GALOIS DESCENT AND SEVERI-BRAUER VARIETIES

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1. Introduction

We say an algebraic object or property over a field k is *arithmetic* if it becomes trivial or vanishes after finite separable base extension. Since such objects or properties owe their existence to the presence of "arithmetic gaps" in k, i.e., the failure of k to be algebraically closed, we view them as responses to specific arithmetic properties of k, and we study them in order to gain insight into the *arithmetic complexity* of k, which consists of the features of k responsible for the existence and relative abundance of arithmetic objects and properties.

Since the objects to be studied become trivial after a finite separable base extension, they become trivial over a finite *Galois* extension K/k. Thus our goal is to characterize isomorphism classes of objects defined over k that become isomorphic when base-extended to K, often to some trivial object. These are called *twisted forms*, a loosening of *isomorphism class*. In the ideal outcome each twisted form is obtained from the (trivial) K-object as a fixed object under a *Galois action* on the K-object. We then say that the twisted forms are obtained by *Galois descent*. Ideally, we obtain the following statement.

Meta-Theorem *Let $\mathscr{C}$: $\mathsf{Fields} \to \mathsf{Cat}$ be a functor that takes a field k to a concrete category $\mathscr{C}_k$, and a morphism $k \to K$ to the "extension" functor $E : \mathscr{C}_k \to \mathscr{C}_K$. Let K/k a finite Galois extension with group G. Let $\mathscr{C}_K[G]$ be the category whose objects are pairs (W, α) for W an object of $\mathscr{C}_K$ and α a* Galois *G-action on V, and whose morphisms are G-equivariant morphisms in $\mathscr{C}_K$. Then E maps into $\mathscr{C}_K[G]$, and the G-action defines a "fixed object" functor $F : \mathscr{C}_K[G] \to \mathscr{C}_k$, so that we have a category equivalence*

$$\mathscr{C}_k \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{C}_K[G]$$

For each object V in $\mathscr{C}_k$, the set of twisted forms *of $[V]$*

$$\mathrm{TF}_{K|k}(V) = \{[V'] : V' \in \mathscr{C}_k \text{ and } E(V') \simeq E(V)\}$$

admits a pointed-set isomorphism

$$\mathrm{H}^1(G, \mathrm{Aut}_{\mathscr{C}_K}(E(V)) \longrightarrow \mathrm{TF}_{K|k}(V)$$

taking a class $[c]$ *to the class* $[F({}_cE(V))] \in \mathrm{TF}_{K|k}(V)$*, where* ${}_cE(V)$ *is* $E(V)$ *with "twisted"* G*-action* $\sigma \star_c x = c_\sigma {}^\sigma x$*. The inverse takes a class* $[V'] \in \mathrm{TF}_{K|k}(V)$ *to the class* $[c]$ *defined by* $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$*, for any* $\phi \in \mathrm{Isom}_{\mathscr{C}_K}(E(V), E(V'))$*.*

This summary is aimed at students with a graduate-level background in algebra, and for some topics a course in algebraic geometry, including basics on schemes and projective varieties. We prove the meta-theorem in categories of vector spaces with tensor (e.g., quadratic spaces, algebras, commutative algebras, and central simple algebras), quasi-projective varieties, quasi-coherent sheaves, and locally free sheaves of fixed rank. We use this background to work out the basic facts about Severi-Brauer varieties, as presented in Artin's classic paper [2].

All of this material is "classical". We rely on the sources [4] and [5] (k-rationality), [8] (tensors), [10] (Galois descent), [15] (Galois descent and torsors), and [2] (Severi-Brauer varieties). Serre gives a polished account of Galois descent in [15] and an introduction to Severi-Brauer varieties in [14], and Artin gives the canonical in-depth account of Severi-Brauer varieties in [2]. Jahnel treats most of our topics in detail, and kindly specializes Grothendieck's theory of faithfully flat descent ([9]) to Galois descent. Gille-Szamuely also give a thorough treatment of all of these subjects (and more) in the valuable resource [8]. Seminal founding papers include Weil ([16]), Châtelet ([6]), Amitsur ([1]), and others; see [2] and [10] for historical background and bibliography.

2. Concrete Categories with Left G-Action

Recall a category is *concrete* if it admits a faithful ("forgetful") functor to the category Set. Examples include the categories of vector spaces, rings, algebras, central simple algebras, and quadratic forms, and these extend to schemes, sheaves, and projective varieties. If $\mathscr{C}$ is a category we write $\mathrm{Hom}_{\mathscr{C}}(V, W)$, $\mathrm{Isom}_{\mathscr{C}}(V, W)$, and $\mathrm{Aut}_{\mathscr{C}}(V)$ for the sets of morphisms, isomorphisms, and automorphisms, respectively.

2.1. Left G-Objects.

Definition 2.1. Let G be a group, and let $\mathscr{C}$ be a concrete category. We say a left G-action on an object V of $\mathscr{C}$ is *in* $\mathscr{C}$ if there is a group homomorphism

$$\alpha : G \longrightarrow \mathrm{Aut}_{\mathscr{C}}(V)$$

Denote by $\mathscr{C}[G]$ the category whose objects are pairs (V, α), where V is an object of $\mathscr{C}$ and α is a left G-action on V in $\mathscr{C}$, and whose morphisms are G-equivariant $\mathscr{C}$-morphisms. Let $[(V, \alpha)]$ and $[V]$ denote the isomorphism classes in $\mathscr{C}[G]$ and $\mathscr{C}$, respectively. Set

$$[V]/G = \{[(V', \alpha')] : [V'] = [V]\},$$

the G-isomorphism classes in $\mathscr{C}[G]$ that are isomorphic in $\mathscr{C}$. The set $[V]/G$ has distinguished element $[(V, \alpha)]$, making it a pointed set.

Two actions α and β of G on an object V in $\mathscr{C}$ are *equivalent* if $[(V,\alpha)] = [(V,\beta)]$ in $\mathscr{C}[G]$, i.e., if there is an automorphism $b \in \mathrm{Aut}_{\mathscr{C}}(V)$ such that $\beta(\sigma) = b^{-1}\cdot\alpha(\sigma)\cdot b$ for all $\sigma \in G$, i.e., a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{b} & V \\ {\scriptstyle\beta(\sigma)}\uparrow & & \uparrow{\scriptstyle\alpha(\sigma)} \\ V & \xrightarrow{b} & V \end{array}$$

We often suppress the action notation, writing $\sigma(x)$ or $\sigma \star x$ instead of $\alpha(\sigma)(x)$, and consider that two objects V and V' in $\mathscr{C}[G]$ may be isomorphic but not G-isomorphic. For V, V' in $\mathscr{C}[G]$ the set $\mathrm{Isom}_{\mathscr{C}}(V,V')$ admits a left G-action defined by ${}^{\sigma}\phi = \sigma\cdot\phi\cdot\sigma^{-1}$.

3. Galois Cohomology

We follow [15, Section I.5] and introduce the first cohomology set and principal homogeneous spaces in the category of left G-sets.

3.1. Cohomology Sets.

Let G be a finite group and let A be a left G-group, with action written $b \mapsto {}^{\sigma}b$ for $b \in A$. A (left) *cocycle* is a function $c : G \to A$ such that $c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau$. We sometimes write (c_σ) for c. Let $\mathrm{Z}^1(G,A)$ denote the set of cocycles. We say cocycles (c_σ) and (c'_σ) are *cohomologous*, and write $(c_\sigma) \sim (c'_\sigma)$, if there exists an element $b \in A$ such that $c'_\sigma = b^{-1}c_\sigma {}^{\sigma}b$ for all σ. Now define the *degree 1 cohomology set*

$$\mathrm{H}^1(G,A) \overset{\mathrm{df}}{=} \mathrm{Z}^1(G,A)/\sim .$$

The trivial cocycle [1] given by $1_\sigma = e$ gives $\mathrm{H}^1(G,A)$ a pointed-set structure.

3.2. Torsors.

Let G be a finite group, and let A be a left G-group. A *principal homogeneous G-set* over A, or *A-torsor*, is a left G-set P that admits a principal transitive right A-action, such that ${}^{\sigma}(x\cdot a) = {}^{\sigma}x\cdot{}^{\sigma}a$ for all $x \in P$, $\sigma \in G$, and $a \in A$. We say two A-torsors are *isomorphic* if they admit a G- and A-equivariant set bijection. Let

$$A\text{-}\mathrm{Tors}_G$$

denote the pointed set of isomorphism classes of A-torsors, with distinguished element the left G-set A, with the obvious right A-action.

Main Example. For each $c \in \mathrm{Z}^1(G,A)$, let ${}_cA$ denote the set A with "c-twisted" G-action

$$\sigma \star_c x \overset{\mathrm{df}}{=} c_\sigma {}^{\sigma}x$$

and right A-action $x\cdot a = xa$. Then ${}_cA$ is an A-torsor, an "affine" set for A.

Proposition 3.1. *There is a pointed-set isomorphism*

$$g : A\text{-}\mathrm{Tors}_G \longrightarrow \mathrm{H}^1(G,A)$$

taking an A-torsor P to the class $[c]$ of the cocycle c defined by ${}^{\sigma}x = x \cdot c_\sigma$ for some $x \in P$. A different $x \in P$ determines a cohomologous cocycle, inducing a surjection from elements $x' \in P$ to cocycles c' cohomologous to c. The inverse of g is given by $g^{-1}([c]) = {}_cA$.

Proof. This is [15, Proposition 33]. If P is an A-torsor and $x \in P$ then for each $\sigma \in G$ we have ${}^{\sigma}x = x \cdot c_\sigma$ for some $c_\sigma \in A$, and we claim $c = (c_\sigma)$ is in $\mathrm{Z}^1(G, A)$, i.e., $c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau$. Let's check:

$${}^{\sigma\tau}x = x \cdot c_{\sigma\tau} = {}^{\sigma}({}^{\tau}x) = {}^{\sigma}(x \cdot c_\tau) = ({}^{\sigma}x) \cdot {}^{\sigma}c_\tau = x \cdot c_\sigma {}^{\sigma}c_\tau$$

Since P is principal, we conclude $c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau$, as desired. It is easy to see that varying x varies the cocycle in the class $[c]$, so the map g is well-defined, and if $c'_\sigma = b^{-1} c_\sigma {}^{\sigma}b$ then c' arises from $x' = xb$. Therefore each cocycle in the class of $[c]$ arises from some $x \in P$. Since $\mathrm{id} \in A$ is G-invariant, $g(A) = [1]$, so g is a map of pointed sets.

We show

$$g' : \mathrm{H}^1(G, A) \longrightarrow A\text{-Tors}_G$$
$$[c] \longmapsto {}_cA$$

defines an inverse. It is well defined, since if $c \sim c'$ and $c_\sigma = b^{-1} c'_\sigma {}^{\sigma}b$ for some $b \in A$, then left multiplication by b is an A-torsor isomorphism ${}_cA \to {}_{c'}A$:

$$b(\sigma \star_c x) = b c_\sigma {}^{\sigma}x = c'_\sigma {}^{\sigma}b {}^{\sigma}x = \sigma \star_{c'} bx \checkmark$$
$$b(x \cdot a) = (bx) \cdot a \checkmark$$

To compute $g \cdot g'$ we must compute $g({}_cA)$. Fixing $x \in {}_cA$, we obtain a cocycle c' by the formula $\sigma \star_c x = x \cdot c'_\sigma$. Since $\sigma \star_c x = c_\sigma {}^{\sigma}x$ by the definition of the c-twisted action, we have $c'_\sigma = x^{-1} c_\sigma {}^{\sigma}x$ for $x \in A$, and we conclude $g({}_cA) = [c'] = [c]$, proving $g \cdot g' = \mathrm{id}$. To show $g' \cdot g = \mathrm{id}$, suppose $P \in A\text{-Tors}_G$ and $x \in P$ defines the cocycle (c_σ), so that $g(P) = [c]$. Define a function

$$\phi : P \longrightarrow {}_cA$$
$$x \cdot a \longmapsto a$$

We claim ϕ is an A-torsor isomorphism. It is G-equivariant:

$$\phi({}^{\sigma}(x \cdot a)) = \phi(x \cdot c_\sigma {}^{\sigma}a) = c_\sigma {}^{\sigma}a = \sigma \star_c x \checkmark$$

It is A-equivariant: $\phi(x \cdot a) = a = e \cdot a = \phi(x \cdot e) \cdot a = \phi(x) \cdot a \checkmark$. This proves the claim. Finally, since g' clearly maps the distinguished class $[1] \in \mathrm{H}^1(G, A)$ to the distinguished A-torsor A, g' is an isomorphism of pointed sets, and $g' = g^{-1}$, as desired.

□

3.3. **Twisted G-Action.** Let $\mathscr{C}$ and $\mathscr{C}[G]$ be the concrete categories of Definition 2.1. Let (V, α) be an object of $\mathscr{C}[G]$, and set $A = \mathrm{Aut}_{\mathscr{C}}(V)$. Then A is a left G-group via the action ${}^{\sigma}a = \sigma \cdot a \cdot \sigma^{-1}$ for $a \in A$ and $\sigma \in G$, and A acts on V on the left, so that ${}^{\sigma}(a(x)) = {}^{\sigma}a \cdot \sigma(x)$ for $a \in A$, $\sigma \in G$, and $x \in V$.

If $(V, \alpha), (V', \alpha')$ are in $\mathscr{C}[G]$, the induced left G-action and right A-action on $\mathrm{Isom}_{\mathscr{C}}(V, V')$ are given by ${}^{\sigma}\phi = \sigma \cdot \phi \cdot \sigma^{-1}$ and $\phi^a = \phi \cdot a$.

Proposition 3.2. *If (V, α) and (V', α') are in $\mathscr{C}[G]$ and V and V' are $\mathscr{C}$-isomorphic, then $\mathrm{Isom}_{\mathscr{C}}(V, V')$ is an A-torsor.*

Proof. The A-action is G-equivariant: ${}^{\sigma}(\phi \cdot b) = \sigma \cdot \phi \cdot b \cdot \sigma^{-1} = \sigma \cdot \phi \cdot \sigma^{-1} \cdot \sigma \cdot b \cdot \sigma^{-1} = {}^{\sigma}\phi \cdot {}^{\sigma}b$ for $\sigma \in G$, $b \in A$, and $\phi \in \mathrm{Isom}_{\mathscr{C}}(V, V')$. The A-action is principal: If $\phi \in \mathrm{Isom}_{\mathscr{C}}(V, V')$ and $b \in A$ then $\phi \cdot b = \phi$ if and only if $b = \phi \cdot \phi^{-1} = 1$. The A-action is transitive: If $\phi, \phi' \in \mathrm{Isom}_{\mathscr{C}}(V, V')$ then $\phi' = \phi \cdot b$ for $b = \phi^{-1} \cdot \phi'$, and b is in A.

□

Observe that if (V, α) and (V', α') are $\mathscr{C}$-isomorphic then for each $\phi \in \mathrm{Isom}_{\mathscr{C}}(V, V')$ there is a (possibly not commutative) diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V' \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\sigma} \\ V & \xrightarrow[\phi]{} & V' \end{array}$$

for all $\sigma \in G$. The failure of the diagram to commute is encoded in the *cocycle*

$$c_\sigma = \phi^{-1} \cdot {}^{\sigma}\phi \ \in \mathrm{Z}^1(G, A) \tag{3.3}$$

This cocycle is the identity if and only if ϕ is a G-isomorphism.

Definition 3.4. Suppose (V, α) is in $\mathscr{C}[G]$, and $A = \mathrm{Aut}_{\mathscr{C}}(V)$. For each $c \in \mathrm{Z}^1(G, A)$ let $(V, {}_c\alpha)$ (or ${}_cV$ if α is understood) denote the object V with *twisted* G-action given by

$$\sigma \star_c x = c_\sigma \sigma(x)$$

for all $x \in V$.

Proposition 3.5. *Suppose (V, α) and (V', α') are objects in $\mathscr{C}[G]$, V' is $\mathscr{C}$-isomorphic to V, and $\phi \in \mathrm{Isom}_{\mathscr{C}}(V, V')$. Then $[(V', \alpha')] = [(V, {}_c\alpha)]$ for the cocycle $c \in \mathrm{Z}^1(G, A)$ of* (3.3)*.*

Proof. We check that the diagram

$$\begin{array}{ccc} {}_cV & \xrightarrow{\phi} & V' \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\sigma} \\ {}_cV & \xrightarrow[\phi]{} & V' \end{array}$$

commutes for all $\sigma \in G$, proving $(V, {}_c\alpha)$ and (V', α') are G-isomorphic via ϕ.

□

Theorem 3.6. *Fix $(V,\alpha) \in \mathscr{C}[G]$, put $A = \mathrm{Aut}_{\mathscr{C}}(V)$, and let $[V]/G$ be the pointed set of Definition 2.1. Let $[(V,\alpha)]$, A, and $[1]$ be distinguished elements of $[V]/G$, A-Tors_G, and $\mathrm{H}^1(G,A)$, respectively. Then we have pointed-set isomorphisms*

$$[V]/G \xrightarrow{\ f\ } A\text{-}\mathrm{Tors}_G \xrightarrow{\ g\ } \mathrm{H}^1(G,A)$$

where f, g, and $h = g\cdot f$ and their inverses are as follows:

(*a*) *$f([(V',\alpha')]) = \mathrm{Isom}_{\mathscr{C}}(V,V')$;*
(*b*) *$g(P) = [c]$ for $c \in \mathrm{Z}^1(G,A)$ defined by ${}^{\sigma}x = x\cdot c_\sigma$ for any $x \in P$, and $g^{-1}([c]) = {}_cA$;*
(*c*) *$h([(V',\alpha')]) = [c]$ for $c \in \mathrm{Z}^1(G,A)$ defined by $c_\sigma = \phi^{-1}\cdot {}^{\sigma}\phi$ for any $\phi \in \mathrm{Isom}_{\mathscr{C}}(V,V')$, and $h^{-1}([c]) = [(V, {}_c\alpha)]$.*

Proof. The map g and its inverse were established in Proposition 3.1.

