

Alternating Forms and the Brauer Group of a Field Containing the Roots of Unity

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ABSTRACT. We compute $H^2(G, \mathbb{Q}/\mathbb{Z})$ for any proabelian group G , using a natural isomorphism with the group $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ of continuous alternating forms. We use this to establish an effective behavioral ideal for the Brauer group $\text{Br}(F)$ of a field F of characteristic zero that contains all roots of unity. We show this ideal is exact in several interesting cases.

INTRODUCTION.

Let F be a field of characteristic 0 that contains μ , the group of all roots of unity. Our aim is to establish a *behavioral ideal* for the division algebras and Brauer group $\text{Br}(F)$ of F . This ideal consists of *a priori* bounds for a given Brauer class's most important invariants, as well as explicit computations of the fundamental functorial maps inflation, restriction, and corestriction, with respect to abelian extensions. We do this by exploiting a natural isomorphism $\text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Q}/\mathbb{Z})$ where G is any proabelian group and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ is the group of continuous alternating forms on G with values in \mathbb{Q}/\mathbb{Z} . We compute the theory of $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ using simple linear algebra techniques, developed and applied over \mathbb{Q}/\mathbb{Z} . We define and compute the exponent, index, and canonical form of an alternating form, and we compute inflation, restriction, and corestriction maps. Our method here is based on the recognition that $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ is computable from $G^* = H^1(G, \mathbb{Q}/\mathbb{Z})$ as a type of exterior product. We thus obtain a detailed and computationally effective theory of $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ for any proabelian group G . Letting G be a Galois group over F , we then use Kummer Theory to associate elements in the d -torsion subgroup of G^* with elements in $F^\times / F^{\times d}$, for each $d \in \mathbb{N}$. This allows us to compute the composite $\text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Br}(F)$ in terms of symbol classes, converting our theory of alternating forms on G into a theory of symbol classes in $\text{Br}(F)$. When G is F 's absolute abelian Galois group, the map is surjective, and we obtain our behavioral ideal for $\text{Br}(F)$.

In certain interesting and nontrivial cases the ideal is realized, and we obtain explicit index formulas and functorial maps, and a simple algorithm that determines a Brauer class's representative division algebra. This is the case, for example, when $F = \mathbb{C}((x_1)) \cdots ((x_n))$, the field of iterated power series, or any strictly henselian field (see also [B], [TW]). There we obtain a complete theory, including the symbol decomposition of $\text{Br}(F)$ with respect to the x_i , transformation formulas for symbols under restriction and corestriction, the relative Brauer group for finite Kummer extensions, canonical decompositions into disjoint reduced symbols, and an explicit index formula. The algorithm for placing a class in canonical form also determines a presentation for the class's representative division algebra.

Similarly complete results hold for the subgroup of $\text{Br}(F)$ generated by symbols $[(x_i, x_j)_{\omega_d}]$, whenever the elements x_i form part of a transcendence basis for F over some subfield. In general, however, we obtain only upper bounds for the index of a class, and a "disjoint reduced symbol

2000 *Mathematics Subject Classification.* Primary 16K50; Secondary 20J06.

presentation” that is only a candidate for representative division algebra. We retain the explicit descriptions of restriction and corestriction with respect to finite abelian extensions in all cases.

To make explicit calculations in this theory we assume G is torsion-free, we identify a basis of “Frobenius generators” for G , and we map them to “prime elements” in F .

Example. Here is how our theory explicitly calculates index, representative division algebra, and restriction and corestriction in the Brauer group. Let $F = \mathbb{C}(x_1, x_2, x_3, x_4)$, the function field in four variables. Our theory computes the part of the Brauer group that is expressible in terms of the x_i . This is a subgroup, which we call $\text{Br}^\circ(\Omega/F)$, of the relative Brauer group $\text{Br}(\Omega/F)$, where Ω is the (infinite) Kummer extension obtained by taking all possible roots of the x_i . Let $[(x_i, x_j)_{\omega_d}^r]$ denote the Brauer class of the r -th tensor power of the symbol algebra of degree d defined by x_i, x_j , and a primitive root of unity ω_d . Let $\alpha \in \text{Br}(F)$ be the element

$$\alpha = [(x_1, x_2)_{\omega_4}^{-1}] + [(x_1, x_3)_{\omega_{20}}^9] + [(x_1, x_4)_{\omega_{20}}^{-1}] + [(x_2, x_3)_{\omega_{20}}] + [(x_2, x_4)_{\omega_5}^{-1}] + [(x_3, x_4)_{\omega_4}].$$

α 's coefficients define the alternating matrix

$$\mathcal{A} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{9}{20} & -\frac{1}{20} \\ \frac{1}{4} & 0 & \frac{1}{20} & -\frac{1}{5} \\ -\frac{9}{20} & -\frac{1}{20} & 0 & \frac{1}{4} \\ \frac{1}{20} & \frac{1}{5} & -\frac{1}{4} & 0 \end{pmatrix} \in \text{Alt}_4(\mathbb{Q}/\mathbb{Z}).$$

Although this matrix is associated directly to the given Brauer class, in our theory it is the matrix of an alternating form on the Galois group $G_{\Omega/F} \simeq \widehat{\mathbb{Z}}^4$, given by specifying a basis of generators for G , which are “Frobenius” with respect to the “primes” x_i . The index of α is given by the formula

$$\begin{aligned} \text{ind}(\alpha) &= \text{lcm}\{|a_{ij}|, |a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}|\} \\ &= \text{lcm}\{|1/4|, |1/20|, |1/5|, |1/40|\} = 40, \end{aligned}$$

computed by taking the least common multiples of the orders in \mathbb{Q}/\mathbb{Z} of the subpfaffians of the alternating matrix \mathcal{A} . To find the division algebra underlying α , we “2-block-diagonalize” \mathcal{A} using the standard algorithm. If

$$\mathcal{P} = \begin{pmatrix} 3 & 0 & -1 & 0 \\ -2 & 5 & 1 & 1 \\ 5 & 8 & -2 & 2 \\ 15 & 7 & -6 & 2 \end{pmatrix} \in \text{GL}_4(\mathbb{Z}); \quad \mathcal{Q} = \mathcal{P}^{-1} = \begin{pmatrix} 4 & -2 & 3 & -2 \\ -4 & 4 & -5 & 3 \\ 11 & -6 & 9 & -6 \\ 17 & -17 & 22 & -13 \end{pmatrix}$$

then

$$\mathcal{A}' = \mathcal{P}^* \mathcal{A} \mathcal{P} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{20} \\ 0 & 0 & -\frac{1}{20} & 0 \end{pmatrix} \in \text{Alt}_4(\mathbb{Q}/\mathbb{Z}),$$

where \mathcal{P}^* is \mathcal{P} 's transpose. This shows α has *canonical form*

$$\alpha = [(x'_1, x'_2)_{\omega_2}] + [(x'_3, x'_4)_{\omega_{20}}]$$

with representative division algebra $D \simeq (x'_1, x'_2)_{\omega_2} \otimes (x'_3, x'_4)_{\omega_{20}}$. The x'_i are algebraically independent over \mathbb{C} , and $x'_i = x_1^{q_{i1}} x_2^{q_{i2}} x_3^{q_{i3}} x_4^{q_{i4}}$, with exponents taken from \mathcal{Q} 's i -th row.

Now let K/F be a finite subextension of Ω/F , defined by $K = F(y_1, y_2, y_3, y_4)$, where $y_i = x_1^{q'_{i1}} x_2^{q'_{i2}} x_3^{q'_{i3}} x_4^{q'_{i4}}$, and the q'_{ij} are given by the matrix

$$\mathcal{Q}' = (q'_{ij}) = \begin{pmatrix} \frac{7}{15} & -\frac{2}{15} & -\frac{1}{5} & -\frac{7}{15} \\ -\frac{8}{15} & \frac{11}{30} & \frac{4}{5} & \frac{61}{30} \\ -\frac{2}{15} & -\frac{1}{30} & \frac{1}{5} & \frac{19}{30} \\ -\frac{2}{15} & -\frac{1}{30} & \frac{1}{5} & -\frac{11}{30} \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Q}).$$

It is important that the y_i form a basis for the subgroup they generate in $F^\times / F^{\times 30}$, so that \mathcal{Q} 's inverse is defined over \mathbb{Z} . Here,

$$\mathcal{P}' = (\mathcal{Q}')^{-1} = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 0 & 2 & -7 & -1 \\ 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in \mathrm{M}_4(\mathbb{Z}).$$

It follows that $[K : F] = |\det(\mathcal{P}')| = 30$. Let $\mathcal{A} \in \mathrm{Alt}_4(\mathbb{Q}/\mathbb{Z})$ and $\alpha \in \mathrm{Br}(F)$ be as above. The matrices \mathcal{P}' and \mathcal{Q}' , which represent the field extension K/F , are used to perform functorial operations on the alternating matrix \mathcal{A} representing α . For example, $\beta := \mathrm{res}(\alpha) \in \mathrm{Br}(K)$ is determined by the matrix

$$\mathcal{B} = \mathcal{P}'^* \mathcal{A} \mathcal{P}' = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{5} & \frac{11}{20} \\ \frac{1}{4} & 0 & \frac{17}{20} & \frac{3}{20} \\ -\frac{1}{5} & -\frac{17}{20} & 0 & -\frac{1}{20} \\ -\frac{11}{20} & -\frac{3}{20} & \frac{1}{20} & 0 \end{pmatrix} \in \mathrm{Alt}_4(\mathbb{Q}/\mathbb{Z}).$$

Explicitly,

$$(*) \quad \beta = \mathrm{res}(\alpha) = [(y_1, y_2)_{\omega_4}^{-1}] + [(y_1, y_3)_{\omega_5}] + [(y_1, y_4)_{\omega_{20}}^{11}] + [(y_2, y_3)_{\omega_{20}}^{17}] + [(y_2, y_4)_{\omega_{20}}^3] + [(y_3, y_4)_{\omega_{20}}^{-1}]$$

where each $y_i = \prod_j x_j^{q'_{ij}}$ is given by \mathcal{Q}' . The index of β is

$$\mathrm{ind}(\beta) = \mathrm{lcm}\{|1/4|, |1/5|, |1/20|, |1/80 - 3/100 + 187/400|\} = \mathrm{lcm}\{4, 5, 20, |9/20|\} = 20.$$

This degree 30 scalar extension K/F reduces the index of α by a factor of two.

