Gerbe patching and a Mayer-Vietoris sequence

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Patching after Harbater, Hartmann & Krashen
Patching after HHK

- Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a finite inverse factorization system i.e.
  - a finite inverse system of fields and inclusions
  - the inverse limit is a field $F$
  - $I = I_v \sqcup I_e$ such that for $k \in I_e$, there are exactly two $i, j \in I_v$ with $i, j < k$
  - these are the only relations
  - associated graph $\Gamma$ with vertices from $I_v$ and edges from $I_e$
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- A patching problem of vector spaces over $\mathcal{F}$ is a collection of finite dimensional vector spaces $V_i$ over $F_i$ for all $i \in I_v$ together with isomorphisms $\phi_k : V_i \otimes F_i F_k \hom V_j \otimes F_j F_k$ whenever $i, j < k$
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- Let $\text{PP}(\mathcal{F})$ denote the category of patching problems
- There is a natural functor $\beta : \text{Vect}(F) \rightarrow \text{PP}(\mathcal{F})$
- We say that patching holds over $\mathcal{F}$ if $\beta$ is an equivalence
Patching after HHK

- Let $T$ be a cdvr with uniformizer $t$
- Let $K$ be its fraction field and $k$ its residue field
- Let $F$ be a one variable function field over $K$
- Let $\hat{X}$ be a normal model of $F$ (normal, connected, projective $T$-curve)
- Let $X$ be the special fiber of $\hat{X}$
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For a closed point $p \in X$:
- $R_p$ = local ring of $\hat{X}$ at $p$
- $\hat{R}_p$ = its completion
- $F_P$ = fraction field of $\hat{R}_p$
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- Let $U \subset X$ be contained in an irreducible component and not meeting any other
  - $R_U =$ subring of $F$ of rational functions regular on $U$
  - $\hat{R}_U =$ completion of $R_U$ at $t$
  - $F_U =$ fraction field of $\hat{R}_U$
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- Let $b$ be a branch at some $p$ (i.e. height one prime ideal in $\hat{R}_p$ containing $t$)
  - $\hat{R}_b$ = completion of the localization of $\hat{R}_p$ at $b$
  - $F_b$ = fraction field of $\hat{R}_b$
Patching after HHK

- Let $\mathcal{P}$ be a finite subset of closed points of $X$ containing all points where irreducible components meet
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- Let \( \mathcal{U} \) be the set of connected components of the complements of \( \mathcal{P} \).
Patching after HHK

- Let $\mathcal{P}$ be a finite subset of closed points of $X$ containing all points where irreducible components meet.
- Let $\mathcal{U}$ be the set of connected components of the complements of $\mathcal{P}$.
- Let $\mathcal{B}$ be the set of branches on $U \in \mathcal{U}$ at $p \in \mathcal{P}$ for $p \in \overline{U}$.
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For $p$ in the closure of $U$: $F \subset F_U$, $F_p \subset F_b$. 
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For $p$ in the closure of $U$: $F \subset F_U, F_p \subset F_b$.

With these relations, we get a finite inverse factorization system $\mathcal{F}$ with $I_{\mathcal{V}} = \mathcal{P} \sqcup \mathcal{U}$ and $I_{\mathcal{V}} = \mathcal{B}$ and limit $F$. 

Theorem (Harbater, Hartmann, Krashen) With the notation from above, patching holds for vector spaces over $F$. 

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- For $p$ in the closure of $U$: $F \subset F_{\mathcal{U}}, F_{p} \subset F_{\mathcal{B}}$
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Theorem (Harbater, Hartmann, Krashen)

With the notation from above, patching holds for vector spaces over $\mathcal{F}$. 
From now on: $\mathcal{F} = \{F_1, F_2, F_0\}$ with

and $F_1 \cap F_2 = F$
Patching of torsors and Galois cohomology
Let $G$ be a linear algebraic group over $F$. Let $\text{TPP}(G, \mathcal{F})$ be the category of torsor patching problems. We have a natural functor

$$\beta': \text{torsors}(G, F) \rightarrow \text{TPP}(G, \mathcal{F})$$

**Theorem (HHK)**

*If patching holds for vector spaces, then it holds for $G$-torsors. In particular, patching for $G$-torsors holds over arithmetic curves.*
A Mayer–Vietoris sequence