We show f is well-defined. Suppose V', V'' are in $[V]$, and (V'',α'') is in $[(V',\alpha')]$. Then any G-isomorphism $\psi : V' \to V''$ induces a map $\psi^* : \mathrm{Isom}_{\mathscr{C}}(V,V') \to \mathrm{Isom}_{\mathscr{C}}(V,V'')$ defined by $\psi^*(\phi) = \psi\cdot\phi$, which we claim is an A-torsor isomorphism. It is clearly a set-isomorphism, since ψ is an isomorphism, and it is G-equivariant since ψ is G-equivariant: For if $\phi \in \mathrm{Isom}_{\mathscr{C}}(V,V')$ we compute

$$\psi^*({}^{\sigma}\phi) = \psi\cdot {}^{\sigma}\phi = \psi\cdot(\sigma\cdot\phi\cdot\sigma^{-1}) = \sigma\cdot(\psi\cdot\phi)\cdot\sigma^{-1} = {}^{\sigma}\psi^*(\phi)$$

Showing A-equivariance is trivial, and we conclude $\mathrm{Isom}_{\mathscr{C}}(V,V')$ and $\mathrm{Isom}_{\mathscr{C}}(V,V'')$ are isomorphic A-torsors. This shows f is well defined. Since $f([(V,\alpha)]) = \mathrm{Isom}_{\mathscr{C}}(V,V) = A$, f is a map of pointed sets.

We show f is onto. If $P \in A$-Tors_G then $P = {}_cA$ for some $c \in \mathrm{Z}^1(G,A)$ by Proposition 3.1. It is trivial to check that ${}_cA = \mathrm{Isom}_{\mathscr{C}}(V, {}_cV)$, hence $P = f([(V,{}_c\alpha])$, hence f is onto.

We show f is 1-1. Suppose $f([(V',\alpha')]) = f([(V'',\alpha'')])$ via an A-torsor isomorphism $\theta : \mathrm{Isom}_{\mathscr{C}}(V,V') \to \mathrm{Isom}_{\mathscr{C}}(V,V'')$, and ϕ is any element in $\mathrm{Isom}_{\mathscr{C}}(V,V')$. We claim $\psi = \theta(\phi)\cdot\phi^{-1} : V' \to V''$ is a G-isomorphism. It is an isomorphism as a composition of isomorphisms. Since θ is A-equivariant and $\mathrm{Isom}_{\mathscr{C}}(V,V')$ is an A-torsor, we compute for any $\phi' \in \mathrm{Isom}_{\mathscr{C}}(V,V')$,

$$\psi^*(\phi') = \psi\cdot\phi' = \theta(\phi)(\phi^{-1}\cdot\phi') = \theta(\phi')$$

hence $\psi^* = \theta$. Since θ is G-equivariant we have $({}^{\sigma}\theta)(\phi') = \theta(\phi')$, i.e., $\sigma\cdot\psi\cdot\sigma^{-1}\cdot\phi' = \psi\cdot\phi'$, since $\theta = \psi^*$. Since ϕ' is a bijection, we conclude ${}^{\sigma}\psi = \psi$ for all σ, i.e., ψ is G-equivariant. This shows f is 1-1, hence f is a pointed-set isomorphism.

We compute $h = g\cdot f$. Suppose $[(V',\alpha')] \in [V]/G$. Then $f([(V',\alpha')]) = \mathrm{Isom}_{\mathscr{C}}(V,V')$ by (i), and so $h([(V',\alpha')]) = [c]$ where c is defined by ${}^{\sigma}\phi = \phi\cdot c_\sigma$ for any $\phi \in \mathrm{Isom}_{\mathscr{C}}(V,V')$. Then $c_\sigma = \phi^{-1}\cdot {}^{\sigma}\phi$, as desired.

It remains to compute h^{-1}. But if $h([(V',\alpha')]) = [c]$ then $f([(V',\alpha')]) = {}_cA = \mathrm{Isom}_{\mathscr{C}}(V, {}_cV)$, hence $[(V',\alpha')] = [(V, {}_c\alpha)]$, so $h^{-1}([c]) = [(V, {}_c\alpha)]$. This completes the proof.

□

4. Twisted Forms, Rationality, and Galois Descent

Definition 4.1. Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be a functor that takes a field k to a concrete category $\mathscr{C}_k$, and a morphism $k \to K$ of fields to the *scalar extension functor*

$$E : \mathscr{C}_k \longrightarrow \mathscr{C}_K$$

If V is an object in $\mathscr{C}_k$ we write V_K for V's scalar extension to K. For each $V \in \mathscr{C}_k$ the *twisted forms of V relative to K* is the pointed set of isomorphism classes

$$\mathrm{TF}_{K|k}(V) = \{[V'] : V' \in \mathscr{C}_k \text{ and } V'_K \simeq V_K \text{ in } \mathscr{C}_K\}$$

with distinguished element $[V]$.

Conversely, we say an object W of $\mathscr{C}_K$ has a *k-structure* V if there exists an object V in $\mathscr{C}_k$ such that $V_K \simeq W$. A morphism $g : W \to W'$ between objects with k-structures V and V' has a *k-structure* if there is a morphism $f : V \to V'$ such that $f_K = g$. If W and g have k-structures we say they are *defined over* k, and *rational over* k (see [4] and [5]).

Let G be a finite group. Recall a field extension K/k is *G-Galois* if G acts by field automorphisms on K and k is the field of fixed points. We want to extend this notion to $\mathscr{C}_K$, defining a functor

$$F : \mathscr{C}_K[G] \longrightarrow \mathscr{C}_k$$

that assigns to each object W in $\mathscr{C}_K[G]$ a G-fixed object V in $\mathscr{C}_k$ and to each G-equivariant morphism a morphism of the G-fixed objects. If $\mathscr{C}$ is covariant then every G-invariant map $W' \to W$ should factor through a morphism $i : V \to W$, and we think of V as a *subobject* of W whose elements are *G-fixed points*. If $\mathscr{C}$ is contravariant then every G-invariant $W \to W'$ should factor through a morphism $p : W \to V$, and we think of V as a *quotient* of W, and the fibers of p as *G-orbits*.

The main goal of this paper is to describe situations in which we have the following theorem:

Theorem 4.2. *The scalar extension and fixed point functors define a category equivalence*

$$\mathscr{C}_k \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{C}_K[G]$$

Suppose V is an object of $\mathscr{C}_k$. Then there is a pointed-set isomorphism

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_{\mathscr{C}_K}(V_K)) &\longrightarrow \mathrm{TF}_{K|k}(V) \\ [c] &\longmapsto [({}_cV_K)^G] \end{aligned}$$

whose inverse takes a class $[V'] \in \mathrm{TF}_{K|k}(V)$ *to the class* $[c]$ *defined by* $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$*, for any* $\phi \in \mathrm{Isom}_{\mathscr{C}_K}(V_K, V'_K)$*, as in Theorem 3.6.*

Definition 4.3. In the situation of Theorem 4.2 we say the G-action is *Galois*, and an object V in $\mathscr{C}_k$ in the image of F is obtained from $\mathscr{C}_K$ by *Galois descent*.

For Theorem 4.2 to be realized we require that all extended objects in $\mathscr{C}_K$ be in $\mathscr{C}_K[G]$ and that all objects of $\mathscr{C}_k$ be obtained by Galois descent. We then require that E and F be inverse, and that for each object V in $\mathscr{C}_k$ they induce a category equivalence $\mathrm{TF}_{K|k}(V) \longleftrightarrow [V_K]/G$. Then we invoke Theorem 3.6, which relates $[V_K]/G$ to $\mathrm{H}^1(G, \mathrm{Aut}_{\mathscr{C}_K}(V_K)$, and shows that each object of $\mathscr{C}_k$ is the fixed object V_K under the twisted G-actions defined by elements $c \in \mathrm{Z}^1(G, \mathrm{Aut}_{\mathscr{C}_K}(V_K))$.

5. Vector Spaces with Tensors

We will investigate twisted forms and Galois descent on the concrete categories of vector spaces, algebras, and more generally vector spaces with the additional structure given by a multilinear "tensor". We now define these objects, using [5] as reference.

5.1. Algebras.

A K-algebra is a K-vector space V equipped with a (K-bilinear and associative) multiplication map $\Phi : V \otimes_K V \to V$. We prove below that there is a natural K-vector space isomorphism $\mathrm{Hom}_K(V \otimes_K V, V) \simeq V \otimes_K V^* \otimes_K V^*$, consequently we will write

$$\Phi \in V \otimes V^{*\otimes 2}$$

and call Φ a *tensor of type* $(1, 2)$.

Proposition 5.1. *There is a natural* K*-vector space isomorphism*

$$\mathrm{Hom}_K(V \otimes_K V, V) \simeq V \otimes_K V^* \otimes_K V^*$$

Proof. We have a canonical isomorphism $V^{*\otimes 2} \to \mathrm{Hom}_K(V \otimes_K V, K) = (V^{\otimes 2})^*$ by the universal property of the tensor product. The canonical isomorphism $V \simeq \mathrm{Hom}_K(K, V)$ then defines a K-balanced pairing $(V, (V^{\otimes 2})^*) \to \mathrm{Hom}_K(V^{\otimes 2}, V)$ by composition of functions, hence a unique K-linear map $V \otimes (V^{\otimes 2})^* \to \mathrm{Hom}_K(V^{\otimes 2}, V)$, hence a K-map

$$\begin{aligned} T : V \otimes V^{*\otimes 2} &\longrightarrow \mathrm{Hom}_K(V^{\otimes 2}, V) \\ v \otimes \phi \otimes \psi &\longmapsto (x \otimes y \mapsto \phi(x)\psi(y)v) \end{aligned}$$

If $\{e_i\}$ and $\{\phi_j\}$ are dual bases for V and V^* then $\{e_i \otimes e_j\}$ is a basis for $V^{\otimes 2}$, and since $T(e_k \otimes \phi_i \otimes \phi_j)(e_{i'} \otimes e_{j'}) = \delta_{ii'}\delta_{jj'}e_k$ we find $\{T(e_k \otimes \phi_i \otimes \phi_j)\}$ is a basis for $\mathrm{Hom}_K(V^{\otimes 2}, V)$, which shows that T is an isomorphism. The inverse of T is given by

$$\begin{aligned} T^{-1} : \mathrm{Hom}_K(V^{\otimes 2}, V) &\longrightarrow V \otimes V^{*\otimes 2} \\ \Phi &\longmapsto \sum_{i,j,k} a_{ijk} e_k \otimes \phi_i \otimes \phi_j \end{aligned}$$

where the constants a_{ijk} are determined by the expression $\Phi(e_i \otimes e_j) = \sum_k a_{ijk} e_k$.

$\square$

If A and B are K-algebras represented by (V, Φ) and (W, Ψ), then a K-algebra homomorphism $\varphi : A \to B$ is map $f : (V, \Phi) \to (W, \Psi)$, where $f : V \to W$ is a K-vector space map such that $f \cdot \Phi = \Psi \cdot (f \otimes f)$.

Various K-algebra properties amount to conditions on Φ. To say that $A = (V, \Phi)$ is *associative* is to say that $\Phi \cdot (\mathrm{id}_V \otimes \Phi) = \Phi \cdot (\Phi \otimes \mathrm{id}_V)$. To say that A is *commutative* is to say that $\Phi = \Phi \cdot \tau$, where $\tau : V \otimes_K V \xrightarrow{\sim} V \otimes_K V$ is the canonical transposition isomorphism. To say that A has a *unit element*, see [5, p.432]. To say that A is a *central simple K-algebra of degree n* is to say $\Phi \in V \otimes V^{*\otimes 2}$, and then that (V, Φ) becomes isomorphic to $(M_n(L), \Psi_L)$ over some finite extension L/K, where Ψ_L is the multilinear map describing matrix multiplication.

5.2. **More General Tensors.** More generally, a *tensor of type* (n, m), with $n, m \geq 0$, is an element of $V^{\otimes n} \otimes_K V^{*\otimes m}$. We consider the set of pairs (V, Φ), with V a K-vector space and Φ a tensor of type (n, m). A tensor of type (n, m) on V gives an element of $\mathrm{Hom}_K(V^{\otimes m}, V^{\otimes n})$: If

$$\Phi = \sum_{IJ} a_{IJ} (e_{j1} \otimes \cdots \otimes e_{jn}) \otimes (\phi_{i1} \otimes \cdots \otimes \phi_{im}),$$

then we define $\Phi \in \mathrm{Hom}_K(V^{\otimes m}, V^{\otimes n})$ by

$$\Phi(v_1 \otimes \cdots \otimes v_m) = \sum_{IJ} a_{IJ} \phi_{i1}(v_1) \cdots \phi_{im}(v_m) e_{j1} \otimes \cdots \otimes e_{jn}$$

This correspondence is bijective. We have the following familiar structures.

1. If Φ has type $(0, 0)$, then (V, Φ) is a K-vector space.
2. If Φ has type $(0, 2)$, then (V, Φ) is a K-quadratic space.
3. If Φ has type $(1, 2)$, then (V, Φ) is a K-algebra.

Definition 5.2. The *category of vector spaces with tensor of type* (n, m) is the category whose objects are pairs (V, Φ), where V is a K-vector space and Φ is a tensor of type (n, m) (which we will view as an element of $\mathrm{Hom}_K(V^{\otimes m}, V^{\otimes n})$), and whose morphisms are maps $f : (V, \Phi) \to (W, \Psi)$ given by a morphism $f : V \to W$ of K-vector spaces such that $f \cdot \Phi = \Psi \cdot f$, where f acts on $V^{\otimes m}$ and $V^{\otimes n}$ by acting on each tensor factor.

6. Galois Descent for Vector Spaces.

Definition 6.1. Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field k the category of k-vector spaces with k-linear maps, and to each morphism $k \to K$ the *scalar* *extension functor*

$$\begin{aligned} E : \mathscr{C}_k &\longrightarrow \mathscr{C}_K \\ V &\longmapsto V_K := V \otimes_k K \\ \phi &\longmapsto \phi_K := \phi \otimes_k \mathrm{id}_K \end{aligned}$$

Let K/k be a finite Galois extension with group G. Let W be a K-vector space, and let $\alpha : G \to \mathrm{Aut}_{\mathscr{C}_k}(W)$ be an action in $\mathscr{C}_k$, where W is viewed as a k-vector space via the map $k \to K$. We say the action α is *Galois* (or *semilinear*) if $\sigma(aw) = \sigma(a)\sigma(w)$ for each $a \in K$ and $w \in W$. As in Definition 2.1, let $\mathscr{C}_K[G]$ denote the category whose objects are pairs (W, α) where W is a K-vector space and α is a Galois G-action, and whose morphisms are G-equivariant morphisms.

Every K-vector space admits at least one Galois G-action, given by fixing a basis for W and letting G act on coefficients. In particular the action of G on the right tensor factor of $V \otimes_k K$ is Galois, since for all $a, a' \in K$, $\sigma \in G$, and $v \in V$,

$$\sigma(a'(v \otimes a)) = v \otimes \sigma(a'a) = v \otimes \sigma(a')\sigma(a) = \sigma(a')(v \otimes \sigma(a)) = \sigma(a')\sigma(v \otimes a).$$

Moreover, if $\phi : V \to V'$ is in $\mathscr{C}_k$, then $\phi \otimes \mathrm{id}_K : V \otimes_k K \to V' \otimes_k K$ is a G-equivariant map of K-vector spaces, since for all $\sigma \in G$, $v \in V$, and $a \in K$ we compute

$$\begin{aligned} (\phi \otimes \mathrm{id}_K \cdot \sigma)(v \otimes a) &= \phi \otimes \mathrm{id}_K(v \otimes \sigma(a)) = \phi(v) \otimes \sigma(a) = \sigma(\phi(v) \otimes a) \\ &= (\sigma \cdot \phi \otimes \mathrm{id}_K)(v \otimes a) \end{aligned}$$

Thus the extension functor actually maps into $\mathscr{C}_K[G]$:

$$E : \mathscr{C}_k \longrightarrow \mathscr{C}_K[G]$$

Definition 6.2. Let $\mathscr{C}_K[G]$ be the category of K-vector spaces with Galois G-action, as in Definition 4.3. The *fixed point functor*

$$F : \mathscr{C}_K[G] \longrightarrow \mathscr{C}_k$$

is defined by $W \mapsto W^G$ and $\psi \mapsto \psi|_{W^G}$

We show F is well-defined. If W is an object of $\mathscr{C}_K[G]$ then since the G-action on W is Galois, multiplication by k stabilizes W^G, so W^G is a k-vector space. Then, since any morphism $\psi : W \to W'$ in $\mathscr{C}_K[G]$ is G-equivariant, the restriction of ψ to W^G has image in $(W')^G$. Since F clearly takes the identity morphism to the identity and preserves composition of morphisms, it is a well-defined functor.