Conversely, suppose $\beta \in \mathrm{Br}(K)$ is given by (*). Then $\gamma := \mathrm{cor}(\beta) \in \mathrm{Br}(F)$ is determined by the matrix

$$\mathcal{C} = 30(\mathcal{Q}'^* \mathcal{B} \mathcal{Q}') = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix} \in \mathrm{Alt}_4(\mathbb{Q}/\mathbb{Z}).$$

Thus

$$\gamma = \mathrm{cor}(\beta) = [(x_1, x_2)_{\omega_2}] + [(x_1, x_3)_{\omega_2}] + [(x_1, x_4)_{\omega_2}] + [(x_2, x_3)_{\omega_2}] + [(x_3, x_4)_{\omega_2}]$$

where $x_i = \prod_j y_j^{p_{ij}'}$ is given by \mathcal{P}' . Note $\gamma = \text{cor} \cdot \text{res}(\alpha) = 30\alpha$, as required. The index is

$$\text{ind}(\gamma) = \text{lcm}\{|1/2|, |1/4 - 0 + 1/4|\} = 2.$$

Thus the corestriction reduced β 's index by a factor of 10.

The author thanks Adrian Wadsworth for invaluable, detailed feedback on preliminary drafts, including the improvement of several proofs. The author thanks David Saltman and Emily Hamilton for valuable comments and suggestions. The author also thanks the UCLA and UCSD Math Departments for their warm hospitality during his visits.

0. CONVENTIONS

Write \mathcal{P} for the set of rational primes. If $d \in \mathbb{N}$, let $\text{supp}(d)$ denote d 's prime support. Use \mathfrak{m} to denote a subset of \mathcal{P} . Let $\mathbb{N}(\mathfrak{m})$ denote the set of positive integers supported on \mathfrak{m} .

For each prime p , let \mathbb{Z}_p be the additive group underlying the ring of p -adic integers. If $\mathfrak{m} \subset \mathcal{P}$ is a set of primes, let $\mathbb{Z}_{\mathfrak{m}}$ denote the product $\prod_{\mathfrak{m}} \mathbb{Z}_p$. Let $\widehat{\mathbb{Z}}$ be the profinite completion of \mathbb{Z} , then $\widehat{\mathbb{Z}} = \mathbb{Z}_{\mathfrak{m}}$, where \mathfrak{m} is the set of all primes. Let $\mathbb{Z}[1/p]$ denote the additive group of the localization of \mathbb{Z} with respect to the prime number p , and let $\mathbb{Z}[1/p]/\mathbb{Z}$ denote the quotient, a divisible torsion p -group. Let $\mathbb{Z}[1/\mathfrak{m}]/\mathbb{Z}$ denote the (divisible) direct sum $\coprod_{\mathfrak{m}} \mathbb{Z}[1/p]/\mathbb{Z}$. Note $\mathbb{Z}[1/\mathfrak{m}]/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$ when \mathfrak{m} is the set of all primes.

Each $\mathbb{Z}_{\mathfrak{m}}$ acts diagonally on $\mathbb{Z}[1/\mathfrak{m}]/\mathbb{Z}$ with respect to the primary decomposition. Explicitly, if $x = \prod_{\mathfrak{m}} x_p \in \mathbb{Z}_{\mathfrak{m}}$, where $x_p \in \mathbb{Z}_p$, and $a = \prod_{\mathfrak{m}} a_p \in \mathbb{Z}[1/p]/\mathbb{Z}$, where $a_p \in \mathbb{Z}[1/p]/\mathbb{Z}$, then $x \cdot a = \prod_{\mathfrak{m}} x_p a_p$. In particular, if $x \in \widehat{\mathbb{Z}}$ and $1/d \in \mathbb{Q}/\mathbb{Z}$, then $x/d \in \mathbb{Q}/\mathbb{Z}$ is the product of $x \pmod{d\widehat{\mathbb{Z}}}$ and $1/d$. Extend the action of $\mathbb{Z}_{\mathfrak{m}}$ on $\mathbb{Z}[1/\mathfrak{m}]/\mathbb{Z}$ to all of \mathbb{Q}/\mathbb{Z} via the standard embedding $\mathbb{Z}_{\mathfrak{m}} \rightarrow \widehat{\mathbb{Z}}$, so that the action on the other primary components is trivial.

If G is an abelian group and $d \in \mathbb{N}$, let G^d (or dG) and ${}_dG$ denote the d -power (or d -multiple for additive notation) and d -torsion subgroup of G , respectively. Let $|G|$ denote the cardinality of G , and let $\text{exp}(G)$, the *exponent* of G , denote the smallest number $m \in \mathbb{N}$ such that $G^m = \{1\}$ (or $mG = \{0\}$).

If G is a proabelian group, say a natural number d *divides* $|G|$ if d divides the order of a finite homomorphic image of G . If $\sigma \in G$, say d divides the supernatural number $|\sigma|$ if d divides the order of some finite homomorphic image of σ .

A subset $\underline{x} = \{x_i\}_I$ of a (multiplicative) abelian group G is *p-independent* for a prime p if for every finite subset $\{x_{i_1}, \dots, x_{i_k}\}$, $\prod_{j=1}^k x_{i_j}^{a_j} \in G^{p^r}$ implies $p^r \mid a_j$, for all $j : 1 \leq j \leq k$. Note \underline{x} is *p-independent* if and only if it is independent $\pmod{G^p}$. Say \underline{x} is *m-independent* for some $\mathfrak{m} \subset \mathcal{P}$ if it is *p-independent* for all $p \in \mathfrak{m}$. Say \underline{x} is \mathbb{Z} -independent if it is \mathcal{P} -independent. A subgroup H of an abelian group G is *pure* if whenever an element of H has an n -th root in G , it has an n -th root in H . A *pure subset* of G is a subset that generates a pure subgroup.

1. DUALITY

By Pontrjagin Duality [RZ, 2.9.6], the category of proabelian groups is dual to the category of discrete torsion abelian groups. In particular, there is an exact contravariant functor “ $*$ ” from proabelian groups to discrete torsion abelian groups, sending an object G to $G^* = \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$, the group of continuous homomorphisms into the discrete group \mathbb{Q}/\mathbb{Z} ; and sending a homomorphism f to its *transpose* f^* , defined by composition $f^*(\phi^*) = \phi^* \cdot f$, for $\phi^* \in G^*$. In [RZ], it is shown that there is a natural isomorphism $G \simeq G^{**} = \text{Hom}(G^*, \mathbb{Q}/\mathbb{Z})$.

Suppose G is a proabelian group, and \mathbb{Q}/\mathbb{Z} is the continuous G -module with trivial action. Then $G^* \simeq H^1(G, \mathbb{Q}/\mathbb{Z})$. It will soon be shown that Pontrjagin duality extends to dimension 2, to groups $\tilde{\otimes}^2 G^*$ and $\tilde{\wedge}^2 G^*$ that correspond to the groups of continuous bilinear forms and continuous alternating forms on $G \times G$, with values in \mathbb{Q}/\mathbb{Z} . It will be proved in Theorem 2.4 below that $\tilde{\wedge}^2 G^* = \tilde{\wedge}^2 H^1(G, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$.

To construct the two functors “ $\tilde{\otimes}^2$ ” and “ $\tilde{\wedge}^2$ ” from general torsion abelian groups G^* to torsion abelian groups $\tilde{\otimes}^2 G^*$ and $\tilde{\wedge}^2 G^*$, the usual tensor product definition must be modified to produce a nontrivial object should G^* happen to be divisible. In particular, a direct limit applied to the usual tensor product does not work. Here is the modified definition.

