karate, Erdos992, and ca-GrQc networks, the intersection distance for the top 10 nodes between successive choices of $\beta$ is always less than 0.1. For the DrugUser,
Yeast, Erdos971, Erdos982, and ca-HepTh networks, the intersection distances are somewhat higher for low values of β, but by the time β = 2, they are all equal to 0 as the rankings have converged to those produced by the eigenvector centrality. For the Erdos972 network, this occurs slightly more slowly. The intersection distances between the rankings of the top 10 nodes produced by β = 2 and β = 5 are 0.033 and for all subsequent choices of β are 0. In the case of the Minnesota Road network, the intersection distances between the top 10 ranked nodes never stabilize to 0, as is expected. More detailed results and plots can be found in [7, Appendix B].

For the networks examined, when β < 0.5, the exponential subgraph centrality and total communicability rankings are very close to those produced by degree centrality. When β ≥ 2, they are essentially identical to the rankings produced by eigenvector centrality. Thus, the most additional information about node rankings (i.e. information that is not contained in the degree or eigenvector centrality rankings) is obtained when 0.5 < β < 2. This supports the intuition developed in section 5 of the accompanying paper that “moderate” values of β should be used to gain the most benefit from the use of matrix exponential-based centrality rankings.

2.2. Resolvent subgraph and Katz centrality. In this section we investigate the effect of changes in α on the resolvent subgraph centrality and Katz centrality in the networks listed in Table 2.1, as well as the relationship of these centrality measures to degree and eigenvector centrality. We calculate the scores and node rankings produced by degree and eigenvector centrality, as well as those produced by the resolvent (RC_i(α)) and Katz (K_i(α)) centralities for various values of α. The values of α tested are given by α = 0.01 · 1 / λ_1, 0.05 · 1 / λ_1, 0.1 · 1 / λ_1, 0.25 · 1 / λ_1, 0.5 · 1 / λ_1, 0.75 · 1 / λ_1, 0.9 · 1 / λ_1, 0.95 · 1 / λ_1, and 0.99 · 1 / λ_1.

As in section 2.1, the rankings produced by degree centrality and eigenvector centrality were compared to those produced by resolvent-based centrality measures for all choices of α using the intersection distance method. The results are plotted in Figs. 2.4 and 2.5. The rankings produced by successive choices of α are also compared and these intersection distances are plotted in Fig. 2.6.

Fig. 2.4 shows the intersection distances between the degree centrality rankings and those produced by resolvent subgraph centrality or Katz centrality for the values
Fig. 2.4: The intersection distances between degree centrality and the resolvent subgraph centrality (blue circles) or Katz centrality (blue crosses) rankings of the nodes in the networks in Table 2.1. The x-axis measures $\alpha$ as a percentage of its upper bound, $\frac{1}{\lambda_1}$. The red lines show the intersection distances between degree centrality and eigenvector centrality for each of the networks.

Of $\alpha$ tested. When $\alpha$ is small, the intersection distances between the resolvent-based centrality rankings and the degree centrality rankings are low. For $\alpha = 0.01 \cdot \frac{1}{\lambda_1}$, the largest intersection distance between the degree centrality rankings and the resolvent subgraph centrality rankings is slightly less than 0.2 (for the Minnesota road network). The largest intersection distance between the degree centrality rankings and the Katz centrality rankings is also slightly less than 0.2 (again, for the Minnesota road network). The relatively large intersection distances for the node rankings on the Minnesota road network when $\alpha = 0.01 \cdot \frac{1}{\lambda_1}$ is due to the fact that with both the degree centrality and the resolvent subgraph (or Katz) centrality, there are many nodes with very close scores. Thus, small changes in the score values (induced by small changes in $\alpha$) can lead to large changes in the rankings. As $\alpha$ increases towards $\frac{1}{\lambda_1}$, the intersection distances increase. This increase is more rapid for the Katz centrality rankings than for the resolvent subgraph centrality rankings.