Theorem 6.3. *Let K/k be a finite Galois extension with group G, and let $\mathscr{C}_k$, $\mathscr{C}_K$, and $\mathscr{C}_K[G]$ be the categories of vector spaces in Definition 6.1. Then:*

(i) *The scalar extension and fixed point functors define a category equivalence*

$$\mathscr{C}_k \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{C}_K[G]$$

(ii) *Suppose V is an object of $\mathscr{C}_k$. Then there is an isomorphism of trivial pointed sets*

$$\mathrm{H}^1(G, \mathrm{GL}(V_K)) \longrightarrow \mathrm{TF}_{K|k}(V)$$
$$[c] \longmapsto [({}_cV_K)^G]$$

where ${}_cV_K \in \mathscr{C}_K[G]$ is V_K with twisted Galois G-action $\sigma \star v = c_\sigma \sigma(v)$. The inverse takes a class $[V'] \in \mathrm{TF}_{K|k}(V)$ to the class $[c]$ defined by $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$, for any $\phi \in \mathrm{Isom}_{\mathscr{C}_K}(V_K, V'_K)$. In particular, $\mathrm{H}^1(G, \mathrm{GL}(V_K)) = \{[1]\}$, $\mathrm{TF}_{K|k}(V) = \{[V]\}$, and any Galois G-action on a K-vector space is equivalent to the standard one on the coefficients of some basis.

Proof. For (i), it is obvious that $F \circ E$ equals the identity, since the G-action on $V \otimes_k K$ is via the right tensor factor. Thus every object of $\mathscr{C}_k$ is obtained by Galois descent from its own extension. It remains to show that $E \circ F$ is isomorphic to the identity in $\mathscr{C}_K[G]$, i.e., that each object of $\mathscr{C}_K[G]$ has a k-structure, equal to its fixed point set. We state this as a lemma.

Lemma 6.4. *Let K/k be a finite Galois extension with group G, W is an object of $\mathscr{C}_K[G]$, and $f : W \to W'$ is a morphism in $\mathscr{C}_K[G]$. Then W is G-isomorphic to $W^G \otimes_k K$, and $f \simeq (f|_{W^G}) \otimes_k \mathrm{id}_K$. Thus $E \circ F$ is isomorphic to the identity in $\mathscr{C}_K[G]$.*

Proof. Let $W^G \otimes_k K$ be K-vector space with (left) G-action on the right factor. We will show the canonical map

$$\phi : W^G \otimes_k K \longrightarrow W$$
$$v \otimes a \longmapsto av$$

is a G-isomorphism. Let $\{x_1, \ldots, x_n\}$ be a k-basis for K. Write $G = \{\sigma_1, \ldots, \sigma_n\}$, with $\sigma_1 = e$. Fix $w \in W$, and consider the n elements

$$v_j = \sum_{i=1}^{n} \sigma_i(x_j w) = \sum_{i=1}^{n} \sigma_i(x_j)\sigma_i(w) \in W^G$$

By linear independence of characters [11, VI.5.4], the matrix $(\sigma_i(x_j)) \in M_n(K)$ is invertible, so that we may invert this system, and in particular express $w = \sigma_1(w)$ as a K-linear combination of the v_j. Thus ϕ is onto.

We show ϕ is G-equivariant:

$$\phi(\sigma(v \otimes x)) = \phi(v \otimes \sigma(x)) = \sigma(x)v = \sigma(xv) = \sigma(\phi(v \otimes x)).$$

We show ϕ is 1-1: Since ϕ is G-equivariant, $\ker(\phi)$ is stable under G: If $\phi(\sum v_i \otimes x_i) = 0$ then $\phi(\sigma(\sum v_i \otimes x_i)) = \sigma(\phi(\sum v_i \otimes x_i)) = 0$. Let $\{e_i\}_I$ be a k-basis for W^G, then $\{e_i \otimes 1\}_I$ is a K-basis for $W^G \otimes_k K$. Since $\phi(v \otimes 1) = v$, $\ker(\phi) \cap (W^G \otimes_k 1) = (0)$. Suppose $\ker(\phi) \neq (0)$, and $w \in \ker(\phi)$ is a nonzero element whose expression $w = \sum_{i=1}^m a_i e_i \otimes 1$ involves the fewest number of nonzero $a_i \in K$, of all nonzero elements of $\ker(\phi)$. We may assume that $a_1 = 1$ is nonzero, after reordering and rescaling if necessary. Since $w \notin W^G \otimes_k 1$, we may assume $a_2 \notin k$. Then there exists a $\sigma \in G$ such that $\sigma(a_2) \neq a_2$, hence $\sigma(w) - w \neq 0$, and since $\ker(\phi)$ is stable under G we have $\sigma(w) - w \in \ker(\phi)$. But since the basis e_i is in k, $\sigma(w)$ is supported on the same e_i as is w, so $\sigma(w) - w$ has a shorter expression than w, contradiction. This proves $\ker(\phi) = (0)$, hence that $E \circ F$ is isomorphic to the identity on objects in $\mathscr{C}_K[G]$.

If $f : W \to W'$ is a G-equivariant K-morphism, and ϕ and ϕ' are the maps defined above, then it is immediate that $f|_{W^G} \otimes \mathrm{id}_K : W^G \otimes_k K \to (W')^G \otimes_k K$ is a G-equivariant K-morphism that equals $(\phi')^{-1} \cdot f \cdot \phi : W^G \otimes_k K \to (W')^G \otimes_k K$. Thus $E \circ F$ is isomorphic to the identity in $\mathscr{C}_K[G]$.

□

It remains to prove (ii). The set $\mathrm{TF}_{K|k}(V)$ consists of the one class $[V]$, since if V' is any k-vector space such that $V'_K \simeq V_K$ then the k-dimension of V' equals the K-dimension of V_K, since the scalar extension functor preserves dimension. Since vector spaces are classified up to isomorphism by dimension, we conclude all twisted forms of V are isomorphic to V, hence $\mathrm{TF}_{K|k}(V) = \{[V]\}$.

Since $\mathrm{GL}(V_K)$ is K-linear, we see immediately that the G-action on V_K is Galois if and only if the G-action on ${}_cV_K$ is Galois for each $c \in \mathrm{Z}^1(G, \mathrm{GL}(V_K))$. Since each W' isomorphic to V_K is G-isomorphic to some ${}_cV_K$ by Theorem 3.6, the set $[V_K]/G$ is in $\mathscr{C}_K[G]$, and by Theorem 3.6 again, we have an isomorphism $[V_K]/G \xrightarrow{\sim} \mathrm{H}^1(G, \mathrm{GL}(V_K))$. On the other hand, by (i) the categories $\mathscr{C}_k$ and $\mathscr{C}_K[G]$ are equivalent, and since an equivalence preserves isomorphism classes, we have a pointed-set isomorphism $\mathrm{TF}_{K|k}(V) \xrightarrow{\sim} [V_K]/G$ given by $[V'] \mapsto [V'_K]_G$, whose inverse is the fixed point functor. Composing these isomorphisms yields the induced pointed-set isomorphisms in the statement of Theorem 6.3. In particular, $\mathrm{H}^1(G, \mathrm{GL}(V_K)) = \{[1]\}$. This completes the proof.

□

7. Galois Descent for Vector Spaces with Tensor

We aim to apply descent to vector spaces with the additional structure provided by a tensor. By Theorem 6.3 there is only one k-structure on a given K-vector space, since vector spaces are classified (up to isomorphism) by dimension. Adding structure to the objects and morphisms restricts the automorphism groups and subdivides the isomorphism classes, creating the prospect of a nontrivial theory of twisted forms.

Definition 7.1. Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor of Definition 5.2, taking a field k to the concrete category $\mathscr{C}_k$ of k-vector spaces with tensor of type (n, m), and a morphism

$k \to K$ of fields to the extension functor

$$E : \mathscr{C}_k \longrightarrow \mathscr{C}_K$$

defined by $E(V, \Phi) = (V \otimes_k K, \Phi \otimes_k \mathrm{id}_K)$ and $E(f) = f \otimes_k \mathrm{id}_K$.

Suppose K/k is a finite Galois extension with group G, and (W, Ψ) is an object of $\mathscr{C}_K$. A left G-action on (W, Ψ) in $\mathscr{C}_k$ is *Galois* if it is Galois on W, and Ψ is G-equivariant: if $\Psi \in \mathrm{Hom}_K(W^{\otimes m}, W^{\otimes n})$ then $\sigma \cdot \Psi = \Psi \cdot \sigma$ for each $\sigma \in G$, where σ acts on $W^{\otimes m}$ and $W^{\otimes n}$ by acting on each tensor factor. Let

$$\mathscr{C}_K[G]$$

denote the category whose objects are pairs $((W, \Psi), \alpha)$ with W a K-vector spaces with tensor Ψ of type (n, m) and Galois G-action α, and whose morphisms are G-equivariant morphisms. (If (W, Ψ) is an algebra, i.e., $\Psi \in \mathrm{Hom}_K(W^{\otimes 2}, W)$, then this means each σ is a ring automorphism.) We will omit explicit reference to the action α, and just write (W, Ψ) for a K-vector space with tensor Ψ and some Galois action. If (V, Φ) is in $\mathscr{C}_k$ let (V_K, Φ_K) denote the object (V_K, Φ_K) with standard G-action on K scalars. This action is Galois, so under this convention E takes $\mathscr{C}_k$ into $\mathscr{C}_K[G]$:

$$E : \mathscr{C}_k \longrightarrow \mathscr{C}_K[G]$$

Let

$$F : \mathscr{C}_K[G] \longrightarrow \mathscr{C}_k$$

denote the *fixed point functor*, which takes an object (W, Ψ) to $(W, \Psi)^G := (W^G, \Psi|_{(W^G)^{\otimes m}})$ and a morphism $f : (W, \Psi) \to (W', \Psi')$ to $f|_{(W,\Psi)^G}$.

Fix an object (V, Φ) of $\mathscr{C}_k$. As in Definition 4.1 and Definition 5.2, the set of twisted forms of (V, Φ) is the pointed set

$$\mathrm{TF}_{K|k}(V, \Phi) \overset{\mathrm{df}}{=} \{[(V', \Phi')] : (V', \Phi') \in \mathscr{C}_k,\ (V_K, \Phi_K) \simeq (V'_K, \Phi'_K)\}$$

Theorem 7.2. *Let K/k be a finite Galois extension with group G, and let $\mathscr{C}_k$, $\mathscr{C}_K$, and $\mathscr{C}_K[G]$ be the categories of vector spaces with tensor in Definition 7.1. Then:*

(i) *The scalar extension and fixed point functors define a category equivalence*

$$\mathscr{C}_k \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{C}_K[G]$$

(ii) *Suppose (V, Φ) is an object of $\mathscr{C}_k$. Then there is a pointed-set isomorphism*

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_K(\Phi_K)) &\longrightarrow \mathrm{TF}_{K|k}(V, \Phi) \\ [c] &\longmapsto [({}_c(V_K, \Phi_K))^G] \end{aligned}$$

where $({}_cV_K, {}_c\Phi_K) \in \mathscr{C}_K[G]$ is (V_K, Φ_K) with twisted Galois G-action $\sigma \star v = c_\sigma \sigma(v)$. The inverse takes a class $[(V', \Phi')]$ in $\mathrm{TF}_{K|k}(V, \Phi)$ to the class $[c]$ defined by $c_\sigma = g^{-1} \cdot {}^\sigma g$ for all $\sigma \in G$, for any $g \in \mathrm{Isom}_K(\Phi_K, \Phi'_K)$.

Proof. We start with (i). Because of Theorem 6.3 we only have to check that E and F are inverse on the tensors in each category. But if $(V, \Phi) \in \mathscr{C}_k$ it is obvious that $\Phi_K|_V = \Phi$, hence that $F \circ E$ equals the identity in $\mathscr{C}_k$. On the other hand if $(W, \Psi) \in \mathscr{C}_K[G]$ then it is easy to see the G-isomorphism $(W^G) \otimes_k K \simeq W$ of Theorem 6.3, which is $E \circ F$ on the underlying vector space, induces $(\Psi|_{W^G}) \otimes \mathrm{id}_K \simeq \Psi$. Thus $(W, \Psi)^G \otimes_k K \simeq (W, \Psi)$ in $\mathscr{C}_K[G]$, hence $E \circ F$ is isomorphic to the identity. This proves (i).

For (ii), we note that the set of G-isomorphism classes $[(V_K, \Phi_K)]/G$ is indeed in $\mathscr{C}_K[G]$. For since each c_σ is an automorphism in $\mathscr{C}_K$, $c_\sigma \cdot \Phi_K = \Phi_K \cdot c_\sigma$ by definition of $\mathrm{Aut}_K(\Phi_K)$. Therefore $c_\sigma\sigma \cdot \Phi_K = \Phi_K \cdot c_\sigma\sigma$, since the (untwisted) G-action is Galois. Furthermore since c_σ is K-linear we have the semilinear property $c_\sigma\sigma(av) = \sigma(a)c_\sigma\sigma(v)$ for all $a \in K$, $v \in V_K$. Therefore every twisted G-action on (V_K, Φ_K) is Galois, so $[(V_K, \Phi_K)]/G$ is in $\mathscr{C}_K[G]$.

The functorial equivalence $[(V_K, \Phi_K)]/G \longleftrightarrow \mathrm{TF}_{K|k}(V, \Phi)$ given by F and E now gives the desired pointed-set isomorphism exactly as in Theorem 6.3, and we omit the details.

□

7.1. **Example: Algebras of Finite Type.** Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field k the category $\mathscr{C}_k$ of commutative k-algebras of finite type. We check that the additional hypothesis (finite type) is compatible with Theorem 7.2. Suppose K/k is a finite Galois field extension with group G. If A is an object of $\mathscr{C}_k$ then $B = A_K$ is a K-algebra, and since K/k is finite, B/A is finite, hence B is a K-algebra of finite type. Therefore $E : \mathscr{C}_k \to \mathscr{C}_K[G]$ is well defined. Conversely if B is a K-algebra of finite type and B admits a Galois G-action, then B_k is a k-algebra of finite type and B is integral over $A = B^G$, hence A is a k-algebra of finite type by [3, Proposition 7.8]. Therefore $F : \mathscr{C}_K[G] \to \mathscr{C}_k$ is well defined, and Theorem 7.2 applies to this situation.

7.2. **Example: Étale k-Algebras.** A finite-dimensional commutative algebra A over a field k is a pair (V, Φ) where V is a finite dimensional k-vector space and Φ is a tensor of type $(1, 2)$ that commutes with the transposition of tensor factors in the domain. Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field k the category $\mathscr{C}_k$ of finite-dimensional commutative k-algebras, and k-algebra morphisms.

If K/k is a finite Galois field extension with group G, a left Galois G-action on an object A of $\mathscr{C}_K$ is a group action by ring automorphisms, such that $\sigma(ca) = \sigma(c)\sigma(a)$ for all $c \in K$ and $a \in A$. Let $\mathscr{C}_K[G]$ be the corresponding category of pairs (A, α). Since the action is Galois, the fixed point object A^G is a finite-dimensional commutative k-algebra, so the fixed point functor takes $\mathscr{C}_K[G]$ to $\mathscr{C}_k$, and we are in the situation of Theorem 7.2.

Let K^n denote the split étale K-algebra of degree n. Let $(K^n, \alpha) \in \mathscr{C}_K[G]$ be defined by the standard action ${}^\sigma(a_1, \ldots, a_n) = ({}^\sigma a_1, \ldots, {}^\sigma a_n)$, which is easily checked to be Galois. Then $(K^n)^G = k^n$.

Proposition 7.3. *Let K/k be a finite G-Galois extension, and let K^n be the K-algebra. Then the pointed set of twisted forms $\mathrm{TF}_{K|k}(k^n)$ consists of the étale k-subalgebras of K^n of degree n, which are precisely the k-algebras of the form $A = \prod_{i=1}^m J_i$ for fields J_i/k contained in K, such that $\sum_{i=1}^m [J_i : k] = n$.*

Proof. Let A be a k-subalgebra of K^n. Since k is a field, the ideals of A are finite-dimensional k-vector spaces, hence they satisfy the descending chain condition, hence A is artinian. Since A is artinian it is a direct product of local artinian k-algebras, and since K^n is reduced, each of these is a field extension of k. If J/k is one of A's direct factors, then J's image under one of the standard projections of K^n to K is nonzero, hence J is isomorphic to a subfield of K. Thus $A \simeq \prod_{i=1}^m J_i$ for fields J_i/k contained in K.

By Theorem 7.2 the twisted forms of k^n are the k-algebras of fixed points of K^n under Galois actions, hence they are isomorphic to k-subalgebras of K^n, hence they are of the form $A \simeq \prod_i J_i$, with J_i/k field extensions contained in K, such that $\sum_i [J_i : k] = n$.

Conversely, suppose A is a k-subalgebra of K^n of degree n, so $A \simeq \prod_{i=1}^m J_i$ with J_i/k field extensions contained in K, such that $\sum_i [J_i : k] = n$. Since each J_i is separable we may write $J_i = k[T]/(p_i)$ for a monic irreducible $p_i \in k[T]$, and since K/k is Galois we have $(J_i)_K \simeq K[T]/(p_i) \simeq K^{n_i}$ by the Chinese Remainder Theorem. Therefore $A_K \simeq \prod_i (J_i)_K \simeq \prod_i K^{n_i} \simeq K^n$. Since $A_K \simeq (k^n)_K$, A is a twisted form of k^n.