Let G^* be a torsion abelian group. Let $\mathbb{Z}\langle G^* \times G^* \rangle$ denote the free abelian group generated by the set $G^* \times G^*$. Let \mathcal{R} denote the subgroup generated by the following types of elements:

$$(1.1) \quad \begin{aligned} & (\phi^*, \psi^*) - \left(\frac{m}{d}\phi^*, \frac{n}{d}\psi^*\right); \\ & (\phi^*, \psi_1^*) + (\phi^*, \psi_2^*) - \left(\phi^*, \frac{n_1}{d_1}\psi_1^* + \frac{n_2}{d_2}\psi_2^*\right); \\ & (\phi_1^*, \psi^*) + (\phi_2^*, \psi^*) - \left(\frac{m_1}{c_1}\phi_1^* + \frac{m_2}{c_2}\phi_2^*, \psi^*\right); \end{aligned}$$

where $m = |\phi^*|$, $n = |\psi^*|$, $m_i = |\phi_i^*|$, $n_i = |\psi_i^*|$, $d = \text{gcd}(m, n)$, $d_i = \text{gcd}(m, n_i)$, and $c_i = \text{gcd}(m_i, n)$.

Definition 1.2.

$$\tilde{\otimes}^2 G^* := \frac{\mathbb{Z}\langle G^* \times G^* \rangle}{\mathcal{R}}.$$

Let $\phi^* \otimes \psi^*$ denote the image of (ϕ^*, ψ^*) in $\tilde{\otimes}^2 G^*$. Say the representative $\phi^* \otimes \psi^*$ is *reduced* if $m = n = d$. If $f^* : G^* \rightarrow H^*$ is a homomorphism of torsion abelian groups, let $c = \text{lcm}[m, n]$ and $c' = \text{lcm}[|f^*(\phi^*)|, |f^*(\psi^*)|]$, and let $f^{*\otimes 2}$ be the function defined by

$$\begin{aligned} f^{*\otimes 2} : \tilde{\otimes}^2 G^* &\longrightarrow \tilde{\otimes}^2 H^* \\ \phi^* \otimes \psi^* &\longmapsto \frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)). \end{aligned}$$

The function $f^{*\otimes 2}$ will be shown to be well defined in Lemma 1.4.

Consequences 1.3. Here is a list of consequences of (1.1).

a. For all $r \in \mathbb{N}$, $r(\phi^* \otimes \psi^*) = \phi^* \otimes \left(\frac{rm}{d}\psi^*\right) = \left(\frac{rm}{d}\phi^*\right) \otimes \psi^*$. To see this, induct on r . When $r = 1$, the result holds by the defining relations on $\tilde{\otimes}^2 G^*$. If the result holds for $r - 1$,

$$r(\phi^* \otimes \psi^*) = \phi^* \otimes \psi^* + (r - 1)(\phi^* \otimes \psi^*) = \phi^* \otimes \psi^* + \phi^* \otimes \left(\frac{(r-1)n}{d}\psi^*\right).$$

Since the order of $\frac{(r-1)n}{d}\psi^*$ divides m ,

$$\phi^* \otimes \psi^* + \phi^* \otimes \left(\frac{(r-1)n}{d}\psi^*\right) = \phi^* \otimes \left(\frac{n}{d}\psi^* + \frac{(r-1)n}{d}\psi^*\right) = \phi^* \otimes \left(\frac{rn}{d}\psi^*\right).$$

By the second relation in (1.1), the result is true for all r , by induction. The other equality is proved in the same way.

b. Every $\phi^* \otimes \psi^*$ has a reduced representative by (1.1). By (a), on reduced elements

$$r(\phi^* \otimes \psi^*) = (r\phi^*) \otimes \psi^* = \phi^* \otimes (r\psi^*).$$

In particular, if $d = \gcd(m, n)$, $d(\phi^* \otimes \psi^*) = d\left(\frac{m}{d}\phi^* \otimes \frac{n}{d}\psi^*\right) = 0 \otimes \frac{n}{d}\psi^* = 0$. Thus $|\phi^* \otimes \psi^*|$ divides d , and $\tilde{\otimes}^2 G^*$ is a torsion abelian group. It will be shown in Theorem 2.3 that in fact $|\phi^* \otimes \psi^*| = d = \gcd(|\phi^*|, |\psi^*|)$.

c. If $(r, m) = (r, n) = 1$, then $r(\phi^* \otimes \psi^*) = (r\phi^*) \otimes \psi^* = \phi^* \otimes (r\psi^*)$, whether or not $\phi^* \otimes \psi^*$ is reduced. This follows immediately from (a).

d. If r divides d , then $r(\phi^* \otimes \psi^*) = (r\phi^*) \otimes (r\psi^*)$, whether or not $\phi^* \otimes \psi^*$ is reduced. This follows immediately from (a) and the first relation in (1.1).

e. If $\text{lcm}[n_1, n_2]$ divides m , then $\phi^* \otimes (\psi_1^* + \psi_2^*) = \phi^* \otimes \psi_1^* + \phi^* \otimes \psi_2^*$. This follows immediately from (1.1).

Lemma 1.4. *Suppose $f^* : G^* \rightarrow H^*$ is a homomorphism, and $\phi^*, \psi^* \in G^*$. Let $m = |\phi^*|$, $n = |\psi^*|$, $m' = |f^*(\phi^*)|$, $n' = |f^*(\psi^*)|$, $c = \text{lcm}[m, n]$, $d = \gcd(m, n)$, $c' = \text{lcm}[m', n']$, and $d' = \gcd(m', n')$. Then*

$$\begin{aligned} f^{*\otimes 2} : \tilde{\otimes}^2 G^* &\longrightarrow \tilde{\otimes}^2 H^* \\ \phi^* \otimes \psi^* &\longmapsto \frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)) \end{aligned}$$

is a well defined homomorphism. Moreover,

$$\frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)) = \left(\frac{c}{n'}f^*(\phi^*)\right) \otimes f^*(\psi^*) = f^*(\phi^*) \otimes \left(\frac{c}{m'}f^*(\psi^*)\right).$$

Proof. Define a homomorphism

$$\begin{aligned} f^{*\times 2} : \mathbb{Z}\langle G^* \times G^* \rangle &\longrightarrow \tilde{\otimes}^2 H^* \\ (\phi^*, \psi^*) &\longmapsto \frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)). \end{aligned}$$

Note $m'n' = c'd'$. By (1.3a), $\frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)) = \left(\frac{c}{n'}f^*(\phi^*)\right) \otimes f^*(\psi^*) = f^*(\phi^*) \otimes \left(\frac{c}{m'}f^*(\psi^*)\right)$, as desired. To show that the kernel of $f^{*\times 2}$ contains all elements of the form (1.1), use $cd = mn$, $\text{lcm}[\frac{m}{d}\phi^*, |\psi^*|] = n$, and the result in the preceding sentence to check

$$f^{*\times 2}\left(\frac{m}{d}\phi^*, \psi^*\right) = f^{*\times 2}\left(\frac{c}{n}\phi^*, \psi^*\right) = \left(\frac{n}{n'}f^*\left(\frac{c}{n}\phi^*\right)\right) \otimes f^*(\psi^*) = \left(\frac{c}{n'}f^*(\phi^*)\right) \otimes f^*(\psi^*) = f^{*\times 2}(\phi^*, \psi^*),$$

as desired. Similarly, $f^{*\times 2}(\phi^*, \frac{n}{d}\psi^*) = f^{*\times 2}(\phi^*, \psi^*)$. Thus $\ker(f^{*\times 2})$ contains the first type of element in (1.1). For the second type, let $n_i = |\psi_i^*|$, $d_i = \gcd(m, n_i)$, and $c_i = \text{lcm}[m, n_i]$. Compute on one hand

$$f^{*\times 2}((\phi^*, \psi_1^*) + (\phi^*, \psi_2^*)) = f^*(\phi^*) \otimes (\frac{c_1}{m'} f^*(\psi_1^*)) + f^*(\phi^*) \otimes (\frac{c_2}{m'} f^*(\psi_2^*)).$$

Since $|\frac{c_i}{m'} f^*(\psi_i^*)|$ divides $|f^*(\phi^*)|$, this is $f^*(\phi^*) \otimes (\frac{c_1}{m'} f^*(\psi_1^*) + \frac{c_2}{m'} f^*(\psi_2^*))$, by (1.3e). On the other hand, since $\text{lcm}[d_1, d_2]$ divides m and $c_i d_i = m n_i$,

$$f^{*\times 2}(\phi^*, \frac{n_1}{d_1}\psi_1^* + \frac{n_2}{d_2}\psi_2^*) = f^*(\phi^*) \otimes \frac{m}{m'} f^*(\frac{n_1}{d_1}\psi_1^* + \frac{n_2}{d_2}\psi_2^*) = f^*(\phi^*) \otimes (\frac{c_1}{m} f^*(\psi_1^*) + \frac{c_2}{m} f^*(\psi_2^*)).$$

Thus $\ker(f^{*\times 2})$ contains the second type of element in (1.1). Similarly, it contains the third type. By the universal property of free abelian groups, $f^{*\times 2}$ extends to the homomorphism of the lemma. ■