In Fig. 2.5, the resolvent subgraph centrality and Katz centrality rankings for various values of $\alpha$ are compared to the eigenvector centrality rankings on the networks described in Table 2.1. For small values of $\alpha$, the intersection distances tend to be large. As $\alpha$ increases, the intersection distances decrease for both resolvent subgraph centrality and Katz centrality on all of the networks examined. This decrease is faster for the (row sum-based) Katz centrality rankings than for the (diagonal-based) resolvent subgraph centrality rankings. The network with the highest intersection
Fig. 2.5: The intersection distances between eigenvector centrality and the resolvent subgraph centrality (blue circles) or Katz centrality (blue crosses) rankings of the nodes in the networks in Table 2.1. The $x$-axis measures $\alpha$ as a percentage of its upper bound, $\frac{1}{\lambda_1}$. The red reference lines show the intersection distance between eigenvector centrality and degree centrality.

distances between the eigenvector centrality rankings and those based on the matrix resolvent, and slowest decrease of these intersection distances as $\alpha$ increases, is the Minnesota road network. As was the case when matrix exponential-based scores were examined, this is expected due to this network’s small spectral gap. The node rankings in networks with large relative spectral gaps (karate, Erdos971, as-735) converge to the eigenvector centrality rankings most quickly.

The intersection distance between the resolvent subgraph and Katz centrality rankings produced by successive choices of $\alpha$ are plotted in Fig. 2.6. All of these intersection distances are extremely small (the largest is $< 0.08$), indicating that the rankings do not change much as $\alpha$ increases. However, as $\alpha$ increases, the rankings corresponding to successive values of $\alpha$ tend to be slightly less similar to each other. The exception to this is the Katz centrality rankings for the as-735 network which become more similar as $\alpha$ increases.

Again, if the analysis is restricted to the top 10 nodes, the intersection distances between the rankings produced by successive choices of $\alpha$ are very small. For the karate, Erdos971, Erdos982, Erdos992, ca-GrQc, and Minnesota road networks, the intersection distances between the top 10 ranked nodes for successive choices of $\alpha$ are always less than or equal to 0.1 and often equal to zero. For the ca-HepTh network, the top 10 ranked nodes are exactly the same for all choices of $\alpha$. For the DrugUser, Yeast, and Erdos972 networks, they are always less than 0.2. Detailed results can be
found in [7].

For the eleven networks examined, the resolvent subgraph and Katz centrality rankings tend to be close to the degree centrality rankings when $\alpha < 0.5 \cdot \frac{1}{\lambda_1}$. It is interesting to note that as $\alpha$ increases, these rankings stay close to the degree centrality rankings until $\alpha$ is approximately one half of its upper bound. Additionally, the resolvent based rankings are close to the eigenvector centrality rankings when $\alpha > 0.9 \cdot \frac{1}{\lambda_1}$. Thus, the most information is gained by using resolvent based centrality measures when $0.5 \cdot \frac{1}{\lambda_1} \leq \alpha \leq 0.9 \cdot \frac{1}{\lambda_1}$. This supports the intuition from section 5 of the accompanying paper that “moderate” values of $\alpha$ provide the most additional information about node ranking beyond that provided by degree and eigenvector centrality.

It is worth noting that similar conclusions have been obtained for the choice of the damping parameter $\alpha$ used in the PageRank algorithm; see [2, 3].

3. Numerical experiments on directed networks. In this section, we examine the relationship between the exponential and resolvent-based broadcast centrality measures with the out-degrees and the dominant right eigenvectors of two real world directed networks. A similar analysis can be done on the relationship between the receive centrality measures and the in-degrees and dominant left eigenvectors. For the experiments we use two networks from the University of Florida Sparse Matrix Collection [5]. As before, the rankings are compared using the intersection distance method. The first network we examine is wb-cs-Stanford, a network of hyperlinks between the Stanford CS webpages in 2001. It is in the Gleich group of the UF collection. The second network is the wiki-Vote network, which is a network of who votes for whom in elections for Wikipedia editors to become administrators. It is in the SNAP group of the UF collection.

Since our theory applies to strongly connected networks with irreducible adjacency matrices, our experiments were performed on the largest strongly connected component of the above networks. Basic data on these strongly connected components can be found in Table 3.1. In both of the networks examined, the two largest eigenvalues of the largest strongly connected component are real. Both networks are simple.
Table 3.1: Basic data on the largest strongly connected component of the real-world directed networks examined.

<table>
<thead>
<tr>
<th>Graph</th>
<th>$n$</th>
<th>$nnz$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gleich/wb-cs-Stanford</td>
<td>2759</td>
<td>13895</td>
<td>35.618</td>
<td>12.201</td>
</tr>
<tr>
<td>SNAP/wiki-Vote</td>
<td>1300</td>
<td>39456</td>
<td>45.145</td>
<td>27.573</td>
</tr>
</tbody>
</table>

3.1. Total communicability. As in section 2.1, we examine the effect of changing $\beta$ on the broadcast total communicability rankings of nodes in the networks, as well as their relation to the rankings obtained using the out-degrees and dominant right eigenvectors of the networks. The measures were calculated for the networks described in Table 3.1. To examine the sensitivity of the broadcast total communicability rankings, we calculate the scores and rankings for various choices of $\beta$. The values of $\beta$ tested are: 0.1, 0.5, 1, 2, 5, 8 and 10.