□

We compute $\mathrm{Aut}_K(K^n) = S_n$ in $\mathscr{C}_K$ the group of K-linear ring automorphisms of K^n, which are completely determined by their action on the n orthogonal idempotents. The induced left action ${}^{\sigma}b = \sigma \cdot b \cdot \sigma^{-1}$ is evidently trivial, since S_n permutes the components while G acts componentwise. Since S_n is a trivial G-group, the cocycle condition reads $c_{\sigma\tau} = c_\sigma c_\tau$, hence $\mathrm{Z}^1(G, S_n) = \mathrm{Hom}(G, S_n)$. Two homomorphisms $c, c' : G \to S_n$ are cohomologous if $c'_\sigma = b^{-1} c_\sigma b$ for some fixed $b \in S_n$. Thus $\mathrm{H}^1(G, S_n)$ is the set of homomorphisms up to conjugacy, with distinguished element represented by the trivial homomorphism.

8. Galois Descent for Modules

Definition 8.1. Let A be a commutative k-algebra of finite type, and let $\mathsf{Mod} : \mathsf{Fields}/k \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field K containing k the category $\mathsf{Mod}_{\mathsf{A_K}}$ of A_K-modules and A_K-module homomorphisms, and to each morphism $k \to K$ the *scalar extension functor*

$$E : \mathsf{Mod}_{\mathsf{A}} \longrightarrow \mathsf{Mod}_{\mathsf{A_K}}$$

defined by $E(M) = M \otimes_A A_K = M \otimes_k K$ and $\phi \longmapsto \phi \otimes_k \mathrm{id}_K$.

Suppose K/k is a finite Galois extension with group G, and A_K admits a left Galois G-action, as in Definition 7.1. Every A_K-module N may be viewed as an A-module N_A via the map $A \to A_K$. We say a left G-action on N_A in $\mathsf{Mod}_{\mathsf{A}}$ is *Galois* (or *semilinear*) if

$\sigma(an) = \sigma(a)\sigma(n)$ for all $a \in A_K$ and $n \in N$. Let $\mathsf{Mod}_{\mathsf{A_K}}[G]$ denote the category whose objects are pairs (N, α) where N is an A_K-module and α is a Galois G-action, and whose morphisms are G-equivariant A_K-module morphisms. If M is an object in Mod_A then M_K admits a Galois G-action via the standard action on the K scalars, so that E takes Mod_A into $\mathsf{Mod}_{\mathsf{A_K}}[G]$:

$$E : \mathsf{Mod}_\mathsf{A} \longrightarrow \mathsf{Mod}_{\mathsf{A_K}}[G]$$

Let

$$F : \mathsf{Mod}_{\mathsf{A_K}}[G] \longrightarrow \mathsf{Mod}_\mathsf{A}$$

denote the *fixed point functor*, which takes N to N^G.

Note that a Galois action on an A_K-module N is also a Galois action on the K-vector space N as in Definition 6.1.

We show F is well-defined. If N is an object of $\mathsf{Mod}_{\mathsf{A_K}}[G]$ then since the G-action is Galois, multiplication by A stabilizes N^G, so N^G is an object of $\mod A$. Since any morphism $\psi : N \to N'$ in $\mathsf{Mod}_{\mathsf{A_K}}[G]$ is G-equivariant, we have $\psi|_{N^G} : N^G \to (N')^G$, hence $\psi|_{N^G}$ is in Mod_A. The remaining requirements are easy to check.

Theorem 8.2. *Suppose K/k is a finite Galois extension with group G, and A is a k-algebra of finite type. Let Mod_A and $\mathsf{Mod}_{\mathsf{A_K}}[G]$ be the categories of A and A_K-modules-with-Galois-action defined above. Then:*

(i) *The scalar extension and fixed point functors define a category equivalence*

$$\mathsf{Mod}_\mathsf{A} \underset{F}{\overset{E}{\rightleftarrows}} \mathsf{Mod}_{\mathsf{A_K}}[G]$$

(ii) *Suppose M is an object of Mod_A. Then there is a pointed-set isomorphism*

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_{A_K}(M_K) &\longrightarrow \mathrm{TF}_{K|k}(M) \\ [c] &\longmapsto [({}_cM_K)^G] \end{aligned}$$

whose inverse takes a class $[M'] \in \mathrm{TF}_{K|k}(M)$ to the class $[c]$ defined by $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$, for any $\phi \in \mathrm{Isom}_{A_K}(M_K, M'_K)$.

Proof. Since (a) A_k and A_K-modules are k and K-vector spaces, (b) the G-action on A_K-modules is Galois with respect to the K-vector space structure, and (c) E and F are well-defined on Mod_A and $\mathsf{Mod}_{\mathsf{A_K}}[G]$, all that remains for (i) is to check that $E \circ F$ is isomorphic to the identity in $\mathsf{Mod}_{\mathsf{A_K}}[G]$. The latter amounts to showing the K-vector space isomorphism $\mu : (N^G) \otimes_A A_K \to N$ given by $n \otimes a \mapsto an$ is a G-equivariant A_K-module isomorphism for any object N in $\mathsf{Mod}_{\mathsf{A_K}}$, which is immediate, and that for each G-equivariant morphism $\phi \in \mathrm{Hom}_{A_K}(N, N')$ we have a commutative diagram $\phi \circ \mu = \mu' \circ \phi|_{N^G}$ in $\mathsf{Mod}_{\mathsf{A_K}}[G]$, also easy to check.

For (ii) we show that the twisted actions are Galois in $\mathsf{Mod}_{\mathsf{A_K}}$. For each $c_\sigma \in \mathrm{Aut}_{A_K}(M_K)$ we have $c_\sigma(am) = ac_\sigma(m)$, hence $\sigma \star_c am = \sigma(a)c_\sigma \cdot \sigma(m)$, as required. Thus if N admits a Galois G-action, so it is an object of $\mathsf{Mod}_{\mathsf{A_K}}[G]$, then the A_K-modules ${}_cN$ are also in $\mathsf{Mod}_{\mathsf{A_K}}[G]$, hence $[N]/G$ is in $\mathsf{Mod}_{\mathsf{A_K}}[G]$. The rest of the proof now proceeds as before, and we omit the details.

□

9. Galois Descent for Affine Schemes

9.1. **Affine K-Schemes.** Theorem 7.2 (and Example 7.1) is a theory of descent and twisted forms for algebras of finite type over fields, and their modules. Let $\mathscr{C} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field K the category $\mathscr{C}_K$ of finitely generated commutative K-algebras. The *prime spectrum functor* defines a contravariant category isomorphism

$$\begin{aligned} \mathrm{Spec} : \mathscr{C}_K &\longrightarrow \mathscr{C}_K^\circ \\ B &\longmapsto Y = \mathrm{Spec}\, B \end{aligned}$$

where $\mathscr{C}^\circ : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ is the functor that assigns to each field K the category $\mathscr{C}_K^\circ$ of affine K-schemes of finite type. We will use this equivalence to extend Theorem 7.2 to the category of affine k-schemes.

For each morphism $k \to K$ in Fields the scalar extension functor

$$\begin{aligned} E : \mathscr{C}_k^\circ &\longrightarrow \mathscr{C}_K^\circ \\ X &\longmapsto X_K = X \times_k \mathrm{Spec}\, K \\ \phi &\longmapsto \phi_K = \phi \times_k \mathrm{id}_{\mathrm{Spec}\, K} \end{aligned}$$

is given by the fiber product, so that $(\mathrm{Spec}\, A) \times_k \mathrm{Spec}\, K = \mathrm{Spec}\,(A \otimes_k K)$. The functor Spec takes the scalar extension map $A \to A_K$ to the projection morphism $X_K \to X$, given by intersecting prime ideals of A_K with A. We say an object Y of $\mathscr{C}_K^\circ$ is *rational over k* or *defined over k* if $Y \simeq X \times_k \mathrm{Spec}\, K$ for some k-scheme X, which is then a *k-structure* for Y, and a morphism $\psi : Y \longrightarrow Y'$ of objects with k-structures in $\mathscr{C}_K^\circ$ is *defined over k* if the map arises from a morphism of k-structures via E.

9.1.1. *Opposite Actions.* Since Spec is contravariant, it reverses all morphisms. Let $\mathrm{Aut}_{\mathscr{C}_K^\circ}(Y)$, $\mathrm{Hom}_{\mathscr{C}_K^\circ}(Y', Y)$, and $\mathrm{Isom}_{\mathscr{C}_K^\circ}(Y', Y)$ denote the opposite groups, acting on the left, so that if $a^\circ, b^\circ \in \mathrm{Aut}_{\mathscr{C}_K^\circ}(Y)$ equal Spec of $a, b \in \mathrm{Aut}_{\mathscr{C}_K}(B)$, then $(ab)^\circ = a^\circ * b^\circ = b^\circ \cdot a^\circ$, where the operation "$*$" is the product in the opposite group, and the operation "$\cdot$" is composition of functions. A left G-action $\alpha : G \to \mathrm{Aut}_{\mathscr{C}_k}(B)$ on B in $\mathscr{C}_K[G]$ induces a left G°-action

$$\begin{aligned} \alpha^\circ : G^\circ &\longrightarrow \mathrm{Aut}_{\mathscr{C}_k^\circ}(Y_k) \\ \sigma^\circ &\longmapsto (\alpha(\sigma))^\circ \end{aligned}$$

on Y_k in $\mathscr{C}_K^\circ[G]$, where G° is the opposite group, and Y_k is the k-scheme structure on Y obtained via the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$.

The left action of G on $\operatorname{Aut}_{\mathscr{C}_K}(B)$ induces a left action of G° on $\operatorname{Aut}_{\mathscr{C}_K^\circ}(Y)$, given by ${}^{\sigma^\circ}\phi^\circ = \sigma^\circ * \phi^\circ * (\sigma^\circ)^{-1}$. The isomorphism $G \to G^\circ$ determined by Spec takes σ to $\sigma^\circ = \sigma^{-1}$, so in terms of composition of functions we have a left action of G on $\operatorname{Aut}_{\mathscr{C}_K^\circ}(Y)$,

$$ {}^{\sigma}(\phi^\circ) = \sigma \cdot \phi^\circ \cdot \sigma^{-1} \tag{9.1}$$

Thus we have a well-defined set $\mathrm{H}^1(G, \operatorname{Aut}_{\mathscr{C}_K^\circ}(Y))$. The map $\operatorname{Aut}_{\mathscr{C}_K}(B) \to \operatorname{Aut}_{\mathscr{C}_K^\circ}(Y)$ induced by Spec takes an automorphism ϕ to ϕ°, and we compute

$$({}^{\sigma}\phi)^\circ = \sigma^\circ * \phi^\circ * (\sigma^{-1})^\circ = \sigma \cdot \phi^\circ \cdot \sigma^{-1} = {}^{\sigma}(\phi^\circ)$$

Thus $\phi \mapsto \phi^\circ$ is G-equivariant under (9.1). It now follows that Spec induces a pointed-set isomorphism

$$\mathrm{H}^1(G, \operatorname{Aut}_{\mathscr{C}_K}(B)) \xrightarrow{\sim} \mathrm{H}^1(G, \operatorname{Aut}_{\mathscr{C}_K^\circ}(Y))$$

that preserves the cocycle condition: if $d = c^\circ$ then $d_{\sigma\tau} = d_\sigma * {}^{\sigma}d_\tau$. Moreover, if $c_\sigma = \phi^{-1} \cdot {}^{\sigma}\phi$ for some $\phi \in \operatorname{Isom}_{\mathscr{C}_K}(B, B')$, as in the derivation of cocycles in Theorem 7.2, then $d_\sigma = (\phi^\circ)^{-1} * {}^{\sigma}(\phi^\circ) = {}^{\sigma}(\phi^\circ) \cdot (\phi^\circ)^{-1}$. Thus the twisted action on Y induced by Spec is

$$\sigma \star_d y = d_\sigma * \sigma = \sigma \cdot d_\sigma$$

9.1.2. *Galois Action.* Suppose K/k is a finite Galois extension with group G, B is an object of $\mathscr{C}_K[G]$, and $Y = \operatorname{Spec} B$. By Definition 7.1, a *Galois* G-action is given by a homomorphism $\alpha : G \to \operatorname{Aut}_{\mathscr{C}_k}(B)$ such that the structure map $K \to B$ is G-*equivariant*, yielding for all $\tau \in G$ a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\tau} & B \\ \uparrow & & \uparrow \\ K & \xrightarrow{\tau} & K \end{array}$$

Thus the corresponding left action $\alpha^\circ : G^\circ \longrightarrow \operatorname{Aut}_{\mathscr{C}_k^\circ}(Y_k)$ is *Galois* if the structure map $Y \to \operatorname{Spec} K$ is G-equivariant, i.e., for all $\tau \in G$ we have a commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\tau^\circ} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xleftarrow{\tau^\circ} & \operatorname{Spec} K \end{array}$$

If X is an object in $\mathscr{C}_k^\circ$ then the induced left action of G° on $E(X) = X_K$ is Galois, and if $\phi : X \to X'$ is a morphism then $E(\phi) = \phi \times_k \mathrm{id}_K$ is G-equivariant. Therefore E takes $\mathscr{C}_k^\circ$ into $\mathscr{C}_K^\circ[G]$:

$$E : \mathscr{C}_k^\circ \longrightarrow \mathscr{C}_K^\circ[G]$$

9.1.3. *Scheme Quotient.* Let Y be a K-scheme, and let Y_k denote its k-scheme structure. The *scheme quotient* Y/G of Y by a group G is a k-scheme X together with a G-invariant k-morphism $p : Y_k \to X$ such that X represents the functor $\mathrm{Hom}_{\mathscr{C}_k^\circ}(Y_k, -)^G$ on the category of affine k-schemes with trivial G-action, so that for each such scheme Z there is a bijection $\mathrm{Hom}_{\mathscr{C}_k^\circ}(X, Z) \longrightarrow \mathrm{Hom}_{\mathscr{C}_k^\circ}(Y_k, Z)^G$ given by composition with p.

Suppose $Y = \operatorname{Spec} B$ admits a Galois G-action, $X = \operatorname{Spec} B^G$, $p : Y \to X$ is the map induced by $B^G \to B$, and $f : Y_k \to Z = \operatorname{Spec} C$ is a G-invariant morphism of affine k-schemes. Then f is induced by a morphism $C \to B^G$, hence f factors through p. Thus

$$Y/G = \operatorname{Spec}(B^G)$$

Let

$$F : \mathscr{C}_K^\circ[G] \longrightarrow \mathscr{C}_k^\circ$$

denote the functor sending an object Y in $\mathscr{C}_K^\circ$ to the object Y/G in $\mathscr{C}_k^\circ$ and a morphism $\phi : Y \to Y'$ to the induced morphism $\phi/G : Y/G \to Y'/G$. We call it the *quotient functor.*

Theorem 9.2. *Let K/k be a finite Galois extension with group G, let G° denote the opposite group, and let $\mathscr{C}_k^\circ$ and $\mathscr{C}_K^\circ[G]$ denote the categories of affine k-schemes and affine K-schemes with Galois G-action.*

(i) *The scalar extension and quotient functors define a category equivalence*

$$\mathscr{C}_k^\circ \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{C}_K^\circ[G]$$

(ii) *Suppose X is an object of $\mathscr{C}_k^\circ$. There is a pointed-set isomorphism*

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_{\mathscr{C}_K^\circ}(X_K)) &\longrightarrow \mathrm{TF}_{K|k}(X) \\ [d] &\longmapsto [({}_dX_K)/G] \end{aligned}$$

where the (left) G-action on ${}_dX_K$ is given by $\sigma \star_d y = \sigma \cdot d_\sigma(y)$. The inverse takes a class $[X']$ to the cocycle $d_\sigma = {}^\sigma\psi \cdot \psi^{-1}$ for any $\psi \in \mathrm{Isom}_{\mathscr{C}_K^\circ}(X'_K, X_K)$. If $X_K = \operatorname{Spec} B$ then ${}_dX_K = \operatorname{Spec} {}_cB$ and $({}_dX_K)/G = \operatorname{Spec}(({}_cB)^G)$, where $c = (c_\sigma) \in \mathrm{Z}^1(G, \mathrm{Aut}_{\mathscr{C}_K}(B))$ is defined by $c_\sigma^\circ = d_\sigma$.

Proof. Both (i) and (ii) are immediate by Theorem 7.2 and the translations into the opposite category given by the discussion in Section 9.1.

□

10. Galois Action on Structure Sheaves Over Affine Schemes

We discuss the locally-ringed-space-theoretic aspects of a Galois action on an affine scheme. We first prove a lemma that applies in a more general setting.

Lemma 10.1. *Suppose K/k is finite, and X is a k-scheme. Then the projection $p : X_K \to X$ is finite, faithfully flat, open, and closed.*

Proof. Since $k \to K$ is finite and flat, p is finite and flat by base change. Thus p is faithfully flat, open, and closed by [13, I.2.11,14].