Definition 1.5. Define $\text{alt} : \tilde{\otimes}^2 G^* \longrightarrow \tilde{\otimes}^2 G^*$ by $\phi^* \otimes \psi^* \mapsto \phi^* \otimes \psi^* - \psi^* \otimes \phi^*$, and set

$$\tilde{\wedge}^2 G^* := \text{alt}(\tilde{\otimes}^2 G^*)$$

Let $\phi^* \wedge \psi^* = \text{alt}(\phi^* \otimes \psi^*) = \phi^* \otimes \psi^* - \psi^* \otimes \phi^*$. If $f^* : G^* \rightarrow H^*$ is a homomorphism of torsion abelian groups, assume the notation of Definition 1.2, and set

$$\begin{aligned} f^{*\wedge 2} : \tilde{\wedge}^2 G^* &\longrightarrow \tilde{\wedge}^2 H^* \\ \phi^* \wedge \psi^* &\longmapsto \frac{c}{c'}(f^*(\phi^*) \wedge f^*(\psi^*)). \end{aligned}$$

It is easily checked that alt is a well defined homomorphism. The map $f^{*\wedge 2}$ is the restriction of $f^{*\otimes 2}$ to $\tilde{\wedge}^2 G^*$. For compute

$$\begin{aligned} f^{*\otimes 2}(\phi^* \wedge \psi^*) &= f^{*\otimes 2}(\phi^* \otimes \psi^* - \psi^* \otimes \phi^*) = f^{*\otimes 2}(\phi^* \otimes \psi^*) - f^{*\otimes 2}(\psi^* \otimes \phi^*) \\ &= \frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*)) - \frac{c}{c'}(f^*(\psi^*) \otimes f^*(\phi^*)) \\ &= \frac{c}{c'}(f^*(\phi^*) \wedge f^*(\psi^*)) = f^{*\wedge 2}(\phi^* \wedge \psi^*). \end{aligned}$$

Theorem 1.6. $\tilde{\otimes}^2 : (G^*, f^*) \mapsto (\tilde{\otimes}^2 G^*, f^{*\otimes 2})$ and $\tilde{\wedge}^2 : (G^*, f^*) \mapsto (\tilde{\wedge}^2 G^*, f^{*\wedge 2})$ are covariant functors from (divisible) torsion abelian groups to (divisible) torsion abelian groups.

Proof. Suppose G^* is divisible, $s \in \mathbb{N}$, and $\phi^* \otimes \psi^* \in \tilde{\otimes}^2 G^*$. Let $m = |\phi^*|$, $n = |\psi^*|$, and $d = |\phi^* \otimes \psi^*|$. By (1.3b), d divides $\gcd(m, n)$. Factor $s = ru$, where $r \in \mathbb{N}$ is the divisor of s whose prime support also supports d , so that $(u, d) = 1$. Choose a number $v : uv \equiv 1 \pmod{d}$. Since G^* is divisible, there exist r -th roots $\frac{1}{r}\phi^*, \frac{1}{r}\psi^* \in G^*$ of ϕ^* and ψ^* , respectively, and since r 's prime support divides d , $|\frac{1}{r}\phi^*| = mr$ and $|\frac{1}{r}\psi^*| = nr$. Now compute, using (1.3), $sv(\frac{1}{r}\phi^* \otimes \frac{1}{r}\psi^*) = uv(\phi^* \otimes \psi^*) = \phi^* \otimes \psi^*$. Therefore

$$(1.7) \quad \frac{1}{s}(\phi^* \otimes \psi^*) := v(\frac{1}{r}\phi^* \otimes \frac{1}{r}\psi^*)$$

is an s -th root of $\phi^* \otimes \psi^*$, of order dr . Thus if G^* is divisible, $\tilde{\otimes}^2 G^*$ is divisible, and similarly, $\tilde{\wedge}^2 G^*$ is divisible.

Since $\tilde{\wedge}^2 G^* \leq \tilde{\otimes}^2 G^*$ and $f^{*\wedge 2}$ is the restriction of $f^{*\otimes 2}$, $\tilde{\wedge}^2$ is a functor if $\tilde{\otimes}^2$ is. Therefore it remains to show $\tilde{\otimes}^2$ preserves identity homomorphisms, and composition of homomorphisms.

If f^* is the identity map on G^* , then by Definition 1.2 $f^{*\otimes 2}(\phi^* \otimes \psi^*) = \frac{c}{c'}(\phi^* \otimes \psi^*)$. Since $c = c' = \text{lcm}[|\phi^*|, |\psi^*|]$, $f^{*\otimes 2}$ is the identity on $\tilde{\otimes}^2 G^*$.

Suppose $f^* : G^* \rightarrow H^*$ and $g^* : H^* \rightarrow J^*$ are homomorphisms, and $\phi^*, \psi^* \in G^*$. Let $c = \text{lcm}[|\phi^*|, |\psi^*|]$, $c' = \text{lcm}[|f^*(\phi^*)|, |f^*(\psi^*)|]$, and $c'' = \text{lcm}[|(g^* \cdot f^*)(\phi^*)|, |(g^* \cdot f^*)(\psi^*)|]$. Since $f^{*\otimes 2}$ and $g^{*\otimes 2}$ are homomorphisms,

$$\begin{aligned} g^{*\otimes 2} \cdot f^{*\otimes 2}(\phi^* \otimes \psi^*) &= g^{*\otimes 2}\left(\frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*))\right) = \frac{c}{c'} g^{*\otimes 2}(f^*(\phi^*) \otimes f^*(\psi^*)) \\ &= \frac{c}{c'} \frac{c''}{c'}((g^* \cdot f^*)(\phi^*) \otimes (g^* \cdot f^*)(\psi^*)) = \frac{c}{c''}((g^* \cdot f^*)(\phi^*) \otimes (g^* \cdot f^*)(\psi^*)) \\ &= (g^* \cdot f^*)^{\otimes 2}(\phi^* \otimes \psi^*). \end{aligned}$$

This completes the proof. ■

Remark 1.8. Composing the functors $\tilde{\otimes}^2$ and $\tilde{\wedge}^2$ with the Pontrjagin Duality functor yields contravariant functors on the category of proabelian groups, defined by $(G, f) \mapsto (\tilde{\otimes}^2 G^*, f^{*\otimes 2})$ and $(G, f) \mapsto (\tilde{\wedge}^2 G^*, f^{*\wedge 2})$.

2. BILINEAR FORMS, ALTERNATING FORMS, AND COHOMOLOGY

The groups $\tilde{\otimes}^2 G^*$ and $\tilde{\wedge}^2 G^*$ are defined abstractly on any torsion abelian group G^* . It is shown in this section that if G is proabelian and G^* is its dual, then $\tilde{\otimes}^2 G^*$ and $\tilde{\wedge}^2 G^*$ are naturally isomorphic to the groups of continuous bilinear and alternating forms on G , defined below. It is then shown that the group of alternating forms is naturally isomorphic to the Galois cohomology group $H^2(G, \mathbb{Q}/\mathbb{Z})$. These isomorphisms reduce the theory of these groups to the theory of G^* . In particular, they allow the computation of restriction, inflation, and corestriction in Theorem 2.8 and Theorem 2.10, in terms of corresponding maps on character groups.

Definition 2.1. Suppose G is a proabelian group, and \mathbb{Q}/\mathbb{Z} is the continuous G -module with trivial action.

a. Denote by $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ the group of *continuous bilinear forms* on G with values in \mathbb{Q}/\mathbb{Z} . If $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, set $\text{rad}_{\mathbf{b}}(G) = \{\sigma \in G : \mathbf{b}(\sigma, G) = \mathbf{b}(G, \sigma) = 0\}$, the *radical of G with respect to \mathbf{b}* . Since \mathbf{b} is continuous, $\text{rad}_{\mathbf{b}}(G) \leq G$ is an open subgroup.

b. Denote by $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$ the group of *continuous symmetric bilinear forms*, the subgroup of $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ consisting of \mathbf{b} such that $\mathbf{b}(\sigma, \tau) = \mathbf{b}(\tau, \sigma)$ for all $\sigma, \tau \in G$.

c. Denote by $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ the group of *continuous alternating bilinear forms*, the subgroup of $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ consisting of \mathbf{a} such that $\mathbf{a}(\sigma, \sigma) = 0$ for all $\sigma \in G$. Note that $\mathbf{a}(\sigma, \tau) = -\mathbf{a}(\tau, \sigma)$, and $\text{rad}_{\mathbf{a}}(G) = \{\sigma \in G : \mathbf{a}(\sigma, G) = 0\}$.

