Generic Point.

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We define and prove the existence of generic points of schemes, and prove that the irreducible components of any scheme correspond bijectively to the scheme's generic points, and every open subset of an irreducible scheme contains that scheme's unique generic point. All of this material is standard, and [Liu] is a great reference.

Let X be a scheme. Recall X is irreducible if its underlying topological space is irreducible. A (nonempty) topological space is irreducible if it is not the union of two proper distinct closed subsets. Equivalently, if the intersection of any two nonempty open subsets is nonempty. Equivalently, if every nonempty open subset is dense.

Since X is a scheme, there can exist points that are not closed. If $x \in X$, we write $\overline{\{x\}}$ for the closure of x in X. This scheme is irreducible, since an open subset of $\overline{\{x\}}$ that doesn't contain x also doesn't contain any point of the closure of x, since the compliment of an open set is closed. Therefore every open subset of $\overline{\{x\}}$ contains x, and is (therefore) dense in $\overline{\{x\}}$.

Definition. ([Liu, 2.4.10]) A point x of X *specializes* to a point y of X if $y \in \overline{\{x\}}$. A point $\xi \in X$ is a *generic point of* X if ξ is the only point of X that specializes to ξ.

Ring theoretic interpretation. If $X = \operatorname{Spec} A$ is an affine scheme for a ring A, so that every point x corresponds to a unique prime ideal $\mathfrak{p}_x \subset A$, then x specializes to y if and only if $\mathfrak{p}_x \subset \mathfrak{p}_y$, and a point ξ is generic if and only if $\mathfrak{p}_\xi$ is minimal among prime ideals of A.

Every scheme contains a generic point. We first show every affine scheme has a generic point, i.e., every commutative ring with 1 has a minimal prime ideal. This is an exercise in Zorn's lemma. The set $\mathcal{S}$ of multiplicatively closed subsets not containing 0 in a ring A is nonempty (it contains $\{1\}$), partially ordered by inclusion, and the union of the subsets in a chain, being multiplicatively closed, serves as an upper bound. Therefore there is a maximal multiplicatively closed subset T not containing 0 in A by Zorn's lemma. The set $\mathcal{I}$ of ideals disjoint from T in A is also nonempty (contains (0)) and partially ordered, and the union of the ideals in a chain in $\mathcal{I}$ is an ideal disjoint from T, hence $\mathcal{I}$ has a maximal element $\mathfrak{p}$ by Zorn's lemma, and it is easy to show $\mathfrak{p}$ is prime. Since the complement of any prime ideal is multiplicatively closed, $\mathfrak{p}$ is a minimal prime of A, by the maximality of T. Thus $\operatorname{Spec} A$ has a generic point ξ, corresponding to $\mathfrak{p}$.

Now if X is a scheme, and $U = \operatorname{Spec} A$ is a nonempty open subset of X, then by the above U has a unique generic point ξ, corresponding to a minimal prime of A. If $\xi' \in X$ specializes to ξ, so $\xi \in \overline{\{\xi'\}}$, then $\xi' \in U$ since U is open, hence there is a prime ideal $\mathfrak{p}_{\xi'}$ of A corresponding to ξ', such that $\mathfrak{p}_{\xi'} \subset \mathfrak{p}_\xi$, and $\xi' = \xi$ by the minimality of $\mathfrak{p}_\xi$. Therefore ξ is a generic point of X.

Generic point criterion for irreducibility. We show a scheme X is irreducible if and only if it has a single generic point. First we note that every point of an affine scheme $U = \operatorname{Spec} A$ is contained in the closure of some generic point, that is, every prime ideal $\mathfrak{q}$ of A contains a minimal prime. For we can always embed the multiplicative set $A - \mathfrak{q}$ in a maximal multiplicative set using Zorn's lemma, and apply the same trick as we used above to show the existence of a generic point. It follows that if ξ is the only generic point of U, then $U = \overline{\{\xi\}}$. If ξ is not the only generic point, then the closure of the open set $U - \overline{\{\xi\}}$ is nonempty ($U - \overline{\{\xi\}}$ contains the other generic point), and consists of those points corresponding to primes of A that contain a prime that doesn't contain $\mathfrak{p}_\xi$. Thus the closure of $U - \overline{\{\xi\}}$ is not all of U, since it does not contain ξ. Therefore if ξ is not the only generic point of U, then $U - \overline{\{\xi\}}$ is a nonempty open set that is not dense in U. We conclude that $U = \operatorname{Spec} A$ is irreducible if and only if it has a unique generic point ξ, and then $U = \overline{\{\xi\}}$.

Now if X is a scheme, then it has a generic point ξ, contained in some affine open set U. If X is irreducible, then U is dense in X, hence U is also irreducible, so U is the closure of ξ (in U), as we just showed. Therefore $X = \overline{\{\xi\}}$. If also $X = \overline{\{\xi'\}}$, then $\xi = \xi'$ by definition of generic point, so ξ is unique. Conversely, if X has a unique generic point ξ, then we easily show $X = \overline{\{\xi\}}$, since X is covered by affine open sets, and since ξ is contained in every open subset of X, every nonempty open subset of X is dense, hence X is irreducible. We conclude a scheme X is irreducible if and only if it has a unique generic point.

Bijective correspondence between generic points and irreducible components. An irreducible component Z of a scheme X is an irreducible closed subscheme that is maximal among all irreducible closed subschemes. Since Z is irreducible, Z equals the closure of its generic point ξ. As we pointed out above, ξ is contained in $\overline{\{\xi'\}}$ for some generic point ξ' of X. Since $\overline{\{\xi'\}}$ is an irreducible closed subscheme of X, we conclude $\xi = \xi'$ by maximality of $\overline{\{\xi\}}$, hence ξ is a generic point of X. Conversely, if ξ is a generic point of X, then $\overline{\{\xi\}}$ is obviously an irreducible component of X, with generic point ξ. Thus we have our bijective correspondence.

Philosophy of generic points. See [Liu, Chapter 2, Remark 4.15]. The following observation underlies the technique of using generic points in algebraic geometry. Suppose X is an irreducible scheme, with generic point ξ. Suppose that the set of points of X satisfying a property of schemes is inherently an open subset U of X. Then, as we have seen, U is nonempty if and only if it contains ξ, i.e., if and only if ξ satisfies the property. Thus we only have to test the generic point. We often say a scheme is "generically (place adjective here)" to mean that the adjective applies at the generic point, hence on a dense open subset.

Reference.

[Liu] Liu, Q.: *Algebraic Geometry and Arithmetic Curves* (translated by Reinie Erné), Oxford University Press, New York, 2002.