□

10.1. **k-Topology.** Let K/k be a finite field extension, let $X = \operatorname{Spec} A$ be an affine k-scheme of finite type, let $Y = X_K = \operatorname{Spec} A_K$, and let $p : Y \to X$ be the projection. If $U \subset X$ is an open set then $p^{-1}(U) = U_K$ is an open subset of X_K, since p is continuous. We call U_K a *k-open* set. Similarly if $Z \subset X$ is closed, we call Z_K a *k-closed* set. We call the collection of k-open sets of X_K the *k-topology* on X_K. It is generated by the *basic* k-open sets $D(f)_K := \operatorname{Spec} A[1/f]_K = \operatorname{Spec} A_K[1/f]$, for $f \in A$. Since p is closed, by Lemma 10.1, any open subset of X_K contains a k-open set, hence a basic k-open set. However, the k-open sets do not in general generate the topology of X_K, because not every open set of X_K is a union of k-open sets. For example if a prime ideal $\mathfrak{p}$ of A factors into prime ideals $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ in A_K then every k-open set either contains all or none of the set $\{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$; the k-open sets are unions of fibers.

10.2. **Galois Action.** Suppose K/k is a finite Galois extension with group G, Y is an affine K-scheme of finite type admitting a left Galois G°-action $\alpha^\circ : G^\circ \to \operatorname{Aut}_{\mathscr{C}_k^\circ}(Y)$, $X = Y/G$, and $p : Y \to X$ is the projection. From the basic theory of locally ringed spaces each automorphism $\sigma^\circ : Y \to Y$ of schemes is equivalent to a homeomorphism on the underlying topological space Y together with a structure sheaf isomorphism $\sigma : \mathrm{O}_Y \to \sigma^\circ_* \mathrm{O}_Y$.

Proposition 10.2. *The Galois action on Y is equivalent to a morphism*

$$G \longrightarrow \operatorname{Aut}_{\mathrm{O}_X}(p_* \mathrm{O}_Y)$$

under which the map $K \to p_* \mathrm{O}_Y$ is G-equivariant.

Proof. Since Y is affine, each morphism $\sigma : \mathrm{O}_Y \to \sigma^\circ_* \mathrm{O}_Y$ is completely determined by $p_*(\sigma) : p_* \mathrm{O}_Y \to p_* \sigma^\circ_* \mathrm{O}_Y$. In fact, σ is determined already by its action on $B = p_* \mathrm{O}_Y(X)$. The action is Galois if and only if the structure map $K \to B$ is G-equivariant, and this is equivalent to the G-equivariance of $K \to p_* \mathrm{O}_Y$. Thus the Galois G-action on Y is equivalent to a morphism $G \to \operatorname{Aut}_{\mathrm{O}_X}(p_* \mathrm{O}_Y)$ under which $K \to p_* \mathrm{O}_Y$ is G-equivariant.

□

We investigate the extent to which global sections of O_Y are defined over k. If $U \subset X$ is affine then trivially $\mathrm{O}_Y(U_K) = \mathrm{O}_X(U)_K$ and $\mathrm{O}_Y(U_K)^G = \mathrm{O}_X(U)$, by Theorem 9.2. If $U \subset X$ is not affine then we obtain the same result, but only because $p : Y \to X$ is *flat*:

Proposition 10.3. *Suppose $Y = \operatorname{Spec} B$ is an object of $\mathscr{C}_K^\circ[G]$, $X = Y/G$, and $p : Y \to X$ is the quotient map. If $U \subset X$ is any open set, not necessarily affine, then $\mathrm{O}_Y(U_K)$ has the*

k-structure $\mathrm{O}_X(U)$, hence admits a Galois G-action uniquely determined by the action on B, and $\mathrm{O}_Y(U_K)^G = \mathrm{O}_X(U)$. Thus we have a natural sheaf isomorphism

$$\mathrm{O}_X \xrightarrow{\sim} (p_*\mathrm{O}_Y)^G$$

where $(p_*\mathrm{O}_Y)^G$ denotes the subsheaf of $p_*\mathrm{O}_Y$ that associates to each open set U of X the k-algebra $\mathrm{O}_Y(U_K)^G$.

Proof. Since Y is in $\mathscr{C}_K^\circ[G]$ we have a Galois G-action on B. Let $A = B^G$, so $X = \operatorname{Spec} A$. By definition if $\{D(f_i)_K : f_i \in A\}$ is a (finite) cover of U_K by basic k-open sets then $\mathrm{O}_Y(U_K)$ is the equalizer of the pair of maps $\prod_i B[1/f_i] \rightrightarrows \prod_{i<j} B[1/f_i f_j]$. This map has the k-structure $\prod_i A[1/f_i] \rightrightarrows \prod_{i<j} A[1/f_i f_j]$, whose equalizer is $\mathrm{O}_X(U)$, since O_X is a sheaf. Since p is flat we conclude that $\mathrm{O}_Y(U_K)$ equals $\mathrm{O}_X(U)_K$. Therefore by Theorem 7.2 $\mathrm{O}_Y(U_K)$ admits a left Galois action, which is evidently uniquely determined by the action on B, and we have $\mathrm{O}_Y(U_K)^G = \mathrm{O}_X(U)$. It follows immediately that the canonical morphism $\mathrm{O}_X \to p_*\mathrm{O}_Y$ actually maps isomorphically onto the subsheaf $(p_*\mathrm{O}_Y)^G$.

□

Remark 10.4. The familiar Galois action on algebras carries over to global sections on the structure sheaf: If $U \subset Y$ is any open set, $a \in K$, and $x \in \mathrm{O}_Y(U)$, then $\sigma(ax) = \sigma(a)\sigma(x)$. For if $\{D(g_i)\}$ is a finite cover of U then the restriction from U to each basic open set induces an injective K-algebra homomorphism $\mathrm{O}_Y(U) \subset \prod_i B[1/g_i]$. Thus $x = (x_i) \in \mathrm{O}_Y(U)$, and

$$\sigma(ax) = (\sigma(ax_i)) = \sigma(a)(\sigma(x_i)) \in \sigma_*^\circ \mathrm{O}_Y(U) \subset \prod_i B[1/\sigma(g_i)]$$

The next result gives a criterion for a localization $T^{-1}B$ to admit a Galois G-action extending a Galois action on B. We won't actually use this result in the sequel.

Proposition 10.5. *Suppose B is an object of $\mathscr{C}_K[G]$ and $T \subset B$ a multiplicative subset. Then $T^{-1}B$ admits a Galois G-action extending the action on B if and only if the saturation $\hat{T}$ of T is stable under the G-action on B, in which case $(T^{-1}B)^G = (\hat{T}^G)^{-1}B^G$, and $((\hat{T}^G)^{-1}B^G)_K = T^{-1}B$.*

Proof. We may assume T is saturated, then its complement is the union of prime ideals of B that do not intersect T. Suppose $T^{-1}B$ admits a Galois G-action extending the action on B. If $\mathfrak{P}$ is a prime ideal of B that does not intersect T then $T^{-1}\mathfrak{P}$ is a prime of $T^{-1}B$, and since the Galois action on algebras preserves prime ideals, G must take $\mathfrak{P}$ to another prime ideal not intersecting T. Thus G stabilizes T. Conversely if G stabilizes T then $T^{-1}B$ admits the obvious Galois G-action extending that on B.

For the second statement, suppose that $T^{-1}B$ admits a Galois G-action extending the action on B, $A = B^G$, and $S = T^G = T \cap A$. Then clearly $(T^{-1}B)^G$ contains $S^{-1}A$, and $(S^{-1}A)_K$ is contained in $T^{-1}B$. But a prime $\mathfrak{P}$ of B intersects T if and only if $\mathfrak{P}$ intersects S: For if $f \in \mathfrak{P} \cap T$ then the product of the conjugates of f under G is in $\mathfrak{P}$, since $\mathfrak{P}$ is an ideal, and in T, since T is stable under G. Since the product of conjugates is fixed by G, it is in $T \cap A = S$, hence $\mathfrak{P} \cap S$ is nonempty. On the other hand it is obvious that a prime $\mathfrak{P}$ of B intersects S if and only if $\mathfrak{P} \cap A$ intersects S, and we conclude $\mathfrak{P} \cap T$ is empty if and only if $\mathfrak{P} \cap S$ is empty. This shows that T is in the saturation of S, hence that $S^{-1}B = T^{-1}B$. Now since the tensor product commutes with localization, and $A_K = B$, we have $(S^{-1}A)_K = T^{-1}B$. Now since $F \circ E = \mathrm{id}_{\mathscr{C}_k}$ by Theorem 7.2, we conclude $(T^{-1}B)^G = S^{-1}A$.

□

11. Galois Descent of Quasi-Coherent Sheaves Over Affine Schemes

Definition 11.1. Let k be a field, let X be a k-scheme of finite type, and let

$$\mathsf{QCoh} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$$

be the functor that assigns to each field K containing k the category $\mathsf{QCoh}_{\mathsf{X}_\mathsf{K}}$ of quasi-coherent O_{X_K}-modules, and to each morphism $k \to K$ the scalar extension

$$E : \mathsf{QCoh}_\mathsf{X} \longrightarrow \mathsf{QCoh}_{\mathsf{X}_\mathsf{K}}$$

given on objects by $E(\mathscr{M}) = p^*(\mathscr{M})$ and on morphisms $E(\phi) = p^*(\phi)$, where $p : X_K \to X$ is the projection. We say these objects are *defined over* k. We sometimes substitute the notations $\mathscr{M}_K$ and ϕ_K.

11.0.1. *Sheafification.* If B is a fixed k-algebra of finite type and $Y = \operatorname{Spec} B$, the process of module "sheafification" defines a category equivalence

$$\mathsf{Mod}_\mathsf{B} \longrightarrow \mathsf{QCoh}_\mathsf{Y}$$

associating to each B-module N the uniquely determined O_Y-module $\mathscr{N}$ whose restriction to each basic open set $D(g)$ is the $B[1/g]$-module $N[1/g] = N \otimes_B B[1/g]$, and to each B-morphism $\phi : N \to N'$ the uniquely determined O_Y-morphism $\tilde{\phi} : \mathscr{N} \to \mathscr{N}'$ whose restriction to each $D(g)$ is the $B[1/g]$-module morphism $N[1/g] \to N'[1/g]$ induced by $\tilde{\phi}$.

Since these categories are equivalent we have a theory of Galois descent and twisted forms for quasi-coherent sheaves over affine schemes, by Theorem 8.2.

We recall how the module properties used in Theorem 8.2 "sheafify", so that we can correctly translate the theorem. Suppose K/k is a finite Galois extension with group G, and $Y = \operatorname{Spec} B$ is an affine K-scheme of finite type that admits a left Galois G°-action, with quotient the affine k-scheme $Y/G = X = \operatorname{Spec} A$, where $A = B^G$. Let $p : Y \to X$ be the quotient map. Then $\mathrm{O}_Y = p^*\mathrm{O}_X$ is the sheafification of the K-algebra $B = A \otimes_k K$, and if $\mathscr{M}$ is the O_X-module corresponding to the A-module M then $p^*\mathscr{M}$ is the O_Y-module corresponding to $M \otimes_A B = M \otimes_k K$. If $\mathscr{N}$ is the quasi-coherent O_Y-module corresponding to the B-module N then $p_*\mathscr{N}$ is the O_X-module corresponding to N–viewed as an A-module via the map $A \to B$.

11.0.2. *Galois Action.* If $\mathscr{N}$ has k-structure $\mathscr{M}$, then as in Proposition 10.3, since $k \to K$ is flat, for any k-open U_K on X_K we have $\mathscr{N}(U_K) = \mathscr{M}(U)_K$, so that each $\mathscr{N}(U_K)$ has a k-structure. A *left Galois action of G on $\mathscr{N}$* is a homomorphism

$$\alpha : G \longrightarrow \mathrm{Aut}_{\mathrm{O}_X}(p_*\mathscr{N})$$

extending the Galois action on O_Y. If $\mathscr{N} = \tilde{N}$ for a B-module N and $D(f) \subset X$ is a basic open set, then the action of σ on $p_*\mathscr{N}(D(f))$ has the form $\sigma : N[1/f] \to N[1/f]$. Let

$$(p_*\mathscr{N})^G$$

denote the sheaf that assigns to each $U \subset X$ the module of fixed points $\mathscr{N}(U_K)^G$. It is easy to see that $(p_*\mathscr{N})^G = (N^G)\tilde{}$, the sheafification of the A-module N^G Let

$$\mathsf{QCoh}_\mathsf{Y}[G]$$

denote the category whose objects are pairs $(\mathscr{N}, \alpha)$, where α is a Galois G-action on $\mathscr{N}$, and whose morphisms are G-equivariant morphisms. Let

$$F : \mathsf{QCoh}_\mathsf{Y}[G] \longrightarrow \mathsf{QCoh}_\mathsf{X}$$

denote the functor that assigns to each object $(\mathscr{N}, \alpha)$ the O_X-module $(p_*\mathscr{N})^G$, and to each G-equivariant morphism its k-structure.

Theorem 11.2. *Suppose K/k is a finite Galois extension with group G, X is a k-scheme of finite type, and $p : X_K \to X$ is the projection, and QCoh_X, $\mathsf{QCoh}_{\mathsf{X_K}}[G]$ are the categories defined above. Then:*

(i) *The scalar extension and fixed point functors define a category equivalence*

$$\mathsf{QCoh}_\mathsf{X} \underset{F}{\overset{E}{\rightleftarrows}} \mathsf{QCoh}_{\mathsf{X_K}}[G]$$

(ii) *Suppose $\mathscr{M}$ is a quasi-coherent O_X-module. Then there is a pointed-set isomorphism*

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_{\mathrm{O}_{X_K}}(\mathscr{M}_K) &\longrightarrow \mathrm{TF}_{K|k}(\mathscr{M}) \\ [c] &\longmapsto [(p_*({}_c\mathscr{M}_K))^G] \end{aligned}$$

whose inverse takes a class $[\mathscr{M}'] \in \mathrm{TF}_{K|k}(\mathscr{M})$ to the class $[c]$ defined by $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$, for any $\phi \in \mathrm{Isom}_{\mathrm{O}_{X_K}}(\mathscr{M}_K, \mathscr{M}'_K)$.

Proof. We omit the proof, as it is a straightforward consequence of Theorem 8.2. □

12. Galois Descent for Schemes

12.1. **k-Structures.** Let $\mathscr{D} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$ be the functor that assigns to each field K the category $\mathscr{D}_K$ of schemes of finite type over K, and to each morphism $k \to K$ in Fields the fiber product functor $E : \mathscr{D}_k \to \mathscr{D}_K$ taking an object X to $X_K = X \times_k K$ and a morphism ϕ to $\phi \otimes_k \mathrm{id}_K$. We say the scheme X_K is *rational over k* (or *defined over k*).

The *k-topology* on X_K is the collection $\{U_K\}$ of open sets defined over k under the canonical morphism $p : X_K \to X$. We say U_K is *k-open*, and its complement is *k-closed*. The morphism p is finite, hence faithfully flat, open, and closed, by Lemma 10.1. Since p is closed, every open set of X_K contains a k-open set, hence an affine k-open set. If $U_K = p^{-1}(U)$ is k-open then $\mathrm{O}_{X_K}(U_K)$ has the k-structure $\mathrm{O}_X(U)$, since p is flat. The proof of this fact is identical to that of the analogous statement in Proposition 10.3, since $\mathrm{O}_{X_K}(U_K)$ is the equalizer with respect to a G-stable affine cover of U_K. If $\phi_K : X'_K \to X_K$ is defined over k then ϕ_K is continuous with respect to the k-topologies, and if $U_K \subset X_K$ is k-open with preimage U'_K then the induced map $\mathrm{O}_{X_K}(U_K) \to \mathrm{O}_{X'_K}(U'_K)$ is defined over k.

12.2. **Galois Action.** Suppose K/k is a finite Galois extension with group G, and Y is a separated K-scheme of finite type, which is an object of $\mathscr{D}_K$. A left action of G° on Y is a homomorphism

$$\alpha^\circ : G^\circ \longrightarrow \mathrm{Aut}_{\mathscr{D}_k}(Y_k)$$

where Y_k is the k-scheme formed by composing $Y \to \operatorname{Spec} K$ with $\operatorname{Spec} K \to \operatorname{Spec} k$. We say the action is *Galois* if

(a) Y admits a G-stable affine cover, and
(b) the structure map $Y \to \operatorname{Spec} K$ is G-equivariant, i.e., for each $\sigma^\circ \in G^\circ$ we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\sigma^\circ} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow[\sigma^\circ]{} & \operatorname{Spec} K \end{array}$$

The G-stable affine cover $\{V_i\}$ of Y allows us to define affine schemes $\{U_i = V_i/G\}$, which glue together to form a quotient scheme $X = Y/G$.

If Y and Y' are K-schemes admitting left Galois G°-actions, the induced right G°-action on $\mathrm{Hom}_{\mathscr{D}_K}(Y', Y)$ is defined by $\phi^{\sigma^\circ} = (\sigma^\circ)^{-1} * \phi * \sigma^\circ$, or

$${}^\sigma\phi = \sigma \cdot \phi \cdot \sigma^{-1}$$

in composition of functions, as in (9.1), where σ^{-1} acts as σ°. We say ϕ is *G-equivariant* if $\phi \in \mathrm{Hom}_{\mathscr{D}_K}(Y, X)^G$. Let

$$\mathscr{D}_K[G]$$

denote the category whose objects are pairs (Y, α°), where Y is an object of $\mathscr{D}_K$ and α° is a left Galois G°-action on Y, and whose morphisms are G-equivariant morphisms.