It will be shown in Theorem 2.3 that all bilinear forms arise from the following construction. Suppose $\phi^*, \psi^* \in G^*$. Let $m = |\phi^*|$, $n = |\psi^*|$, and $d = \gcd(m, n)$. There is a canonical injection

$$\nu : \left(\frac{1}{m}\mathbb{Z}/\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined by $\nu\left(\frac{a}{m} \otimes \frac{b}{n}\right) = \frac{ab}{d}$. Suppose $\phi^*, \psi \in G^*$. Use ν to define bilinear forms $\nu_*(\phi^* \otimes \psi^*)$ and $\nu_*(\phi^* \wedge \psi^*)$ on G with values in \mathbb{Q}/\mathbb{Z} by the rule

$$(2.2) \quad \begin{aligned} \nu_*(\phi^* \otimes \psi^*)(\sigma, \tau) &= \nu(\phi^*(\sigma) \otimes \psi^*(\tau)) \\ \nu_*(\phi^* \wedge \psi^*)(\sigma, \tau) &= (\nu_*(\phi^* \otimes \psi^*) - \nu_*(\psi^* \otimes \phi^*))(\sigma, \tau) \end{aligned}$$

for all $\sigma, \tau \in G$. The bilinearity of these functions follows from the bilinearity of the tensor product, and the continuity follows from that of ϕ^* and ψ^* . Since $\phi^* \wedge \psi^*(\sigma, \sigma) = 0$, $\phi^* \wedge \psi^*$ is alternating.

Theorem 2.3. *Suppose G is a proabelian group. Then the function*

$$\nu_* : \tilde{\otimes}^2 G^* \xrightarrow{\sim} \text{Bil}(G, \mathbb{Q}/\mathbb{Z}),$$

given by (2.2), is a natural isomorphism, functorial in G , whose restriction to $\tilde{\wedge}^2 G^*$ is an isomorphism

$$\nu_* : \tilde{\wedge}^2 G^* \xrightarrow{\sim} \text{Alt}(G, \mathbb{Q}/\mathbb{Z}).$$

Consequently $|\phi^* \otimes \psi^*| = \gcd(|\phi^*|, |\psi^*|)$. If $f : H \rightarrow G$ is a map of proabelian groups, the maps $\nu_*(f^{*\otimes 2})$ and $\nu_*(f^{*\wedge 2})$ are the usual transpose maps $(f \times f)^*$ on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$.

Proof. To show ν_* is a well defined homomorphism, it suffices to check that the kernel of the map $(\phi^*, \psi^*) \mapsto \nu_*(\phi^* \otimes \psi^*)$ on $\mathbb{Z}\langle G^* \times G^* \rangle$ contains the relations (1.1). Suppose $\phi^*, \psi^*, \psi_1^*, \psi_2^* \in G^*$. Set $\psi_3^* = \frac{n_1}{d_1}\psi_1^* + \frac{n_2}{d_2}\psi_2^*$. Let $m = |\phi^*|$, $n = |\psi^*|$, $n_i = |\psi_i^*|$, $c = \text{lcm}[m, n]$, $d = \gcd(m, n)$, and $d_i = \gcd(m, n_i)$. Since d_1, d_2 both divide m , $d_3 = n_3$. Suppose $\phi(\sigma) = a/m$, $\psi(\tau) = b/n$, and $\psi_i(\tau) = b_i/n_i$. Compute $\psi_3^*(\tau) = b_3/n_3 = \frac{b_1}{d_1} + \frac{b_2}{d_2}$.

It is immediate from the definition that $\nu_*(\phi^* \otimes \psi^*) = \nu_*\left(\frac{m}{d}\phi^* \otimes \frac{n}{d}\psi^*\right)$, in light of the natural isomorphism $\frac{1}{m}\mathbb{Z}/\mathbb{Z} \otimes \frac{1}{n}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{d}\mathbb{Z}/\mathbb{Z} \otimes \frac{1}{d}\mathbb{Z}/\mathbb{Z}$ defined by $\frac{a}{m} \otimes \frac{b}{n} \mapsto \frac{a}{d} \otimes \frac{b}{d}$, which commutes with ν . Thus the first relation holds in $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. By definition, and since $d_3 = n_3$ divides m ,

$$\nu_*(\phi^* \otimes \psi_1^*)(\sigma, \tau) + \nu_*(\phi^* \otimes \psi_2^*)(\sigma, \tau) = \frac{ab_1}{d_1} + \frac{ab_2}{d_2} = \frac{ab_3}{d_3} = \nu_*(\phi^* \otimes (\frac{n_1}{d_1}\psi_1^* + \frac{n_2}{d_2}\psi_2^*))(\sigma, \tau).$$

Thus the second relation in (1.1) holds. Similarly, the last relation holds. Thus ν_* is a well defined homomorphism.

The bilinear form $\nu_*(\phi^* \otimes \psi^*)$ is clearly of order d . For since $|\phi^*| = m$ and $|\psi^*| = n$, there exist elements σ and τ in G such that $\phi^*(\sigma) = 1/m$, $\psi^*(\tau) = 1/n$, hence $\nu_*(\phi^* \otimes \psi^*)(\sigma, \tau) = 1/d$.

To show ν_* is functorial, suppose $f : H \rightarrow G$ is a homomorphism of proabelian groups, with transpose $f^* : G^* \rightarrow H^*$, and functorial map $f^{*\otimes 2}$ on $\tilde{\otimes}^2 G^*$. The usual transpose $(f \times f)^*$ is defined on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ by composition with $f \times f : H \times H \rightarrow G \times G$. Suppose $(f \times f)(\sigma', \tau') = (\sigma, \tau)$. Compute on one hand

$$(f \times f)^*(\nu_*(\phi^* \otimes \psi^*))(\sigma', \tau') = \nu_*(\phi^* \otimes \psi^*)(\sigma, \tau) = \frac{ab}{d}.$$

For the other hand, let $m' = |f^*(\phi^*)|$, $n' = |f^*(\psi^*)|$, $c' = \text{lcm}[m', n']$, and $d' = \text{gcd}(m', n')$. Then $f^*(\phi^*)(\sigma') = \phi^*(\sigma) = a/m = a'/m'$, where $a' = am'/m \in \mathbb{Z}$. Similarly $f^*(\psi^*)(\tau') = b'/n'$, where $b' = bn'/n \in \mathbb{Z}$. Therefore compute

$$\nu_*(f^{*\otimes 2}(\phi^* \otimes \psi^*))(\sigma', \tau') = \nu_*\left(\frac{c}{c'}(f^*(\phi^*) \otimes f^*(\psi^*))\right)(\sigma', \tau') = \frac{c}{c'} \frac{a'b'}{d'}.$$

Since $cd = mn$ and $c'd' = m'n'$, $a'b' = abc'd'/cd \in \mathbb{Z}$. Therefore $a'b'c/c'd' = (abc'd'/cd)/c'd' = ab/d$. Conclude

$$\nu_*(f^{*\otimes 2}(\phi^* \otimes \psi^*))(\sigma', \tau') = (f \times f)^*(\nu_*(\phi^* \otimes \psi^*))(\sigma', \tau').$$

Since (σ, τ) is arbitrary, this shows $(f \times f)^*(\nu_*(\phi^* \otimes \psi^*)) = \nu_*(f^{*\otimes 2}(\phi^* \otimes \psi^*))$. Since $\tilde{\otimes}^2 G^*$ is generated by the elements $\phi^* \otimes \psi^*$, this shows $(f \times f)^* \cdot \nu_* = \nu_* \cdot f^{*\otimes 2}$, i.e., ν_* is functorial.

Now to show ν_* is an isomorphism. Suppose $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. Assume first that G is finite, then G has a basis $\{\phi_i\}$, and \mathbf{b} is completely determined by its values on the basis $\{(\phi_i, \phi_j)\}$ for $G \times G$. If $d_{ij} = \text{gcd}(|\phi_i|, |\phi_j|)$, then $\mathbf{b}(\phi_i, \phi_j) = a_{ij}/d_{ij}$ for some a_{ij} . Let $\{\phi_i^*\}$ be the basis for G^* dual to $\{\phi_i\}$, defined by $\phi_i^*(\phi_j) = \delta_{ij}/|\phi_i|$, where δ_{ij} is the Kronecker delta function. By (2.2), $\nu_*(\phi_i^* \otimes \phi_j^*)(\phi_k, \phi_l) = \delta_{ik}\delta_{jl}/d_{ij}$, hence $\mathbf{b} = \nu_*(\sum_{i,j} a_{ij}(\phi_i^* \otimes \phi_j^*))$. Thus ν_* is surjective.

To show that ν_* is injective, it is enough to show ν_* 's restriction to each p -primary component of G is injective. First note that the $\phi_i^* \otimes \phi_j^*$ span $\tilde{\otimes}^2 G^*$, where G has p -power order. For if $\phi^* \otimes \psi^*$ is a reduced element of $\tilde{\otimes}^2 G^*$, with $\phi^* = \sum a_i \phi_i^*$ and $\psi^* = \sum b_j \phi_j^*$, then by the independence of the ϕ_i^* , $|a_i \phi_i^*|$ and $|b_j \phi_j^*|$ divide $|\psi^*| = |\phi^*|$. Therefore by (1.3e), $\phi^* \otimes \psi^* = \sum_{i,j} a_i \phi_i^* \otimes b_j \phi_j^*$. Write $a_i = u_i p^{r_i}$, $b_j = v_j p^{s_j}$, then by (1.3c, d), $a_i \phi_i^* \otimes b_j \phi_j^* = u_i v_j (p^{r_i} \phi_i^* \otimes p^{s_j} \phi_j^*) = u_i v_j p^{r_k} (\phi_i^* \otimes \phi_j^*)$, where $p^{r_k} = |\phi_i^* \otimes \phi_j^*| / |p^{r_i} \phi_i^* \otimes p^{s_j} \phi_j^*|$. Therefore $\phi^* \otimes \psi^*$ is a linear combination of the $\phi_i^* \otimes \phi_j^*$, so the $\phi_i^* \otimes \phi_j^*$ span $\tilde{\otimes}^2 G^*$. The $\nu_*(\phi_i^* \otimes \phi_j^*)$ are clearly independent, and since $|\phi_i^* \otimes \phi_j^*|$ divides $|\nu_*(\phi_i^* \otimes \phi_j^*)| = \text{gcd}(|\phi_i^*|, |\phi_j^*|)$, the linear combination expressing \mathbf{b} is unique. Thus ν_* is injective.