**Theorem (HHK)**

*Let $G$ be a $LAG$. There is an exact sequence*

$$1 \to H^0(F, G) \to H^0(F_1, G) \times H^0(F_2, G) \to H^0(F_0, G) \to H^1(F, G) \to H^1(F_1, G) \times H^1(F_2, G) \to H^1(F_0, G)$$
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1 \to H^0(F, G) \to H^0(F_1, G) \times H^0(F_2, G) \to H^0(F_0, G) \cong H^1(F, G) \to H^1(F_1, G) \times H^1(F_2, G) \to H^1(F_0, G)
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- The map \( H^0(F_0, G) \to H^1(F, G) \) is defined via \( g_0 \mapsto (G, G, g_0) \).
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- Local-global principle for torsors holds if $H^1(F, G) \to H^0(F_1, G) \times H^0(F_2, G)$ is injective
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- local-global for $G$-torsors is equivalent to surjectivity of $H^0(F_1, G) \times H^0(F_2, G) \to H^0(F_0, G)$
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- Local-global for $G$-torsors is equivalent to surjectivity of $H^0(F_1, G) \times H^0(F_2, G) \to H^0(F_0, G)$
- I.e. for $g_0 \in G(F_0)$, there is $g_1 \in G(F_1)$ and $g_2 \in G(F_2)$ such that $g_0 = g_1 g_2^{-1}$.
Theorem (HHK)

Let $G$ be an abelian linear algebraic group and assume that $\text{char}(k) = 0$ or $\text{char}(k) \nmid |G| < \infty$. There is an exact sequence

\[\cdots \rightarrow H^n(F, G) \rightarrow H^n(F_1, G) \times H^n(F_2, G) \rightarrow H^n(F_0, G) \rightarrow \cdots\]
Non-abelian hypercohomology
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Non-abelian hypercohomology with values in crossed modules was introduced (independently) by Breen and Borovoi.

- Let $G$ be a group sheaf over $F$
- Let $G \to \text{Aut}(G)$ be given by conjugation
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- $H^1(F, G \to \text{Aut}(G)) = \{G - \text{gerbes over } F\} / \text{equivalence}$
- An example of a $G$-gerbe is $BG$, the classifying stack of $G$-torsors. Every $G$-gerbe is locally equivalent to $BG$
Mayer–Vietoris in non-abelian hypercohomology
Theorem

Let $G$ be a $LAG$ and let $\text{char}(k) = 0$. There is an exact sequence

$$1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^{-1}(F_i, G \to \text{Aut}(G)) \to H^{-1}(F_0, G \to \text{Aut}(G))$$

$$\to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G))$$

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1. if $\text{char}(k) = p$, this also holds under additional assumptions on $G$ and $Z(G)$
A Mayer–Vietoris sequence

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Let $G$ be a LAG and let $\text{char}(k) = 0$. There is an exact sequence

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\]

1. if $\text{char}(k) = p$, this also holds under additional assumptions on $G$ and $Z(G)$

2. the first two rows exist and are exact even if $\text{char}(k) = p$. 
Patching for bitorsors

\[ 1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i=1, 2} H^{-1}(F_i, G \to \text{Aut}(G)) \to H^{-1}(F_0, G \to \text{Aut}(G)) \]
\[ \to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1, 2} H^0(F_i, G \to \text{Aut}(G)) \overset{\alpha}{\to} H^0(F_0, G \to \text{Aut}(G)) \]

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- Let \( P \) be a \( G \)-bitorsor, i.e. a right and left \( G \)-torsors where right and left action commute
- Given two \( G \)-bitorsors \( P \) and \( Q \), the wedged product \( P \wedge^G Q \) is again a \( G \)-bitorsor
Patching for bitorsors

\[ 1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^{-1}(F_i, G \to \text{Aut}(G)) \to H^{-1}(F_0, G \to \text{Aut}(G)) \]
\[ \to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \xrightarrow{\alpha} H^0(F_0, G \to \text{Aut}(G)) \]

- Let \( P \) be a \( G \)-bitorsor, i.e. a right and left \( G \)-torsors where right and left action commute
- Given two \( G \)-bitorsors \( P \) and \( Q \), the wedged product \( P \wedge^G Q \) is again a \( G \)-bitorsor
- this allows us to define the map \( \alpha \) via \((P_1, P_2) \mapsto P_1 \wedge^G P_2^{\text{op}}\)
Theorem

Let $G$ be a LAG. Then, patching holds for $G$-bitorsors over $\mathcal{F}$. 