Theorem 12.1. *Let K/k be a finite Galois extension with group G, let G° denote the opposite group, and let $\mathscr{D}_k$ and $\mathscr{D}_K[G]$ denote the categories of k-schemes and K-schemes of finite type with Galois G-action. Then:*

(i) *Then there is a well-defined quotient functor $F : \mathscr{D}_K[G] \to \mathscr{D}_k$, and the scalar extension E and quotient functor F define a category equivalence*

$$\mathscr{D}_k \underset{F}{\overset{E}{\rightleftarrows}} \mathscr{D}_K[G]$$

(ii) *Suppose X is an object of $\mathscr{D}_k$. There is a pointed-set isomorphism*

$$\begin{aligned} \mathrm{H}^1(G, \mathrm{Aut}_{\mathscr{D}_K}(X_K)) &\longrightarrow \mathrm{TF}_{K|k}(X) \\ [d] &\longmapsto [({}_dX_K)/G] \end{aligned}$$

where the (left) G°-action on ${}_dX_K$ is given by $\sigma^\circ \star_d y = \sigma \cdot d_\sigma(y)$. The inverse takes a class $[X']$ to the cocycle $d_\sigma = {}^\sigma\psi \cdot \psi^{-1}$ for any $\psi \in \mathrm{Isom}_{\mathscr{D}_K}(X'_K, X_K)$.

Proof. We first show there is a well-defined quotient functor $F : \mathscr{D}_K[G] \to \mathscr{D}_k$. Suppose Y is an object of $\mathscr{D}_K[G]$. Let $\{V_i\}$ be a G-stable affine cover of Y, and put $V_{ij} = V_i \cap V_j$. By Theorem 9.2 we obtain k-structures $U_i = V_i/G$ for each V_i, and so $(U_i)_K \simeq V_i$. Each V_{ij} is G-stable since V_i and V_j are G-stable, hence each V_{ij} has a k-structure, namely $V_{ij}/G = V_i/G \cap V_j/G$. The V_i are glued together by the identity map $\mathrm{id}_Y : V_{ij} \to V_{ji}$, which trivially satisfies the cocycle condition on triple intersections. Since each V_i and V_{ij} are G-stable and id_Y is G-equivariant, the $U_i = V_i/G$ glue together to give a scheme X over k, whose topology is generated by the basic affine open sets $D(f)$ in each U_i, and whose structure sheaf is defined by

$$\mathrm{O}_X(U) = \varinjlim_{U \supset D(f)} \mathrm{O}_X(D(f))$$

By construction we have $X_K = Y$, hence a projection morphism

$$p : Y \longrightarrow X$$

that is finite, faithfully flat, open, and closed, by Lemma 10.1.

Note that if $U \subset X$ is open then U_K is G-stable as the union of G-stable affine open sets. Let $(p_*\mathrm{O}_Y)^G$ denote the sheaf that assigns to each U the k-algebra $\mathrm{O}_Y(U_K)^G$. Then it is immediate by construction that we have a sheaf isomorphism

$$\mathrm{O}_X \xrightarrow{\sim} (p_*\mathrm{O}_Y)^G$$

It is not hard to show from this that $X = Y/G$ is a scheme quotient. For suppose $f : Y \to Z$ is a G-invariant morphism. If z is a point of Z then the fiber $f^{-1}(z)$ is G-invariant, and if $y \in f^{-1}(z)$ and $p(y) = x$ then we define $g : X \to Z$ by $g(x) = z$. Then $f = g \circ p$ as maps of topological spaces. Since $(f_*\mathrm{O}_Y)^G = g_*(p_*\mathrm{O}_Y)^G = g_*\mathrm{O}_X$, the natural map $\mathrm{O}_Z \to (f_*\mathrm{O}_Y)^G$ factors through $g_*\mathrm{O}_X$, hence g is a morphism of schemes. The map g is uniquely determined. Conversely if $g : X \to Z$ is a morphism then composition with p defines a G-invariant morphism $f : Y \to Z$. Thus we obtain a bijection $\mathrm{Hom}_{\mathscr{D}_k}(X, Z) \to \mathrm{Hom}_{\mathscr{D}_k}(Y_k, Z)^G$, hence $X = Y/G$.

Since Y is of finite type over K, it is quasi-finite, hence X is quasi-finite, hence X is of finite type over k, by Example 7.1.

Suppose $\psi : Y' \to Y$ is a morphism in $\mathscr{D}_K[G]$, and $X' = Y'/G$ and $X = Y/G$ are quotients, respectively. Every point of Y' is contained in a (G-stable) k-open affine open set V' whose image $\psi(V')$ is contained in a k-open subset of Y. Thus $\psi|_{V'}$ is defined over k by the affine case Theorem 9.2, and since the k-structures glue together, all coming from the morphism ψ, we conclude ψ is defined over k.

Now we have a functor

$$F : \mathscr{D}_K[G] \longrightarrow \mathscr{D}_k$$

taking a K-scheme Y of finite type with Galois G-action to a k-scheme X of finite type, such that $X = Y/G$. We have $E \circ F \simeq \mathrm{id}_Y$ by construction of the fiber product, and similarly $F \circ E = \mathrm{id}_X$, hence E and F define the stated category equivalence.

Part (ii) follows from the analogous statement for affine schemes.

□

12.3. **Properties That Descend.** Assume the situation of Theorem 12.1. A property P of an object Y of $\mathscr{D}_K[G]$ *descends* if the object $X = Y/G$ has property P.

Theorem 12.2. *Let X be a k-scheme, let K/k is a finite Galois extension, and set $X_K = X \times_k K$. The following properties descend from X_K to X: reduced, smooth, quasi-projective, projective.*

Proof. If X_K is reduced we may assume $X_K = \mathrm{Spec}\, B$ is affine, and $X = \mathrm{Spec}\, A$ for $A = B^G$. If A is not reduced then it has nonzero nilradical N_A, and since $p^\# : A \to B$ is faithfully flat the module $N_A \otimes_A B$ is nonzero, hence B is not reduced.

If X_K is smooth then by definition $(X_K)_{\bar{K}}$ are regular, where $\bar{K}$ is an algebraic closure of K. But we may assume $\bar{K}$ is also an algebraic closure of k, since K/k is algebraic, hence $(X_K)_{\bar{K}} = X_{\bar{k}}$, hence X is also smooth.

If X_K is projective or quasi-projective then by definition there is an embedding $\psi : X_K \to \mathbb{P}^n(K)$ whose image is either closed or locally closed. We claim this map is G-equivariant, and since $p : X_K \to X$ and $\hat{p} : \mathbb{P}^n(K) \to \mathbb{P}^n(k)$ are open and closed maps, it follows immediately that $\psi = \phi_K$ for a map $\phi : X \to \mathbb{P}^n(k)$ whose image is either closed or locally closed, showing X is projective or quasi-projective.

□

12.4. **Quasi-Projective Varieties.** To apply Theorem 12.1 to quasi-projective varieties, we need to find a G-stable affine cover. We copy the proof in [10].

Lemma 12.3. *Suppose $P = \{p_1, \ldots, p_s\}$ is a finite (possibly empty) set of closed points in $\mathbb{P}^n_K$, and $Z \subsetneq \mathbb{P}^n_K$ is a closed subvariety such that $Z \cap P = \varnothing$. Then there exists a hypersurface $H \subset \mathbb{P}^n_K$ such that $Z \subset H$ and $H \cap P = \varnothing$.*

Proof. We tensor the defining exact sequence $0 \to \mathscr{I}_P \to \mathrm{O}_{\mathbb{P}^n_K} \to \imath_* \mathrm{O}_P \to 0$ with $\mathscr{I}_Z$ over $\mathrm{O}_{\mathbb{P}^n_K}$ to obtain the exact sequence $\mathscr{I}_P \otimes \mathscr{I}_Z \to \mathscr{I}_Z \to \imath_* \mathrm{O}_P \otimes \mathscr{I}_Z \to 0$. Using the hypothesis $Z \cap P = \varnothing$ and looking at stalks, we easily show the sequence is injective on the left, and that the canonical map $\imath_* \mathrm{O}_P \to \imath_* \mathrm{O}_P \otimes \mathscr{I}_Z$ is an isomorphism. Thus for each r we have a short exact sequence

$$0 \to \mathscr{I}_P \otimes \mathscr{I}_Z(r) \to \mathscr{I}_Z(r) \to \imath_* \mathrm{O}_P(r) \to 0.$$

Since $\mathscr{I}_P \otimes_{\mathrm{O}_{\mathbb{P}^n_K}} \mathscr{I}_Z$ is coherent, for $r \gg 0$ we have $\mathrm{H}^1(\mathbb{P}^n_K, \mathscr{I}_P \otimes \mathscr{I}_Z(r)) = 0$ by Serre's Theorem, hence a surjection

$$\Gamma(\mathbb{P}^n_K, \mathscr{I}_Z(r)) \twoheadrightarrow \Gamma(\mathbb{P}^n_K, \imath_* \mathrm{O}_P(r)) = \prod_{i=1}^{s} k(p_i).$$

Thus there exists $f \in \Gamma(\mathbb{P}^n_K, \mathscr{I}_Z(r))$ such that $f(p_i) \neq 0$ for each i, and we take $H = V(f)$. Since f vanishes on Z we have $Z \subset H$, and since $f(p_i) \neq 0$, $H \cap P = \varnothing$.

□

Lemma 12.4. *Let G be a finite group. Any quasi-projective K-variety with a left Galois G°-action has a G°-stable affine cover.*

Proof. Suppose $Y \subset \mathbb{P}^n_K$ is quasi-projective. Let $Z = \overline{Y} \backslash Y$, a (possibly empty) closed subvariety of $\mathbb{P}^n_K$. We will show each $p \in Y$ has a G°-stable affine neighborhood in Y. Let $P = \{\sigma^\circ(p) : \sigma^\circ \in G^\circ\}$. By Lemma 12.3 there exists a hypersurface $H \subset \mathbb{P}^n_K$ such that $Z \subset H$ and $H \cap P = \varnothing$. The basic open set $D = \mathbb{P}^n_K - H$ is an affine variety, and since $Z \subset H$, $Y \cap D$ is a closed subvariety of D, hence an affine variety. Now consider

$$U = \bigcap_{\sigma^\circ \in G^\circ} \sigma^\circ(Y \cap D)$$

Then $\sigma^\circ(U) = U$ for all σ°, and since $\sigma^\circ(p) \in Y \cap D$ for all σ°, $p \in U$. Thus U is a G°-stable affine open subset of Y containing p.

□

Corollary 12.5. *Theorem 12.1 applies to the categories of quasi-projective varieties.*

Proof. This is immediate from Lemma 12.4, Theorem 12.1, and Theorem 12.2. □

13. Galois Descent for Quasi-Coherent Modules

We imitate the notation of Section 11. Let k be a field, let X be a k-scheme of finite type, and let

$$\mathsf{QCoh} : \mathsf{Fields} \longrightarrow \mathsf{Cat}$$

be the functor that assigns to each field K containing k the category $\mathsf{QCoh}_{\mathsf{X_K}}$ of quasi-coherent O_{X_K}-modules, and to each morphism $k \to K$ the scalar extension

$$E : \mathsf{QCoh}_{\mathsf{X}} \longrightarrow \mathsf{QCoh}_{\mathsf{X_K}}$$

given on objects by $E(\mathscr{M}) = p^*(\mathscr{M})$ and on morphisms $E(\phi) = p^*(\phi)$, where $p : X_K \to X$ is the projection. We say these objects are *defined over* k, or k*-rational.* We sometimes substitute the notations $\mathscr{M}_K$ and ϕ_K. If $\mathscr{N}$ has k-structure $\mathscr{M}$, then as in Proposition 10.3, since $k \to K$ is flat, for any k-open U_K on X_K we have $\mathscr{N}(U_K) = \mathscr{M}(U)_K$, so that each $\mathscr{N}(U_K)$ has a k-structure. A *left Galois action of* G *on* $\mathscr{N}$ is a homomorphism

$$\alpha : G \longrightarrow \mathrm{Aut}_{\mathrm{O}_X}(p_*\mathscr{N})$$

extending the Galois action on O_Y. Suppose $\mathscr{N}$ admits a Galois G-action. Let

$$(p_*\mathscr{N})^G$$

denote the sheaf that assigns to each $U \subset X$ the module of fixed points $\mathscr{N}(U_K)^G$. Let

$$\mathsf{QCoh}_{\mathsf{Y}}[G]$$

denote the category whose objects are pairs $(\mathscr{N}, \alpha)$, where α is a Galois G-action on $\mathscr{N}$, and whose morphisms are G-equivariant morphisms. Finally, let

$$F : \mathsf{QCoh}_{\mathsf{Y}}[G] \longrightarrow \mathsf{QCoh}_{\mathsf{X}}$$

denote the functor that assigns to each $(\mathscr{N}, \alpha)$ the O_X-module $(p_*\mathscr{N})^G$, and to each G-equivariant morphism its k-structure.

Theorem 13.1. *Suppose K/k is a finite Galois extension with group G, X is a k-scheme of finite type, $p : X_K \to X$ is the projection, and $\mathsf{QCoh}_{\mathsf{X}}$, $\mathsf{QCoh}_{\mathsf{X_K}}[G]$ are the categories defined above. Then:*

(i) *The scalar extension and fixed point functors define a category equivalence*

$$\mathsf{QCoh}_{\mathsf{X}} \underset{F}{\overset{E}{\rightleftarrows}} \mathsf{QCoh}_{\mathsf{X_K}}[G]$$

(ii) *Suppose $\mathscr{E}$ is a quasi-coherent O_X-module. Then there is a pointed-set isomorphism*

$$\mathrm{H}^1(G, \mathrm{Aut}_{\mathrm{O}_{X_K}}(\mathscr{E}_K) \longrightarrow \mathrm{TF}_{K|k}(\mathscr{E})$$
$$[c] \longmapsto [(p_*({}_c\mathscr{E}_K))^G]$$

whose inverse takes a class $[\mathscr{E}'] \in \mathrm{TF}_{K|k}(\mathscr{E})$ to the class $[c]$ defined by $c_\sigma = \phi^{-1} \cdot {}^\sigma\phi$, for any $\phi \in \mathrm{Isom}_{\mathrm{O}_{X_K}}(\mathscr{E}_K, \mathscr{E}'_K)$.

We omit the proof, which is a straightforward of the result for modules over algebras of finite type.

13.1. **Locally Free Sheaves.** Suppose that $\mathscr{F}$ is a locally free sheaf of rank n and structure map $\imath : \mathrm{O}_Y \to \mathscr{F}$. By definition there exists an open cover $\{U_i\}$ of Y and O_{U_i}-isomorphisms $\phi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathrm{O}_{U_i}^n$, such that $\phi_i \cdot \phi_j^{-1} = g_{ij} \in \mathrm{O}_Y^{*n}(U_i \cap U_j)$. Suppose there is a left G-action on $\mathscr{F}$ and a left G°-action on Y. We claim the action on $\mathscr{F}$ is Galois if and only if the ϕ_i are G-equivariant. For the structure map $\theta_i : \mathrm{O}_{U_i} \to \mathrm{O}_{U_i}^n$ is G-equivariant, so since $\phi_i \cdot \imath_i = \theta_i$, ϕ_i is G-equivariant if and only if $\imath_i$ is as well. It follows that the category equivalence of Theorem 13.1 takes locally free sheaves to locally free sheaves.

Corollary 13.2. *Theorem 13.1 applies to the categories of locally free sheaves.*

14. Application: Severi-Brauer Varieties.

14.1. **Severi-Brauer Varieties and Central Simple Algebras.** A *Severi-Brauer variety of degree* n over a field k is a twisted form P of $\mathbb{P}^{n-1}(k)$ with respect to a finite Galois extension K/k. Since $P_K \simeq \mathbb{P}^{n-1}(K)$, we say P is *split* by K. P is smooth and projective by Theorem 12.2. Since $\mathrm{Aut}(\mathbb{P}^{n-1}(K)) \simeq \mathrm{PGL}_n(K)$, if K/k is finite G-Galois, then by Corollary 12.5 we have a pointed-set bijection between $\mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$ and $\mathrm{H}^1(G, \mathrm{PGL}_n(K))$, and every Severi-Brauer variety may be defined by descent. By convention we identify $\mathrm{PGL}_n(K)$ with the left automorphism group, so $C \in \mathrm{PGL}_n(K)$ induces C^{-t} on the homogeneous coordinate ring, and acts on points as $\lambda_C([a_0, \dots, a_{n-1}] = C^{-t}[a_0, \dots, a_{n-1}]^t$. The standard left Galois G-action on $K[x_0, \dots, x_{n-1}]$ induces a left Galois action on $\mathbb{P}^{n-1}(K)$, and the corresponding left G-action on closed points is ${}^{\sigma}[a_0, \dots, a_{n-1}] = [\sigma(a_0), \dots, \sigma(a_{n-1})]$. Therefore if $c \in \mathrm{Z}^1(G, \mathrm{PGL}_n(K))$,

$$\sigma \star_c [a_0, \dots, a_{n-1}] \;=\; c_\sigma^{-t}[\sigma(a_0), \dots, \sigma(a_{n-1})]^t$$

This G-action on $\mathbb{P}^{n-1}(K)$ produces the quotient $P = ({}_c\mathbb{P}^{n-1}(K))/G$. If $[c] = 1$, we obtain $P = \mathbb{P}^{n-1}(k)$. If $[c] \neq 1$, P has no k-rational points, as we will see below.