The broadcast rankings produced by total communicability for all choices of $\beta$ were compared to those produced by the out-degree rankings and the rankings produced by $x_1$ using the intersection distance method as described in section 2. Plots of the intersection distances for the rankings produced by various choices of $\beta$ with those produced by the out-degrees and right dominant eigenvector can be found in Figs. 3.1 and 3.2. The intersection distances for rankings produced by successive choices of $\beta$ can be found in Figure 3.3.

In Fig. 3.1, the intersection distances between the rankings produced by broadcast total communicability are compared to those produced by the out-degrees of nodes in the network. As $\beta$ approaches 0, the intersection distances decrease for both networks. As $\beta$ increases to 10, the intersection distances initially increase, then stabilize as the rankings converge to those produced by $x_1$.

The intersection distances between the rankings produced by broadcast total communicability are compared to those produced by $x_1$ in Figure 3.2. For both networks, the intersection distances quickly decrease as $\beta$ increases. In the wiki-Vote network, the intersection distances between the compared rankings are 0 by the time $\beta = 0.5$. For the wb-cs-Stanford network, by the time $\beta$ has reached five, the intersection distances between the broadcast total communicability rankings and those produced by $x_1$ have decreased to about 0.04. The rankings then stabilize at this intersection distance. This is due to a group of nodes that have nearly identical total communicability.
Fig. 3.2: The intersection distances (blue circles) between the rankings produced by $x_1$ and the broadcast total communicability rankings of the nodes in the networks in Table 3.1. The red lines show the intersection distances between the $x_1$ rankings and those produced by the out-degrees.

![Graph showing intersection distances between rankings](image)

Fig. 3.3: The intersection distances between the broadcast total communicability rankings produced by successive choices of $\beta$. Each line corresponds to a network in Table 3.1.

![Graph showing intersection distances between rankings and $\beta$](image)

In Fig. 3.3, the intersection distances between the broadcast total communicability rankings for successive choices of $\beta$ are plotted. These intersection distances are slightly lower than those observed in the undirected case, with a maximum of approximately 0.14, which occurs in the wb-cs-Stanford network when $\beta$ increases from 0.01 to 0.05. By the time $\beta = 0.5$, the rankings on the wiki-Vote network have stabilized.
Parameter-dependent centrality measures (supplementary materials)

Fig. 3.4: The intersection distances (blue circles) between the rankings produced by the out-degrees and the broadcast Katz centrality rankings of the nodes in the networks in Table 3.1. Here, the $x$-axis shows $\alpha$ as a percentage of its upper bound, $\frac{1}{\lambda_1}$. The red reference lines show the intersection distance between the rankings produced by $x_1$ and those produced by the out-degrees of the nodes.

and all subsequent intersection distances are 0. For both the broadcast total communicability rankings on the wb-cs-Stanford network, the intersection distances decrease (non-monotonically) as $\beta$ increases until they stabilize at approximately 0.02.

When this analysis is restricted to the top 10 nodes, the intersection distances are extremely small. For the wb-cs-Stanford network, the largest intersection distance between the top 10 ranked nodes for successive choices of $\beta$ is 0.11 (when $\beta$ increases from 0.1 to 0.5). For the wiki-Vote network, the intersection distance between the top 10 total communicability scores is 0.01 when $\beta$ increases from 0.1 to 0.5, and zero otherwise; see [7, Appendix B] for detailed results and plots.

The differences between the out-degree rankings and the broadcast total communicability rankings are greatest when $\beta \geq 0.5$. The differences between the left and right eigenvector based rankings and the broadcast rankings are greatest when $\beta < 2$ (although in the case of the wiki-Vote network, they have converged by the time $\beta = 0.5$). Thus, like in the case of the undirected networks, moderate values of $\beta$ give the most additional ranking information beyond that provided by the out-degrees and the left and right eigenvalues.