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1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^{-1}(F_i, G \to \text{Aut}(G)) \to H^{-1}(F_0, G \to \text{Aut}(G)) \\
\to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G))$$
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\to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G))

Theorem

Let G be a LAG. Then, patching holds for G-bitorsors over \( \mathcal{F} \).

- proof via reduction to left G-torsor case and description of right action given by Breen
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\to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G))
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**Theorem**

Let $G$ be a LAG. Then, patching holds for $G$-bitorsors over $\mathcal{F}$.

- proof via reduction to left $G$-torsor case and description of right action given by Breen
- This proves exactness here
Patching for bitorsors

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\[ \to H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G)) \]

- Note that automorphisms of $G$ as a bitorsor come from $Z(G)$
Patching for bitorsors

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- Note that automorphisms of \( G \) as a bitorsor come from \( Z(G) \)
- Given \( g_0 \in Z(G)(F_0) \), then \((G, G, g_0)\) is a bitorsor patching problem
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- Note that automorphisms of $G$ as a bitorsor come from $Z(G)$
- Given $g_0 \in Z(G)(F_0)$, then $(G, G, g_0)$ is a bitorsor patching problem
- Let $P_{g_0}$ denote a solution
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- Note that automorphisms of \( G \) as a bitorsor come from \( Z(G) \)
- Given \( g_0 \in Z(G)(F_0) \), then \((G, G, g_0)\) is a bitorsor patching problem
- Let \( P_{g_0} \) denote a solution
- This allows us to define the connecting map via \( g_0 \mapsto [P_{g_0}] \)
Local-global principle for bitorsors

1 \rightarrow H^{-1}(F, G \rightarrow Aut(G)) \rightarrow \prod_{i=1,2} H^{-1}(F_i, G \rightarrow Aut(G)) \rightarrow H^{-1}(F_0, G \rightarrow Aut(G))

\rightarrow H^0(F, G \rightarrow Aut(G)) \rightarrow \prod_{i=1,2} H^0(F_i, G \rightarrow Aut(G)) \rightarrow H^0(F_0, G \rightarrow Aut(G))

**Corollary**

Local-global for $G$-bitorsors holds iff $Z(G)$ satisfies factorization.
Local-global principle for bitorsors

\[ 1 \to \mathcal{H}^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} \mathcal{H}^{-1}(F_i, G \to \text{Aut}(G)) \to \mathcal{H}^{-1}(F_0, G \to \text{Aut}(G)) \]
\[ \to \mathcal{H}^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} \mathcal{H}^0(F_i, G \to \text{Aut}(G)) \to \mathcal{H}^0(F_0, G \to \text{Aut}(G)) \]

**Corollary**

*Local-global for G-bitorsors holds iff \( Z(G) \) satisfies factorization.*

- HHK proved that a rational LAG \( H \) satisfies factorization iff \( H \) is connected or \( \Gamma \) is a tree.
Patching for gerbes and local-global principle for gerbes

\[ H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G)) \]

\[ H^1(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^1(F_i, G \to \text{Aut}(G)) \to H^1(F_0, G \to \text{Aut}(G)) \]
Patching for gerbes and local-global principle for gerbes

\[ H^0(F, G \to \text{Aut}(G)) \to \prod_{i=1,2} H^0(F_i, G \to \text{Aut}(G)) \to H^0(F_0, G \to \text{Aut}(G)) \]