A *central simple algebra of degree* n over k is a twisted form of $\mathrm{M}_n(k)$ with respect to some finite K/k. Since $\mathrm{PGL}_n(K)$ is also the automorphism group of $\mathrm{M}_n(K)$, isomorphism classes of central simple k-algebras are also classified by $\mathrm{H}^1(G, \mathrm{PGL}_n(K))$, and may be defined by descent. We say a central simple algebra A and a Severi-Brauer variety P are *associated* if they correspond to the same class in $\mathrm{H}^1(G, \mathrm{PGL}_n(K))$, and write $A = \mathrm{CSA}(P)$ and $P = \mathrm{SBV}(A)$. By Theorem 7.2, each $c \in \mathrm{Z}^1(G, \mathrm{PGL}_n(K))$ defines a central simple k-algebra by the formula

$$({}_c\mathrm{M}_n(K))^G = \{T \in \mathrm{M}_n(K) : c_\sigma {}^{\sigma}T c_\sigma^{-1} = T, \ \forall\, \sigma \in G\}$$

where the G-action on T is via coefficients. This is a nonempty subset of $\mathrm{M}_n(K)$.

14.2. **Cohomology.** For $m, n \in \mathbb{N}$, the tensor product pairing $K^m \times K^n \to K^{mn}$ induce K-linear maps $\mathrm{M}_m \times \mathrm{M}_n \to \mathrm{M}_{mn}$, $\mathrm{GL}_m \times \mathrm{GL}_n \to \mathrm{GL}_{mn}$, and $\mathrm{PGL}_m \times \mathrm{PGL}_n \to \mathrm{PGL}_{mn}$. It is easy to see these maps are G-linear, since G acts via the coefficients, and we obtain a map

$$\begin{array}{rcl} \mathrm{H}^1(G, \mathrm{PGL}_m(K)) \times \mathrm{H}^1(G, \mathrm{PGL}_n(K)) & \longrightarrow & \mathrm{H}^1(G, \mathrm{PGL}_{mn}(K)) \\ ([c], [d]) & \longmapsto & [c \otimes d] \end{array}$$

If $[c]$ and $[d]$ correspond to central simple k-algebras A and B, respectively, then $[c \otimes d]$ corresponds to $A \otimes_k B$, a twisted form of $\mathrm{M}_{mn}(k)$. If P and Q are the corresponding Severi-Brauer varieties, then $[c \otimes d]$ corresponds to the product $P \times_k Q$, a twisted form of $\mathbb{P}^{mn-1}(k)$. This can be seen easily by considering the case of a G-stable affine open set. By induction, if $A = \mathrm{CSA}(P)$ then

$$A^{\otimes n} = \mathrm{CSA}(P^{\times n})$$

Taking $d = \mathrm{I}_n$ yields the map

$$\begin{array}{rcl} \lambda_{m,n} : \mathrm{H}^1(G, \mathrm{PGL}_m(K)) & \longrightarrow & \mathrm{H}^1(G, \mathrm{PGL}_{mn}(K)) \\ [c] & \longmapsto & [c \otimes \mathrm{I}_n] = [c^{\times n}] \end{array}$$

corresponding to $Q = \mathbb{P}^{n-1}(k)$. Then Q has a rational point $q = \operatorname{Spec} k$, and by the universal property of products we have an embedding $P = P \times_k q \to P \times Q$, realizing P as a linear subvariety of $P \times_k Q$ ([2, p.202]).

Since $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = 0$, the short exact sequence

$$1 \longrightarrow \mathrm{G_m} \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1$$

determines an injection

$$\begin{array}{rcl} \delta_n : \mathrm{H}^1(G, \mathrm{PGL}_n(K)) & \longrightarrow & \mathrm{H}^2(G, K^*) \\ [(c_\sigma)] & \longmapsto & [(a_{\sigma,\tau})] \end{array}$$

where $a_{\sigma,\tau} \stackrel{\text{df}}{=} \tilde{c}_\sigma {}^\sigma\tilde{c}_\tau c_{\sigma\tau}^{-1}$, and $\tilde{c}_\sigma$ represents c_σ in $\mathrm{GL}_n(K)$. It is not hard to show the following diagram is commutative

$$\begin{array}{ccc} \mathrm{H}^1(G, \mathrm{PGL}_n) & \xrightarrow{\delta_n} & \mathrm{H}^2(G, K^*) \\ {\scriptstyle \lambda_{m,n}} \downarrow & & \| \\ \mathrm{H}^1(G, \mathrm{PGL}_{mn}) & \xrightarrow[\delta_{mn}]{} & \mathrm{H}^2(G, K^*) \end{array}$$

We say Severi-Brauer varieties P and P' of degrees n and n' are *Brauer equivalent*, and write $[P] = [P']$, if $\delta_n(P) = \delta_{n'}(P')$ in $\mathrm{H}^2(G, K^*)$.

The period of $[P]$ divides n. For it is easy to see the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \mathrm{SL}_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{G_m} & \longrightarrow & \mathrm{GL}_n & \longrightarrow & \mathrm{PGL}_n & \longrightarrow & 1 \end{array}$$

commutes, consequently the map δ_n factors through $\mathrm{H}^2(G, \mu_n) = {}_n\mathrm{H}^2(G, K^*)$.

14.3. **Rational Bundles.** Assume K/k is a finite Galois extension with group G. Recall from Section 13 we call a line bundle on a K-variety P_K with k-structure P *rational* if it descends to a line bundle on P, or equivalently, if it admits a G-action extending the Galois G-action on O_{P_K}. Since $\operatorname{Pic}\mathbb{P}^{n-1}(k) \simeq \mathbb{Z}$ is generated by $\mathrm{O}_{\mathbb{P}^{n-1}(k)}(1)$, scalar extension produces an isomorphism $\operatorname{Pic}\mathbb{P}^{n-1}(k) \simeq \operatorname{Pic}\mathbb{P}^{n-1}(K)$. Thus every line bundle on $\mathbb{P}^{n-1}(K)$ descends to $\mathbb{P}^{n-1}(k)$, which shows the untwisted Galois action of G on $\mathbb{P}^{n-1}(K)$ extends to a Galois action on every line bundle on $\mathbb{P}^{n-1}(K)$. This turns out not to hold in the twisted case.

Proposition 14.1. *Suppose K/k is a finite G-Galois extension and $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$. Then $\mathrm{O}_{P_K}(d)$ is rational if and only if the d-tuple embedding is G-equivariant, if and only if the d-tuple embedding descends to an embedding*

$$\nu_d : P \longrightarrow \mathbb{P}^{N-1}(k) \qquad (N = \tbinom{n+d-1}{d})$$

Proof. The last two statements are equivalent by Theorem 12.1. Since $P_K \to P$ is flat, for any invertible sheaf $\mathscr{L}$ on P we have $\mathscr{L}(P) \otimes_k K = \mathscr{L}_K(P_K)$, as in Proposition 10.3, hence if $\mathrm{O}_{P_K}(d)$ is rational, its global sections descend, so the d-tuple embedding on P_K descends. Conversely, if the d-tuple embedding is defined over k, we have an invertible sheaf $\mathscr{L} \overset{\mathrm{df}}{=} \nu_d^* \mathrm{O}_{\mathbb{P}^{N-1}(k)}(1)$ on P, which becomes $\mathrm{O}_{P_K}(d)$ on P_K.

□

Corollary 14.2. *Let $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$, with $\phi_K : P_K \xrightarrow{\sim} \mathbb{P}^{n-1}(K)$, and let $\mathrm{O}_{P_K}(1) = \phi_K^* \mathrm{O}_{\mathbb{P}^{n-1}(K)}(1)$. Then $\mathrm{O}_{P_K}(1)$ is rational if and only if $P \simeq \mathbb{P}^{n-1}(k)$.*

Remark 14.3. If $P \not\simeq \mathbb{P}^{n-1}(k)$, i.e., the G-action on $\mathbb{P}^{n-1}(K)$ is twisted, then not every line bundle on $\mathbb{P}^{n-1}(K)$ is rational by Corollary 14.2. Despite this, the action of G on $\operatorname{Pic} P_K$ is trivial: Since each σ is an automorphism of P_K, it induces an automorphism on $\operatorname{Pic} P_K$, hence it takes $\mathrm{O}_{P_K}(1)$ to either $\mathrm{O}_{P_K}(1)$ or $\mathrm{O}_{P_K}(-1)$. Since the latter has no global sections, it must go to the former. Alternatively, the action by $\mathrm{PGL}_n(K)$ can be represented by $\mathrm{GL}_{n+1}(K)$ acting on the homogeneous coordinate ring $K[x_0, \ldots, x_{n-1}]$, and the action on $\operatorname{Div} X_K$ can then be traced by the action on forms. As $\mathrm{PGL}_n(K)$ preserves degree on divisors, divisor classes are $\mathrm{PGL}_n(K)$-fixed. Since the Galois action on K also preserves degree, we conclude any twisted action of G on P_K preserves degree. Thus each line bundle $\mathrm{O}_{\mathbb{P}^{n-1}(K)}(d)$ is G-invariant, whether or not it is rational ([2, p.197]).

Severi-Brauer varieties are projective by Theorem 12.2. Alternatively, by Proposition 14.1 we can show every Severi-Brauer variety is projective by observing that every smooth k-variety has an invertible sheaf, hence that a rational bundle exists for every (twisted) G-action on $\mathbb{P}^{n-1}(K)$. We first use the canonical sheaf:

Proposition 14.4. *Let $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$. Then we have an n-tuple embedding*

$$\nu_n : P \longrightarrow \mathbb{P}^{N-1}$$

where $N = \binom{2n-1}{n}$. In particular, P is projective.

Proof. Let Ω^1_X denote the sheaf of Kahler differentials on a nonsingular $n-1$-dimensional k-scheme X of finite type. The canonical sheaf is the wedge product $\omega_X \overset{\mathrm{df}}{=} \bigwedge^n \Omega^1_X \in \operatorname{Pic} X$. We have $\omega_{P_K} \simeq \mathrm{O}_{P_K}(-n)$ by [12, Corollary 6.4.17]. Since the canonical sheaf is stable under (flat) base change ([12, Theorem 6.4.9]), $\omega_P \otimes_k K = \omega_{P_K} \simeq \mathrm{O}_{P_K}(-n)$, hence $\mathrm{O}_{P_K}(n)$ is rational. Since $\omega_P(P) \otimes_k K \simeq \omega_{P_K}(P_K)$, the n-tuple embedding descends.

□

14.4. **Linear Subspaces.** We follow [2] as closely as we can. Suppose $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$. A *twisted linear subvariety* $L \subset P$ is a subvariety that becomes linear when P is split. If $L \subset P$ is a twisted linear subvariety of dimension $d-1$, we call d the "affine dimension" of L. Since $L \times_k K \simeq \mathbb{P}^{d-1}_K$, L is itself a Severi-Brauer variety.

We note as in [2] that the affine dimensions of possible twisted linear subvarieties form a subgroup of $\mathbb{Z}/n$. For if $L, L' \subset P$ are twisted linear subvarieties of affine dimensions d, d', then in general position, $L \cap L'$ has affine dimension 0 if $d + d' \leq n$, and affine dimension $n - (d + d')$ if $d + d' > n$. The set theoretic sum $L + L'$ in general position has affine dimension $d + d'$.

Duality. We write $\mathbb{P}^{n-1}(K) = \mathbb{P}(V)$ for the projective space whose points correspond to linear subspaces of a K-vector space V of dimension n. Let $\{e_i\}$ be a standard basis for V, with dual basis $\{x_i\}$ for V^*, so $K[x_0, \ldots, x_{n-1}]$ is the homogeneous coordinate ring for $\mathbb{P}^{n-1}(K)$. Let $\mathbb{P}^{n-1*}_K = \mathbb{P}(V^*)$ denote the dual projective space, with coordinate ring $K[e_0, \ldots, e_{n-1}]$. Suppose K/k is a finite Galois extension with group G, and V admits a left Galois G-action. The induced left Galois action on $\mathrm{GL}(V)$ is ${}^{\sigma}C = \sigma \cdot C \cdot \sigma^{-1}$, and since the G-action is Galois, we have an induced left action on $\mathrm{PGL}(V)$. The left G action on V induces a left Galois action on V^*, such that ${}^{\sigma}(v^*(w)) = {}^{\sigma}v^*({}^{\sigma}w)$, and hence left actions on $\mathrm{GL}(V^*)$ and $\mathrm{PGL}(V^*)$. Since $\mathrm{H}^1(G, \mathrm{GL}(V)) = 1$, every left Galois G-action on V is given by the standard action on coefficients with respect to some basis, and then the induced action on $\mathrm{GL}_n(K)$ is by coefficients, and the induced action on a dual basis of V^* is again on coefficients.

Lemma 14.5. *(Duality Lemma) The isomorphism* $\mathrm{PGL}(V) \xrightarrow{\sim} \mathrm{PGL}(V^*)$ *induces a bijection*

$$\begin{array}{ccc} \mathrm{H}^1(G, \mathrm{PGL}(V)) & \longrightarrow & \mathrm{H}^1(G, \mathrm{PGL}(V^*)) \\ [c] & \longmapsto & [c^{-t}] \end{array}$$

If P and P^* are corresponding Severi-Brauer varieties, we have an inclusion reversing correspondence between twisted linear subvarieties $L \subset P$ of dimension $d-1$ and those $L^* \subset P^*$ of codimension d.

Proof. The map $W \mapsto \mathrm{Ann}(W)$ sets up an inclusion-reversing 1-1 correspondence between subspaces $W \subset V$ of dimension d and subspaces $W^* \subset V^*$ of codimension d, hence between the corresponding linear subvarieties $L_K \subset P_K$ of dimension $d-1$ and $L_K^* \subset P_K^*$ of codimension d. If $C \in \mathrm{GL}(V)$ stabilizes W, then $C^t \in \mathrm{GL}(V^*)$ stabilizes W^*, hence C^{-t} stabilizes W^*. The map $\mathrm{GL}(V) \to \mathrm{GL}(V^*)$ sending C to C^{-t} is an isomorphism, and since it preserves scalar matrices it induces an isomorphism $\mathrm{PGL}(V) \to \mathrm{PGL}(V^*)$, under which c stabilizes L_K if and only if c^{-t} stabilizes L_K^*. Similarly if σ stabilizes W then $\sigma = \sigma^{-t}$ stabilizes W^*, so if G stabilizes a linear subvariety $L_K \subset P_K$ then it stabilizes $L_K^* \subset P_K^*$.

Since $\sigma^{-t} = \sigma$, the map $\mathrm{GL}(V) \to \mathrm{GL}(V^*)$ is G-equivariant, so $\mathrm{PGL}(V) \to \mathrm{PGL}(V^*)$ is G-linear, and we have the desired bijection $\mathrm{H}^1(G, \mathrm{PGL}(V)) \xrightarrow{\sim} \mathrm{H}^1(G, \mathrm{PGL}(V^*))$. By the compatibility of the induced G and $\mathrm{PGL}(V^*)$ actions, $\sigma\star_c$ stabilizes $L_K \subset P_K$ if and only if $\sigma\star_{c^{-t}}$ stabilizes $L_K^* \subset P_K^*$. Therefore twisted linear subvarieties of dimension $d-1$ in the quotient P correspond to twisted linear subvarieties of codimension d in P^*.

□

Proposition 14.6. *A Severi-Brauer variety P over k is trivial if and only if $P(k) \neq \varnothing$.*

Proof. [2, 3.3]. The result is due to Châtelet. If $p \in P(k)$ then by Lemma 14.5, P^* has a twisted hyperplane, hence an invertible sheaf that scalar extends to $\mathrm{O}_{P_K^*}(1)$. Therefore P^* is trivial by Proposition 14.1, hence it has a k-rational point, and since $(P^*)^* \simeq P$, P has a twisted hyperplane, hence P is trivial. Conversely if $P \simeq \mathbb{P}^{n-1}(k)$ then clearly $P(k) \neq \varnothing$.

□

Remark. Proposition 14.6 says that a Severi-Brauer variety P of degree n is trivial if and only if it shares a twisted linear subvariety with $\mathbb{P}^{n-1}(k)$, i.e., there exists a map $\phi : \mathbb{P}^{n-1}(K) \xrightarrow{\sim} P_K$ that is G-equivariant on a linear subvariety. We will show that similarly two Severi-Brauer varieties P and P' of degree n are equal if and only if they share a twisted linear subvariety, i.e., there exists an isomorphism $\phi : P_K \to P'_K$ that is G-equivariant on a linear subvariety.