3.2. Katz centrality. In this section, we investigate the effect of changes in $\alpha$ on the broadcast Katz centrality rankings of nodes in the networks listed in Table 3.1 and relationship of these centrality measures to the rankings produced by the out-degrees and the dominant right eigenvectors of the network. We calculate the scores and node rankings produced by $K^b_\alpha(\alpha)$ for various values of $\alpha$. The values of $\alpha$ tested are given by $\alpha = 0.01 \cdot \frac{1}{\lambda_1}, 0.05 \cdot \frac{1}{\lambda_1}, 0.1 \cdot \frac{1}{\lambda_1}, 0.25 \cdot \frac{1}{\lambda_1}, 0.5 \cdot \frac{1}{\lambda_1}, 0.75 \cdot \frac{1}{\lambda_1}, 0.9 \cdot \frac{1}{\lambda_1}, 0.95 \cdot \frac{1}{\lambda_1},$ and $0.99 \cdot \frac{1}{\lambda_1}$.

The rankings produced by the out-degrees and the dominant right eigenvectors were compared to those produced by Katz centrality for all choices of $\alpha$ using the intersection distance method, as was done in Section 3.1. The results are plotted in Figs. 3.4 and 3.5.

As $\alpha$ increases from $0.01 \cdot \frac{1}{\lambda_1}$ to $0.99 \cdot \frac{1}{\lambda_1}$, the intersection distances between the scores produced by the broadcast Katz centralities and the out-degrees increase. When $\alpha$ is small, the broadcast Katz centrality rankings are very close to those produced by the out-degrees (low intersection distances). On the wb-cs-Stanford network, when $\alpha = 0.01 \cdot \frac{1}{\lambda_1}$, the intersection distance between the two rankings is approximately 0.06. On the wiki-Vote network, it is approximately 0.01. As $\alpha$ increases, the
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Fig. 3.5: The intersection distances (blue circles) between the rankings produced by \( x_1 \) and the broadcast Katz centrality rankings of the nodes in the networks in Table 3.1. Here, the \( x \)-axis shows \( \alpha \) as a percentage of its upper bound, \( \frac{1}{\lambda_1} \). The red lines show the intersection distance between the rankings produced using \( x_1 \) and the node out-degrees.

Fig. 3.6: The intersection distances between the broadcast Katz centrality rankings produced by successive choices of \( \alpha \). Each line corresponds to a network in Table 3.1.

intersection distances also increase. By the time \( \alpha = 0.99 \cdot \frac{1}{\lambda_1} \), the intersection distance between the two sets of node rankings on the wb-cs-Stanford network is above 0.2 and on the wiki-Vote network it is approximately 0.1.

In Fig. 3.5, the rankings produced by broadcast Katz centrality are compared to those produced by \( x_1 \). Overall, The intersection distances between the two sets of rankings are lower on the wiki-Vote network than they are on the wb-cs-Stanford network. As \( \alpha \) increases from \( 0.01 \cdot \frac{1}{\lambda_1} \) to \( 0.99 \cdot \frac{1}{\lambda_1} \), the intersection distances between the two sets of rankings on the wiki-Vote network decrease from 0.1 to essentially 0. On the wb-cs-Stanford network, they decrease from approximately 0.47 to 0.24.

The intersection distances between the rankings produced by the broadcast Katz centralities for successive values of \( \alpha \) are plotted in Figure 3.6. As was the case for the undirected networks, these rankings are more stable in regards to the choice of \( \alpha \) than the total communicability rankings were in regards to the choice of \( \beta \). Here,
the maximum intersection distance is less than 0.1. When only the top 10 ranked nodes are considered, the intersection distances have a maximum of 0.06 (on the wbc-Stanford network when $\alpha$ increases from $0.25 \cdot \frac{1}{\lambda_1}$ to $0.5 \cdot \frac{1}{\lambda_1}$). For both networks, the intersection distances between the rankings on the top 10 nodes for successive choices of $\alpha$ are quite small (the maximum is 0.18 and the majority are $< 0.1$).

The broadcast Katz centrality rankings are only far from those produced by the out-degrees when $\alpha \geq 0.5 \cdot \frac{1}{\lambda_1}$. They are farthest from those produced by the dominant right eigenvector of $A$ when $\alpha < 0.9 \cdot \frac{1}{\lambda_1}$. Thus, as was seen in the case of undirected networks, the most additional information is gained when moderate values of $\alpha$, $0.5 \cdot \frac{1}{\lambda_1} \leq \alpha < 0.9 \cdot \frac{1}{\lambda_1}$, are used to calculate the matrix resolvent based centrality scores.

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**REFERENCES**