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- exactness here follows from gerbe patching
Patching for gerbes and local-global principle for gerbes

\[
\begin{align*}
\text{H}^0(F,G \to \text{Aut}(G)) & \to \prod_{i=1,2} \text{H}^0(F_i,G \to \text{Aut}(G)) \to \text{H}^0(F_0,G \to \text{Aut}(G)) \\
\text{H}^1(F,G \to \text{Aut}(G)) & \to \prod_{i=1,2} \text{H}^1(F_i,G \to \text{Aut}(G)) \to \text{H}^1(F_0,G \to \text{Aut}(G))
\end{align*}
\]

- exactness here follows from gerbe patching

**Corollary**

*G*-gerbes satisfy local-global principle iff *G*-bitorsors satisfy factorization, i.e. iff for all *G*-bitorsors \( P_0 \) over \( F_0 \), there are *G*-bitorsors \( P_1, P_2 \) over \( F_1 \) and \( F_2 \) such that \( P_0 \cong P_1 \wedge^G P_2^{\text{op}} \) holds.
Let $G$ be a linear algebraic group over $F$ with center $Z$. Then, the local global principle for $G$-gerbes with respect to patching holds if

- if $\text{char}(k) = 0$, $\Gamma$ is a tree and
  - $G$ is a finite constant group scheme with trivial center,
  - $G$ is split, semisimple, adjoint of type $A_1, B_n, C_n, E_7, E_8, F_4$ or $G_2$,
  - $G$ is semisimple such that $G/Z$ admits no outer automorphism and $Z, G/Z$ satisfy bitorsor factorization,
  - $G = SL_1(D)$ where $D$ is a central simple algebra over $F$,
- if $\text{char}(k) = p > 0$, $Z$ has finite order not divided by $\text{char}(k)$ and $G$ and $\Gamma$ are as in the case of $\text{char}(k) = 0$. 
• Let \( P \) be a \( G \)-bitorsor
• Consider the functor

\[
\psi_P : BG \to BG, \\
T \mapsto P \wedge^G T
\]
Let $P$ be a $G$-bitorsor

Consider the functor

$$\psi_P : BG \to BG,$$

$$T \mapsto P \wedge^G T$$

It is an equivalence with quasi-inverse $T \mapsto P^{\text{op}} \wedge^G T$
Automorphisms of $BG$ and bitorsors

- Let $P$ be a $G$-bitorsor
- Consider the functor
  
  $\psi_P : BG \rightarrow BG,$
  
  $T \mapsto P^G T$

- It is an equivalence with quasi-inverse $T \mapsto P^{op}^G T$
- Given an isomorphism of $G$-bitorsors $f : P \rightarrow P'$, we get induced isomorphisms $P^G T \rightarrow P'^G T$
Automorphisms of $BG$ and bitorsors

- Let $P$ be a $G$-bitorsor
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- Given an isomorphism of $G$-bitorsors $f : P \to P'$, we get induced isomorphisms $P \wedge^G T \to P' \wedge^G T$
- These isomorphisms define a natural isomorphism of functors $\alpha_f : \psi_P \Rightarrow \psi_{P'}$
Theorem (Giraud)

There is an equivalence of categories

\[ \text{Bitorsors}(G, F) \simeq \text{Aut}(BG) \]

given by \( P \mapsto \psi_P \) and \( f \mapsto \psi_f \).
A semi-cocyclic description of $G$-gerbes

Let $G$ be a $G$-gerbe over $\text{Spec}(F)$ over the big étale site.
A semi-cocyclic description of $G$-gerbes

Let $G$ be a $G$-gerbe over Spec($F$) over the big étale site.

- Let $Y \to \text{Spec}(F)$ be a cover and $y \in G(Y)$
A semi-cocyclic description of $G$-gerbes

Let $\mathcal{G}$ be a $G$-gerbe over $\text{Spec}(F)$ over the big étale site.

- Let $Y \to \text{Spec}(F)$ be a cover and $y \in \mathcal{G}(Y)$
- We get an equivalence of $G$-gerbes:

$$\mathcal{G}|_Y \to B G|_Y$$

$$p \mapsto \text{Isom}(p, y)$$
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\[
\mathcal{G}|_Y \to B\mathcal{G}|_Y \\
p \mapsto \text{Isom}(p, y)
\]

\[Y^2 = Y \times_F Y \quad \psi: B\mathcal{G}|_{Y^2} \to B\mathcal{G}|_{Y^2}\]
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Let $G$ be a $G$-gerbe over $\text{Spec}(F)$ over the big étale site.