14.5. **Reduction of Structure Group I.** Suppose K/k is a finite Galois extension with group G, and $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$ has a twisted linear subvariety L of dimension $d-1$. Let $\phi : \mathbb{P}^{n-1}(K) \xrightarrow{\sim} P_K$, and fix $\phi^{-1}(L_K) = \mathbb{P}^{d-1}_K \subset \mathbb{P}^{n-1}(K)$. Let $\mathbb{P}^{d-1}_k \subset \mathbb{P}^{n-1}(k)$ be the image in the quotient variety under the standard Galois action on $\mathbb{P}^{n-1}(K)$. There exists a projective transformation $\theta_k \in \mathrm{PGL}_n(k)$ such that $\theta_k^{-1}(\mathbb{P}^{d-1}_k)$ is in standard position in $\mathbb{P}^{n-1}(k)$, occupying the first d coordinates. Scalar extending to K we obtain a new isomorphism $\psi \overset{\mathrm{df}}{=} \phi \cdot \theta : \mathbb{P}^{n-1}(K) \xrightarrow{\sim} P_K$, and $\psi^{-1}(L_K) \subset \mathbb{P}^{n-1}(K)$ is in the same standard position. As $\theta \in \mathrm{PGL}_n(K)$, the corresponding cocycle c, defined by $c_\sigma = \psi^{-1} \cdot {}^\sigma\psi$, is cohomologous to the one defined by $\phi^{-1} \cdot {}^\sigma\phi$, and since θ is defined over k, each c_σ takes $\mathbb{P}^{d-1}_K$ to itself. Thus we may assume the cocycle c defining P from P_K takes values in the subgroup

$$\Gamma \overset{\mathrm{df}}{=} \left(\begin{array}{c|c} \mathrm{GL}_d(K) & * \\ \hline 0 & \mathrm{GL}_{n-d}(K) \end{array} \right)\Big/_{K^*} \leq \mathrm{PGL}_n(K)$$

The extra structure provided by L_K "reduces" the structure group. Define

$$U \overset{\mathrm{df}}{=} \left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array} \right)\Big/_{K^*} \leq \mathrm{PGL}_n(K)$$

$$\bar{\Gamma} \overset{\mathrm{df}}{=} \left(\begin{array}{c|c} \mathrm{GL}_d(K) & 0 \\ \hline 0 & \mathrm{GL}_{n-d}(K) \end{array} \right)\Big/_{K^*} \leq \mathrm{PGL}_n(K)$$

We have a G-stable decomposition $\Gamma = U \ltimes \bar{\Gamma}$, hence we may view $\mathrm{H}^1(G, \bar{\Gamma})$ as a subset of $\mathrm{H}^1(G, \Gamma)$. The group U is connected, since topologically it is a K-vector space, and it is unipotent, since every element is unipotent. Therefore $\mathrm{H}^1(G, U) = 0$ by the additive version of Hilbert 90 ([7, Section 6, Lemma 1]), hence the inclusion $\mathrm{H}^1(G, \bar{\Gamma}) \hookrightarrow \mathrm{H}^1(G, \Gamma)$ is surjective, hence

$$\mathrm{H}^1(G, \Gamma) = \mathrm{H}^1(G, \bar{\Gamma})$$

Thus if $c \in \mathrm{Z}^1(G, \mathrm{PGL}_n(K))$ stabilizes $\mathbb{P}^{d-1}_K \subset \mathbb{P}^{n-1}(K)$, then c is cohomologous to a cocycle with values in $\bar{\Gamma}$, by an element of $\Gamma \leq \mathrm{PGL}_n$. We conclude that whenever there is a twisted linear subvariety $L \subset P$ of affine dimension d, there is a complementary twisted linear subvariety $L' \subset P$ of affine dimension $n-d$, with $L_K \cap L'_K = \varnothing$ and $L_K + L'_K = P_K$. We apply reduction of structure to prove the following important result.

Proposition 14.7. *If $L \subset P$ is a twisted linear subvariety of P, then $[L] = [P]$ in $\mathrm{H}^2(G, K)$.*

Proof. We again follow Artin's outline, though the result belongs to Châtelet. Let L' be a complement of L in P. Consider the G-equivariant diagram

$$\begin{CD}
\mathrm{G_m} @>{\text{diag}}>> \mathrm{G_m}\times\mathrm{G_m} @>{(+,-)}>> \mathrm{G_m} \\
@| @VVV @VVV \\
\mathrm{G_m} @>>> \mathrm{GL}_d\times\mathrm{GL}_{n-d} @>>> \bar{\Gamma} \\
@. @VVV @VVV \\
@. \mathrm{PGL}_d\times\mathrm{PGL}_{n-d} @= \mathrm{PGL}_d\times\mathrm{PGL}_{n-d}
\end{CD}$$

where $(+,-)$ is the map $(x,y)\mapsto x/y$, the map $\mathrm{G_m}\to\bar{\Gamma}$ takes x/y to $(\mathrm{diag}\{x/y\},\mathrm{I}_{n-d})$, and we suppress explicit reference to K. Since the group decompositions are direct and G-stable, they commute with the formation of cohomology, and we have a diagram

$$\begin{CD}
\mathrm{H}^1(G,\mathrm{GL}_d)\times\mathrm{H}^1(G,\mathrm{GL}_{n-d}) @>>> \mathrm{H}^1(G,\bar{\Gamma}) \\
@VVV @VVV \\
\mathrm{H}^1(G,\mathrm{PGL}_d)\times\mathrm{H}^1(G,\mathrm{PGL}_{n-d}) @>{\text{id}}>> \mathrm{H}^1(G,\mathrm{PGL}_d)\times\mathrm{H}^1(G,\mathrm{PGL}_{n-d}) \\
@VVV @VVV \\
\mathrm{H}^2(G,K^*)\times\mathrm{H}^2(G,K^*) @>{(+,-)}>> \mathrm{H}^2(G,K^*)
\end{CD}$$

Let c be a cocycle for P with values in $\bar{\Gamma}$, $[a]$ the class of L in $\mathrm{H}^1(G,\mathrm{PGL}_d)$, and $[a']$ the class of L' in $\mathrm{H}^1(G,\mathrm{PGL}_{n-d})$. Chase:

$$\begin{CD}
@. [c] \\
@. @VVV \\
([a],[a']) @>{\text{id}}>> ([a],[a']) \\
@VVV @VVV \\
([L],[L']) @>{(+,-)}>> [L]-[L']=0
\end{CD}$$

We conclude $[L]=[L']$. It remains to show $[L]=[P]$. The map $(b_\sigma)\mapsto(b_\sigma\otimes\mathrm{I}_n)$ defines an injection $\mathrm{H}^1(G,\mathrm{PGL}_r)\to\mathrm{H}^1(G,\mathrm{PGL}_{rs})$ for any r,s, which commutes with the injection into $\mathrm{H}^2(G,K^*)$. Therefore, we may assume that d divides $n-d$ and $n-d$ divides n, hence that a' is cohomologous via some $b\in\mathrm{PGL}_{n-d}$ to $\lambda_{d,n-d}(a)$, hence that c is cohomologous to $\lambda_{d,n}(a)$. It follows that $[c]$ and $[a]$ have the same image in $\mathrm{H}^2(G,K^*)$, so $[L]=[P]$.

□

Remark 14.8. If P and Q are Severi-Brauer varieties, the following are equivalent.

(i) P embeds (via the Segre map) into the product $P\times_k Q$.
(ii) Q has a rational point.

(iii) $Q \simeq \mathbb{P}^{n-1}(k)$ for some n.
(iv) P is a twisted linear subvariety of $P \times_k Q$.
(v) $[P] = [P \times_k Q]$ in $\mathrm{H}^2(G, K^*)$.

Similarly on the level of algebras we have $[A \otimes_k B] = [A]$ if and only if $B = \mathrm{M}_n(k)$.

14.6. **Index.** The *index* of a central simple k-algebra A is $\mathrm{ind}(A) \stackrel{\mathrm{df}}{=} \sqrt{\dim_k D}$, where D is the k-division algebra underlying A. We would like to recognize this invariant from geometric properties of the corresponding Severi-Brauer variety $P = \mathrm{SBV}(A)$.

Proposition 14.9. *Let P be a Severi-Brauer variety over k, and let $A = \mathrm{CSA}(P)$. Then $\mathrm{ind}(A)$ is the smallest affine dimension of a twisted linear subvariety of P.*

Proof. This is [2, 3.4], again due to Châtelet. Set $n = \mathrm{ind}(A)$, and let $\mathrm{ind}(P)$ denote the smallest affine dimension of a twisted $L \subset P$. By the classical theory of central simple algebras, there is a separable field extension F/k of degree n that splits A, and since A_F is trivial, so is P_F. Pick $x \in P_F(F)$, let K/k be the Galois closure of F/k, and let G be the Galois group. Since x has coordinates in F, there are at most n distinct conjugates $\{\sigma(x)\}$ of x in P_K, and their additive span defines a G-invariant linear subvariety of P_K of affine dimension at most n. Therefore $\mathrm{ind}(P) \leq n$. Conversely if $L \subset P$ is a twisted linear subvariety of degree d, then $[L] = [P]$ by Proposition 14.7, hence $B = \mathrm{CSA}(L) \sim A$, and $\mathrm{ind}(B) = \mathrm{ind}(A) \leq d$. Therefore $n \leq \mathrm{ind}(P)$, hence $\mathrm{ind}(P) = \mathrm{ind}(A)$.

□

14.7. **Reduction of Structure Group II.** We lift more from [2], more or less verbatim. Suppose V and W admit a Galois G-action, and

$$\alpha : \mathrm{PGL}(V) \longrightarrow \mathrm{PGL}(W)$$

is a G-homomorphism defined over K. Let

$$\rho : \mathrm{H}^1(G, \mathrm{PGL}(V)) \longrightarrow \mathrm{H}^1(G, \mathrm{PGL}(W))$$

be the induced map, and if P is a Severi-Brauer variety represented by a class $[c] \in \mathrm{H}^1(G, \mathrm{PGL}(V))$, let P^W be the variety associated to the class $\rho([c])$. Suppose $W = W_1 \oplus W_2$ is a $\mathrm{PGL}(V)$-module decomposition, so that

$$\alpha(\mathrm{PGL}(V)) \subset (\mathrm{GL}(W_1) \times \mathrm{GL}(W_2))/K^* \leq \mathrm{PGL}(W)$$

Then as before we can show $[P^W] = [P^{W_1}] = [P^{W_2}]$. In particular, we have the following representations and corresponding Severi-Brauer varieties ([2, p.202]).

Representations	$W = V^*$	$V^{\otimes d}$	$\mathrm{Sym}^d V$	$\mathrm{Alt}^d V$
Severi-Brauer varieties	$P^W = P^*$	$P^{\times d}$	P^{Sym^d}	P^{Alt^d}

Here

$$\begin{aligned}\mathrm{Sym}^d V &= \{x \in V^{\otimes d} : x = \tau(x)\ \forall \tau \in \mathrm{Transp}(S_d)\} \\ \mathrm{Alt}^d V &= \{x \in V^{\otimes d} : x = -\tau(x)\ \forall \tau \in \mathrm{Transp}(S_d)\}\end{aligned}$$

where $\mathrm{Transp}(S_d)$ is the set of (nontrivial) transpositions of the symmetric group S_d. These are the (G-invariant) spaces of *symmetric* and *skew-symmetric tensors* of order d. This notation is nonstandard in characteristic dividing $d!$, since then it conflicts with the notation for symmetric and skew-symmetric tensor algebras ([5, III.6.3.Remark, III.7.4.Remark]. Clearly $\mathrm{Sym}^d V \cap \mathrm{Alt}^d V = 0$ unless $\mathrm{char} K = 2$, in which case the two subspaces are equal. Thus

$$V^{\otimes d} = \begin{cases} \mathrm{Sym}^d V \oplus \mathrm{Alt}^d V \oplus \cdots & \text{if } \mathrm{char} K \neq 2 \\ \mathrm{Sym}^d V \oplus \cdots & \text{if } \mathrm{char} K = 2 \end{cases}$$

Define the *degree of homogeneity* of α to be $d \in \mathbb{N}$ in the diagram

$$\begin{array}{ccccc} K^* & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ K^* & \longrightarrow & \mathrm{GL}(W) & \longrightarrow & \mathrm{PGL}(W) \end{array}$$

The existence of d is immediate once we prove the middle map exists; this is granted in [2]. By the long exact sequence for nonabelian cohomology, we have

$$[P^W] = d[P] \in \mathrm{H}^2(G, K^*)$$

Proposition 14.10. *Suppose P is a Severi-Brauer variety over k of degree n. Then*

(a) $[P^*] = -[P]$ *in* $\mathrm{H}^2(G, K^*)$.
(b) *If* $d \leq n$ *then* $[P^{\times d}] = [P^{\mathrm{Sym}^d}] = [P^{\mathrm{Alt}^d}] = d[P]$.
(c) *The period of* $A = \mathrm{CSA}(P)$ *divides* n.
(d) *If r is the period of A, then we have an r-tuple embedding*

$$\nu_r : P \longrightarrow \mathbb{P}^{N-1}(k) \qquad (N = \tbinom{n+r-1}{r})$$

Proof. If $W = V^*$ then $\tilde{\alpha}$ sends $C \in \mathrm{GL}(V)$ to $C^{-t} \in \mathrm{GL}(V^*)$, so the degree of homogeneity of the representation $\mathrm{PGL}(V) \to \mathrm{PGL}(V^*)$ is $d = -1$, hence $[P^*] = -[P]$.

If $V = K^n$, and $d \leq n$, then $V \to V^{\otimes d}$ induces $\alpha_d : \mathrm{PGL}(V) \to \mathrm{PGL}(V^{\otimes d})$, whose degree of homogeneity is easily seen to be d. Thus $[P^{\times d}] = d[P]$, and since $\mathrm{PGL}(\mathrm{Sym}^d V)$ and $\mathrm{PGL}(\mathrm{Alt}^d V)$ are direct factors of $\mathrm{PGL}(V^{\otimes d})$, we conclude $[P^{\mathrm{Sym}^d}] = [P^{\mathrm{Alt}^d}] = d[P]$ by reduction of structure. This proves (b). We already showed (c), but we sketch another proof. Since $\mathrm{Alt}^n V = K$, $[P^{\times n}] = [P^{\mathrm{Alt}^n}] = [\mathbb{P}^{n-1}(k)]$, hence $n[P] = 0$ in $\mathrm{H}^2(G, K^*)$. Thus the period of $[P]$ divides the degree of P.

We have $d[P] = 0$ if and only if $P^{\mathrm{Sym}^d} \simeq \mathbb{P}^{N-1}(k)$, where $N = \binom{n+d-1}{d}$; this is equivalent to the rationality of $\mathrm{O}_{P_K}(d)$. Using Proposition 14.1 and taking $d = r$ to be the period of $[P]$, we obtain the r-tuple embedding $\nu_r : P \to \mathbb{P}^{N-1}(k)$.

□

14.8. **Spectral Sequence.** Some of the above results fall out of a spectral sequence. We follow [2]. Suppose K/k is a finite Galois extension with group G. Let $P \in \mathrm{TF}_{K|k}(\mathbb{P}^{n-1}(k))$ be a Severi-Brauer variety of degree n over k, and let $\pi : \mathbb{P}^{n-1}(K) \to P$ be the projection. Set $A = \mathrm{CSA}(P)$. We have a Hochschild-Serre spectral sequence

$$\mathrm{H}^p(G, \mathrm{H}^q(\mathbb{P}^{n-1}(K), \mathrm{G_m})) \Rightarrow \mathrm{H}^{p+q}(P, \mathrm{G_m})$$

where the G-action on H^q can be deduced by applying each σ to an injective resolution for $\mathrm{G_m}$. The associated 5-term sequence yields an exact sequence

$$0 \longrightarrow \operatorname{Pic} P \xrightarrow{\pi^*} \operatorname{Pic} \mathbb{P}^{n-1}(K) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k(P))$$

where $k(P)$ is the function field of P. The map $\operatorname{Pic} \mathbb{P}^{n-1}(K) \to \mathrm{Br}(k)$ sends the generator $\mathrm{O}_{\mathbb{P}^{n-1}(K)}(1)$ to $[A] \in \mathrm{Br}(k)$. Since π^* is injective and $\operatorname{Pic} \mathbb{P}^{n-1}(K)$ is cyclic, we see that $\operatorname{Pic} P$ is cyclic; let $\mathrm{O}_P(1)$ be the "positive" generator, so that

$$\pi^* \mathrm{O}_P(1) = \mathrm{O}_{\mathbb{P}^{n-1}(K)}(r)$$

for some $r > 0$. By definition, r is the smallest positive integer such that $\mathrm{O}_{\mathbb{P}^{n-1}(K)}(r)$ descends, and by the exact sequence, r is also the period of $[A]$ in $\mathrm{Br}(k)$. By Proposition 14.1, with the descent of $\mathrm{O}_{\mathbb{P}^{n-1}(K)}(r)$ comes the descent of the r-tuple embedding,

$$\nu_r : P \to \mathbb{P}^{N-1}(k)$$

where $N = \binom{n+r-1}{r}$, and here $\mathrm{O}_P(1) = \nu_r^* \mathrm{O}_{\mathbb{P}^{N-1}(k)}(1)$.

It is easy to see that restriction from k to $k(P)$ kills $[A]$, since the class $P = \mathrm{SBV}(A)$ has a rational point over $k(P)$, namely, the generic point $\operatorname{Spec} k(P)$. Thus the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(P))$ contains $\langle [A] \rangle$. The exactness at $\mathrm{Br}(k)$ shows the kernel contains nothing more, a remarkable result.

14.9. **Example.** Let (a, b) denote the quaternion algebra over k defined by $a, b \in k^\times$, and let $P = \mathrm{SBV}(a, b)$. Since $(a, b) \in \mathrm{Br}(k)$ has degree 2, we have $\nu_2 : P \hookrightarrow \mathbb{P}^2_k$, and it can be shown that $\nu_2(P) = V(z^2 - ax^2 - by^2) \subset \mathbb{P}^2$, a conic. In Serre's *Course in Arithmetic*, (a, b) is the *Hilbert symbol*, assigned a value of $+1$ if P has a rational point, and -1 otherwise.

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