Now suppose G is a general proabelian group. If $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, then \mathbf{b} factors through $\bar{G} \times \bar{G}$, where $\bar{G} = G/\text{rad}_{\mathbf{b}}(G)$ is finite. That is, \mathbf{b} is in the image of the inflation map $\iota^{*\otimes 2} : \text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. By functoriality, if $\bar{\nu}_*$ denotes the map on $\tilde{\otimes}^2 \bar{G}^*$, then $\iota^{*\otimes 2} \cdot \bar{\nu}_* = \nu_* \cdot \iota^{*\otimes 2}$. Therefore ν_* is surjective by the finite case. To show ν_* is injective, note that inflation $\iota^{*\otimes 2} : \text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ is obviously injective, for any finite quotient \bar{G} of G , and if $\bar{\nu}_*$ denotes the map on $\tilde{\otimes}^2 \bar{G}^*$, then $\bar{\nu}_*$ is injective by the finite case, hence $\iota^{*\otimes 2} \cdot \bar{\nu}_*$ is injective, and by functoriality, $\nu_* \cdot \iota^{*\otimes 2}$ is injective. If $\xi \in \tilde{\otimes}^2 G^*$, then by the construction of $\tilde{\otimes}^2 G^*$, ξ is a finite sum of elementary elements $\phi^* \otimes \psi^*$, so ξ is supported on the dual \bar{G}^* of a finite quotient

\bar{G} of G . Thus ξ is in the image of $\iota^{*\otimes 2} : \tilde{\otimes}^2 \bar{G}^* \rightarrow \tilde{\otimes}^2 G^*$. Therefore since $\nu_* \cdot \iota^{*\otimes 2}$ is injective, $\nu_*(\xi) = 0$ if and only if $\xi = 0$. Thus ν_* is injective. This proves ν_* is an isomorphism.

The restriction of ν_* to the subgroup $\tilde{\wedge}^2 G^*$ is, of course, injective. If $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, where G is finite, then in the above notation, $\mathbf{a}(\phi_i, \phi_j) = -\mathbf{a}(\phi_j, \phi_i) = n_{ij}/d_{ij}$ for some elements $n_{ij}/d_{ij} \in \mathbb{Q}/\mathbb{Z}$, and in that case $\mathbf{a} = \nu_*(\sum_{i < j} n_{ij}(\phi_i^* \wedge \phi_j^*))$. Thus $\nu_*(\tilde{\wedge}^2 G^*) = \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ when G is finite. The general case follows by inflation, as above. This completes the proof. \blacksquare

From now on, suppress the ν_* , and *identify* the groups $\tilde{\otimes}^2 G^*$ and $\tilde{\wedge}^2 G^*$ with $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, respectively, and use the notation $\phi^* \otimes \psi^*$ and $\phi^* \wedge \psi^*$ instead of $\nu_*(\phi^* \otimes \psi^*)$ and $\nu_*(\phi^* \wedge \psi^*)$, and $f^{*\otimes 2}$ and $f^{*\wedge 2}$ instead of $\nu_*(f^{*\otimes 2})$ and $\nu_*(f^{*\wedge 2})$. In particular, let the map alt of Definition 1.5 also denote the surjective map

$$\text{alt} : \text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z}),$$

defined by $\text{alt}(\phi^* \otimes \psi^*) = \phi^* \otimes \psi^* - \psi^* \otimes \phi^* = \phi^* \wedge \psi^*$.

The kernel of alt is $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$, the group of symmetric bilinear forms. Thus there is an exact sequence

$$0 \longrightarrow \text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}} \longrightarrow \text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

If $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, let $\tilde{\mathbf{b}}$ denote the bilinear form defined by $\tilde{\mathbf{b}}(\sigma, \tau) = \mathbf{b}(\tau, \sigma)$. Then $\mathbf{b} + \tilde{\mathbf{b}} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$ and $\mathbf{b} - \tilde{\mathbf{b}} = \text{alt}(\mathbf{b}) \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. It is not hard to show that $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}} \cap \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) = {}_2\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$.

Since G 's action on \mathbb{Q}/\mathbb{Z} is trivial, each $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ satisfies the 2-cocycle condition

$$\partial \mathbf{b}(\rho, \sigma, \tau) := \mathbf{b}(\sigma, \tau) - \mathbf{b}(\rho\sigma, \tau) + \mathbf{b}(\rho, \sigma\tau) - \mathbf{b}(\rho, \sigma) = 0.$$

Therefore naturally $\text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \leq Z^2(G, \mathbb{Q}/\mathbb{Z})$.

Theorem 2.4. *For any proabelian group G , the natural inclusion $\text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \leq Z^2(G, \mathbb{Q}/\mathbb{Z})$ induces a surjection*

$$\text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow H^2(G, \mathbb{Q}/\mathbb{Z}),$$

whose kernel is $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$. This surjection induces a well defined map

$$\begin{aligned} \text{alt} : H^2(G, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \\ [\phi^* \otimes \psi^*] &\longmapsto \phi^* \wedge \psi^* \end{aligned}$$

which is a natural functorial isomorphism.

Proof. Extend alt to $Z^2(G, \mathbb{Q}/\mathbb{Z})$ by setting $\text{alt}(f) = f - \tilde{f}$ for all $f \in Z^2(G, \mathbb{Q}/\mathbb{Z})$, where \tilde{f} is defined by $\tilde{f}(\sigma, \tau) = f(\tau, \sigma)$, for all $\sigma, \tau \in G$. It is straightforward to verify that, because G is abelian and the action of G on \mathbb{Q}/\mathbb{Z} is trivial, $\partial \tilde{f}(\rho, \sigma, \tau) = -\partial f(\tau, \sigma, \rho)$, hence \tilde{f} is a cocycle. Moreover, $f - \tilde{f}$ is indeed an alternating form. For it is clear from the definition that $(f - \tilde{f})(\sigma, \tau) = -(f - \tilde{f})(\tau, \sigma)$, and the alternating sum of the cocycle condition on f applied

successively to the triples (ρ, σ, τ) , (ρ, τ, σ) , and (τ, ρ, σ) shows that $f - \tilde{f}$ is linear on the left, hence bilinear ([AT] Section 1).

The kernel of $\text{alt} : Z^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ is $Z^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}} = \{f : f = \tilde{f}\}$. Since the action of G on \mathbb{Q}/\mathbb{Z} is trivial, a coboundary $f \in B^2(G, \mathbb{Q}/\mathbb{Z})$ is defined by $f(\sigma, \tau) = h(\tau) - h(\sigma\tau) + h(\sigma)$ for some $h : G \rightarrow \mathbb{Q}/\mathbb{Z}$. Therefore since G is abelian, $Z^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$ contains $B^2(G, \mathbb{Q}/\mathbb{Z})$, and $\text{alt} : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ is well defined. Let $H^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$ denote the kernel, equal to $Z^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}/B^2(G, \mathbb{Q}/\mathbb{Z})$. *Claim:* $H^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}} = 0$. For $[f] \in H^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$ if and only if $f = \tilde{f}$, if and only if $[f]$ defines an abelian group extension of G by \mathbb{Q}/\mathbb{Z} . Thus $H^2(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}} = \text{Ext}_{\mathbb{Z}}^1(G, \mathbb{Q}/\mathbb{Z})$. Since \mathbb{Q}/\mathbb{Z} is divisible, the claim follows. Now there is a commutative diagram

$$\begin{array}{ccc} & \text{Bil}(G, \mathbb{Q}/\mathbb{Z}) & \\ \swarrow & & \searrow \\ H^2(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{alt}} & \text{Alt}(G, \mathbb{Q}/\mathbb{Z}). \end{array}$$

Since $\text{alt} : \text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ is surjective, $\text{alt} : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, is surjective, hence bijective. It follows that the natural map $\text{Bil}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$ is surjective, with kernel $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})_{\text{sym}}$. The functoriality in G is obvious. This completes the proof. \blacksquare

Remark 2.5. A similar result appears in [AT]. If G is any group and A is a trivial G -module, then by the Universal Coefficient Theorem, the sequence

$$0 \longrightarrow \text{Ext}_{\mathbb{Z}}^1(H_1(G, \mathbb{Z}), A) \longrightarrow H^2(G, A) \longrightarrow \text{Hom}(H_2(G, \mathbb{Z}), A) \longrightarrow 0$$

is (split) exact. If G is abelian, $H_1(G, \mathbb{Z}) = G$, $H_2(G, \mathbb{Z}) \simeq \tilde{\wedge}^2 G$, and the projection is given by $[f] \mapsto f - \tilde{f}$. This is [AT, Proposition 1.3]. If A is divisible, this is Theorem 2.4. More generally, if A is divisible and G is any group, then $H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$.