- Let $Y \to \text{Spec}(F)$ be a cover and $y \in G(Y)$
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$$Y^3$$

$$\alpha: \psi_{23} \circ \psi_{12} \Rightarrow \psi_{13}$$
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$Y^2 = Y \times_F Y$ \quad $\psi: BG|_{Y^2} \to BG|_{Y^2}$

$Y^3$ \quad $\alpha: \psi_{23} \circ \psi_{12} \Rightarrow \psi_{13}$

$Y^4$ \quad coherence condition on $\alpha$
A semi-cocyclic description of \( G \)-gerbes

Let \( \mathcal{G} \) be a \( G \)-gerbe over \( \text{Spec}(F) \) over the big étale site.

- Let \( Y \to \text{Spec}(F) \) be a cover and \( y \in \mathcal{G}(Y) \)
- We get an equivalence of \( G \)-gerbes:

\[
\mathcal{G}|_Y \to B G|_Y
p \mapsto \text{Isom}(p, y)
\]

\[
Y^2 = Y \times_F Y \quad \psi : B G|_{Y^2} \to B G|_{Y^2} \quad \text{G-bitorsor } P
\]

\[
Y^3 \quad \alpha : \psi_{23} \circ \psi_{12} \Rightarrow \psi_{13}
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\[
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\mathcal{G} \mid_Y \to BG \mid_Y \\
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\]

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$Y^3 \quad \alpha : \psi_{23} \circ \psi_{12} \Rightarrow \psi_{13} \quad f : P_{12} \wedge^G P_{23} \sim P_{13}$

$Y^4 \quad$ coherence condition on $\alpha$
Let $G$ be a $G$-gerbe over $\text{Spec}(F)$ over the big étale site.

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G-bitorsor $P$

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$f: P_{12} \land^G P_{23} \sim P_{13}$

$Y^4$

coherence condition on $\alpha$

coherence condition on $f$
A semi-cocyclic description of $G$-gerbes

**Theorem (Breen)**

A gerbe $\mathcal{G}$ is, up to unique equivalence, determined by $(Y, P, f)$
Gerbe patching

\[ \theta : \mathcal{G}_1|_{F_0} \sim \mathcal{G}_2|_{F_0} \]

\[\mathcal{G}_1 \quad \mathcal{G}_2\]

\[ F \]

\[ F_0 \]

\[ F_1 \]

\[ F_2 \]
Assumption

There is a cover $Y \rightarrow \text{Spec}(F)$ and $y_i \in \mathcal{G}_i(Y_i)$ such that $\theta(y_1)$ and $y_2$ are isomorphic over $Y_0$
Gerbe patching

**Assumption**

There is a cover 
\[ Y \to \operatorname{Spec}(F) \] and 
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1. write \( \mathcal{G}_i = (Y_i, P_i, f_i) \) using \( y_i \)
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1. write $\mathcal{G}_i = (Y_i, P_i, f_i)$ using $y_i$
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   - bitorsor \( P \) over \( Y^2 \)
   - \( f : P_{12} \wedge^G P_{23} \overset{\simeq}{\rightarrow} P_{13} \) over \( Y^3 \)
   - \( f \) satisfies coherence condition on \( Y^4 \)
The gerbe $\mathcal{G} = (Y, P, f)$ solves the patching problem.
Gerbe patching

The gerbe $\mathcal{G} = (Y, P, f)$ solves the patching problem

- The assumption always holds in $\text{char}(k) = 0$ by a theorem of CHHKPS. In $\text{char}(k) = p$, it holds under a technical assumption on $G$ and $Z(G)$.
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**Theorem**

*Under the assumption, there is a 2-equivalence of 2-categories*

\[ \mathcal{G}\text{-Gerbes over } F \simeq \text{G-gerbe patching problems over } \mathbb{F} \]