Functoriality.

Setup 2.6. For the next few results, suppose

$$1 \longrightarrow H \xrightarrow{\iota} G \xrightarrow{\pi} \bar{G} \longrightarrow 1$$

is an exact sequence of proabelian groups, such that \bar{G} has finite exponent r . Since “ $*$ ” is an exact functor, there is a dual exact sequence

$$0 \longrightarrow \bar{G}^* \xrightarrow{\pi^*} G^* \xrightarrow{\iota^*} H^* \longrightarrow 0.$$

The dual $\iota^* : G^* \rightarrow H^*$ is the restriction map, and $\pi^* : \bar{G}^* \rightarrow G^*$ is the inflation. If G is torsion-free, then G^* and H^* are divisible.

For each integer d divisible by r , let j_d denote the composite of exponentiation by d and ι^{-1} ,

$$j_d : G \rightarrow G^d \hookrightarrow H.$$

Its dual is $j_d^* : H^* \rightarrow (G^d)^* \hookrightarrow G^*$. If ι is open and $s = |\bar{G}| < \infty$, then by [S, VII.8.Proposition 7], $j := j_s$ is the classical *transfer homomorphism*, and its dual $j^* : H^* \rightarrow G^*$ is the cohomological corestriction. Since ι^* is restriction, the composition $j_d^* \cdot \iota^* : G^* \rightarrow G^*$ is multiplication by d . On the other hand, since ι^* is surjective, $\iota^* \cdot j_d^*(H^*) = \iota^* \cdot j_d^* \cdot \iota^*(G^*) = \iota^*(dG^*) = d\iota^*(G^*) = dH^*$. Thus $\iota^* \cdot j_d^* : H^* \rightarrow H^*$ is also multiplication by d .

Lemma 2.7. *Assume Setup 2.6. For all $d \in \mathbb{N}$ divisible by r , there is a natural exact sequence,*

$$0 \longrightarrow {}_d\overline{G}^* \xrightarrow{\pi^*} {}_dG^* \xrightarrow{\iota^*} {}_dH^* \xrightarrow{j_d^*} \pi^*(\overline{G}^*).$$

If G is torsion-free, the right arrow j_d^ is surjective. In particular, if G is torsion-free and ι is open, $j^*({}_sH^*) \simeq \overline{G}^*$, where $s = |\overline{G}|$ and j is the transfer.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{G}^* & \longrightarrow & G^* & \longrightarrow & H^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{G}^* & \longrightarrow & G^* & \longrightarrow & H^* \longrightarrow 0 \end{array}$$

where the vertical arrows are multiplication by d . By the Snake Lemma, the sequence

$$0 \longrightarrow {}_d\overline{G}^* \longrightarrow {}_dG^* \longrightarrow {}_dH^* \xrightarrow{j_d^*} \pi^*(\overline{G}^*)/\pi^*(d\overline{G}^*) \longrightarrow G^*/dG^* \longrightarrow 0$$

is exact. Since r divides d , $d\overline{G}^* = \{0\}$, so $\pi^*(\overline{G}^*)/\pi^*(d\overline{G}^*) = \pi^*(\overline{G}^*)$, as desired. If G is torsion-free then G^* is divisible, and then $G^*/dG^* = \{0\}$, i.e., j_d^* is surjective. This completes the proof. ■

Assume Setup 2.6. By [S, Chapter VII, Section 5], ι and π induce the cohomological restriction map res and inflation map inf on $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $H^2(\overline{G}, \mathbb{Q}/\mathbb{Z})$. By the second part of Theorem 2.3, $\iota^{*\otimes 2}$, $\iota^{*\wedge 2}$, $\pi^{*\otimes 2}$, and $\pi^{*\wedge 2}$ are the natural maps on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, $\text{Bil}(\overline{G}, \mathbb{Q}/\mathbb{Z})$, and $\text{Alt}(\overline{G}, \mathbb{Q}/\mathbb{Z})$. Therefore by Theorem 2.4, $[\iota^{*\otimes 2}] = \text{alt}^{-1}(\iota^{*\wedge 2}) = \text{res}$ on $H^2(G, \mathbb{Q}/\mathbb{Z})$, and $[\pi^{*\otimes 2}] = \text{alt}^{-1}(\pi^{*\wedge 2}) = \text{inf}$ on $H^2(\overline{G}, \mathbb{Q}/\mathbb{Z})$. Therefore denote $\iota^{*\otimes 2}$ and $\iota^{*\wedge 2}$ by res , and $\pi^{*\otimes 2}$ and $\pi^{*\wedge 2}$ by inf . In particular, the maps

$$\begin{array}{ccc} \text{res} : \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) & \quad & \text{inf} : \text{Alt}(\overline{G}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \\ \mathbf{a} & \longmapsto & \iota^{*\wedge 2}(\mathbf{a}) & & \mathbf{b} & \longmapsto & \pi^{*\wedge 2}(\mathbf{b}) \end{array}$$

are given explicitly by $\text{res}(\mathbf{a})(\sigma, \tau) = \mathbf{a}(\iota(\sigma), \iota(\tau))$ for each $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, where $\sigma, \tau \in H$, and $\text{inf}(\overline{\mathbf{a}})(\sigma, \tau) = \overline{\mathbf{a}}(\pi(\sigma), \pi(\tau))$ for each $\overline{\mathbf{a}} \in \text{Alt}(\overline{G}, \mathbb{Q}/\mathbb{Z})$, where $\sigma, \tau \in G$. On elementary elements $\phi^* \wedge \psi^* \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ and $\overline{\phi}^* \wedge \overline{\psi}^* \in \text{Alt}(\overline{G}, \mathbb{Q}/\mathbb{Z})$, compute using Definition 1.5, $\text{res}(\phi^* \wedge \psi^*) = \frac{c}{c'}(\iota^*(\phi^*) \wedge \iota^*(\psi^*))$, where $c = \text{lcm}[|\phi^*|, |\psi^*|]$, $c' = \text{lcm}[|\iota^*(\phi^*)|, |\iota^*(\psi^*)|]$ and $\text{inf}(\overline{\phi}^* \wedge \overline{\psi}^*) = \pi^*(\overline{\phi}^*) \wedge \pi^*(\overline{\psi}^*)$.

When $\iota : H \hookrightarrow G$ is open, define the corestriction

$$\text{cor} : \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$$

to be $\text{alt}(\text{cor})$, where $\text{cor} : H^2(H, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological corestriction. This map will be computed in Theorem 2.10.

Theorem 2.8 “Inflation and Restriction”. *Assume Setup 2.6, with G a general proabelian group. The inflation is injective, and $\text{res} \cdot \text{inf} = 0$ on $\text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z})$, $\text{H}^2(\bar{G}, \mathbb{Q}/\mathbb{Z})$, and $\text{Alt}(\bar{G}, \mathbb{Q}/\mathbb{Z})$. If G is torsion-free the restriction surjects onto $\text{Bil}(H, \mathbb{Q}/\mathbb{Z})$, $\text{H}^2(H, \mathbb{Q}/\mathbb{Z})$, and $\text{Alt}(H, \mathbb{Q}/\mathbb{Z})$. The kernel of the restriction in $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ has exponent dividing the maximum order of all subgroups of \bar{G} of rank at most two, so divides $|\bar{G}|$.*

Proof. Since by Definition 2.1, $\text{Alt}(\bar{G}, \mathbb{Q}/\mathbb{Z})$ is a subgroup of $\text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z})$, and since by Theorem 2.4, $\text{alt} : \text{H}^2(\bar{G}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(\bar{G}, \mathbb{Q}/\mathbb{Z})$ is a natural isomorphism, it suffices to prove the first two statements on $\text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z})$. Inflation is obviously injective on bilinear forms. To see that $\text{res} \cdot \text{inf} = 0$, suppose $\mathbf{b} = \text{inf}(\mathbf{a})$, where $\mathbf{a} \in \text{Bil}(\bar{G}, \mathbb{Q}/\mathbb{Z})$, and compute $\text{res}(\mathbf{b})(\sigma, \tau) = \mathbf{b}(\iota(\sigma), \iota(\tau)) = \mathbf{a}(\pi \cdot \iota(\sigma), \pi \cdot \iota(\tau)) = \mathbf{a}(\bar{1}, \bar{1}) = 0$ for all $\sigma, \tau \in H$, as desired.

Suppose G is torsion-free, then so is H , as $H \hookrightarrow G$. Then G^* and H^* are divisible, and by Theorem 1.6 and Theorem 2.3, $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Bil}(H, \mathbb{Q}/\mathbb{Z})$ are divisible, and it follows that $\text{res}(\text{Bil}(G, \mathbb{Q}/\mathbb{Z}))$ is divisible. Suppose $\mathbf{b} \in \text{Bil}(H, \mathbb{Q}/\mathbb{Z})$. For any d divisible by $r = \exp(\bar{G})$, $(\iota^{*\otimes 2} \cdot j_d^{*\otimes 2})(\mathbf{b}) = (j_d \cdot \iota)^{*\otimes 2}(\mathbf{b}) = d^2 \mathbf{b} \in \text{Bil}(H, \mathbb{Q}/\mathbb{Z})$ is in the image of restriction $\iota^{*\otimes 2} = \text{res}$. Thus the quotient $\text{Bil}(H, \mathbb{Q}/\mathbb{Z})/\text{res}(\text{Bil}(G, \mathbb{Q}/\mathbb{Z}))$ has bounded exponent, hence is trivial by the divisibility of each group. Thus restriction is surjective. The same argument works on $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, and by Theorem 2.4 this proves it also for $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$.

Suppose $\mathbf{a} \in \ker(\text{res}) \leq \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, and $\sigma, \tau \in G$. Let k be the order of $\pi(\sigma)$, and let l the order of $\pi(\tau)$ modulo $\langle \pi(\sigma) \rangle \leq \bar{G}$. Then kl is the order of the subgroup $\langle \pi(\sigma), \pi(\tau) \rangle \leq \bar{G}$, $\sigma^k \in H$, and $\tau^l = \sigma^m \rho$ for some number m and element $\rho \in \iota(H)$. Now compute using the bilinearity of \mathbf{a} , $kl\mathbf{a}(\sigma, \tau) = \mathbf{a}(\sigma^k, \sigma^m) + \mathbf{a}(\sigma^k, \rho)$. Since \mathbf{a} is alternating, $\mathbf{a}(\sigma^k, \sigma^m) = 0$, and since $\text{res}(\mathbf{a}) = 0$, $\mathbf{a}(\sigma^k, \rho) = 0$. Therefore $kl\mathbf{a}(\sigma, \tau) = 0$. If n is the maximum order of all subgroups of \bar{G} of rank at most two, then $kl \mid n$, and therefore $n\mathbf{a}(\sigma, \tau) = 0$ for all $\sigma, \tau \in G$, hence $n\mathbf{a} = 0$. By Theorem 2.4, the same result holds on $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$. This completes the proof. ■

Remark 2.9. The last statement of Theorem 2.8 does not hold for $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. For suppose $G = \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$ and $H = 2\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$. By Theorem 2.8, the kernel of the restriction on $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ has exponent dividing 2, since $|\bar{G}| = 2$. But if $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ is defined on the standard basis $\{\phi_1, \phi_2\}$ of G by $\mathbf{b}(\phi_i, \phi_j) = \delta_{i1}\delta_{j1}/4$, then $\text{res}(\mathbf{b}) = 0$, and $|\mathbf{b}| = r^2 = 4$. In general, the kernel of the restriction on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ has exponent dividing $r^2 = \exp(\bar{G})^2$, since if $\text{res}(\mathbf{b}) = 0$ then $r^2\mathbf{b}(\sigma, \tau) = \mathbf{b}(\sigma^r, \tau^r) = 0$ for all $\sigma, \tau \in G$.

Theorem 2.10 “Corestriction”. *Assume Setup 2.6, with $\iota : H \rightarrow G$ an open embedding of torsion-free proabelian groups, $s = |\bar{G}|$, and $j : G \rightarrow H$ the transfer. For each \mathbf{a} in $\text{Alt}(H, \mathbb{Q}/\mathbb{Z})$ and $\mathbf{b} \in \text{Bil}(H, \mathbb{Q}/\mathbb{Z})$, let $\frac{1}{s}\mathbf{a}$ and $\frac{1}{s}\mathbf{b}$ be any elements such that $s \cdot \frac{1}{s}\mathbf{a} = \mathbf{a}$ and $s \cdot \frac{1}{s}\mathbf{b} = \mathbf{b}$. Then*

$$\begin{aligned} \text{cor} : \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \text{cor} : \text{H}^2(H, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{H}^2(G, \mathbb{Q}/\mathbb{Z}) \\ \mathbf{a} &\longmapsto j^{*\wedge 2}(\tfrac{1}{s}\mathbf{a}) & \mathbf{b} &\longmapsto [j^{*\otimes 2}](\tfrac{1}{s}\mathbf{b}). \end{aligned}$$

The corestriction on reduced elementary elements $\phi^* \wedge \psi^*$, $[\phi^* \otimes \psi^*]$, where $\phi^*, \psi^* \in H^*$ both have order d , is computed as follows. As in (1.7), let r be the factor of s that shares d 's prime support,

so that $s = ru$ for some $u : (u, d) = 1$, and let $\frac{1}{r}\phi^*$ and $\frac{1}{r}\psi^*$ be r -th roots. Let v be such that $uv \equiv 1 \pmod{d}$. Then

$$\text{cor}(\phi^* \wedge \psi^*) = v \frac{dr}{c'} (j^*(\frac{1}{r}\phi^*) \wedge j^*(\frac{1}{r}\psi^*)) \quad \text{cor}([\phi^* \otimes \psi^*]) = v \frac{dr}{c'} [j^*(\frac{1}{r}\phi^*) \otimes j^*(\frac{1}{r}\psi^*)],$$

where $c' = \text{lcm}[|j^*(\frac{1}{r}\phi^*)|, |j^*(\frac{1}{r}\psi^*)|]$.

Proof. By Theorem 2.4, it suffices to prove the theorem for alternating forms. Since H is torsion-free, $\text{Alt}(H, \mathbb{Q}/\mathbb{Z})$ is divisible by Theorem 1.6, so $\frac{1}{s}\mathbf{a}$ exists. To show the given map is well defined, suppose $\mathbf{c} \in \text{Alt}(H, \mathbb{Q}/\mathbb{Z})$ has order dividing s . Since G is torsion-free, by Theorem 2.8 there exists an element $\mathbf{b} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ such that $\text{res}(\mathbf{b}) = \mathbf{c}$, and $|\mathbf{b}|$ divides $|\mathbf{c}|s$. In particular, $s^2\mathbf{b} = 0$. By Theorem 1.6 and Theorem 2.3, $\text{Alt}(-, \mathbb{Q}/\mathbb{Z})$ is a covariant functor on torsion abelian groups, therefore $j^{*\wedge 2} \iota^{*\wedge 2} = (j^* \iota^*)^{\wedge 2}$. Since $j^* \iota^*$ is multiplication by s , it follows by an explicit computation using Definition 1.6 that $j^{*\wedge 2} \iota^{*\wedge 2}$ is multiplication by s^2 . Therefore $j^{*\wedge 2}(\mathbf{c}) = j^{*\wedge 2} \iota^{*\wedge 2}(\mathbf{b}) = 0$, and the given map is well defined. Since $j^{*\wedge 2}$ is a homomorphism and $s(\frac{1}{s}(\mathbf{a} + \mathbf{a}')) = \mathbf{a} + \mathbf{a}' = s\frac{1}{s}\mathbf{a} + s\frac{1}{s}\mathbf{a}'$, the map $\mathbf{a} \mapsto j^{*\wedge 2}(\frac{1}{s}\mathbf{a})$ is a homomorphism.

Now to show the given map is the corestriction. Since $\text{res} = \iota^{*\wedge 2}$ is a surjective homomorphism, and $\text{cor} \cdot \text{res}$ is multiplication by s , it suffices to prove that if $\mathbf{b} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, then $j^{*\wedge 2}(\frac{1}{s} \iota^{*\wedge 2}(\mathbf{b})) = s\mathbf{b}$. Since $\iota^{*\wedge 2}$ is a homomorphism, $\frac{1}{s} \iota^{*\wedge 2}(\mathbf{b})$ may be replaced by $\iota^{*\wedge 2}(\frac{1}{s}\mathbf{b})$, where $\frac{1}{s}\mathbf{b}$ is any s -th root of \mathbf{b} . It has already been seen that $j^{*\wedge 2} \iota^{*\wedge 2} = (j^* \iota^*)^{\wedge 2}$ is multiplication by s^2 , so $j^{*\wedge 2} \iota^{*\wedge 2}(\frac{1}{s}\mathbf{b}) = s^2(\frac{1}{s}\mathbf{b}) = s\mathbf{b}$, as desired. Therefore $\text{cor}(\mathbf{a}) = j^{*\wedge 2}(\frac{1}{s}\mathbf{a})$.

Finally, compute using (1.7) and Theorem 2.3,

$$\text{cor}(\phi^* \wedge \psi^*) = v j^{*\wedge 2}(\frac{1}{r}\phi^* \wedge \frac{1}{r}\psi^*) = v \frac{c}{c'} (j^*(\frac{1}{r}\phi^*) \wedge j^*(\frac{1}{r}\psi^*)),$$

where $c = \text{lcm}[|\frac{1}{r}\phi^*|, |\frac{1}{r}\psi^*|] = dr$, and $c' = \text{lcm}[|j^*(\frac{1}{r}\phi^*)|, |j^*(\frac{1}{r}\psi^*)|]$. This completes the proof. \blacksquare

Remarks 2.11. a. Theorem 2.10 can be used to compute the corestriction for $\iota : H \rightarrow G$ an open embedding of arbitrary proabelian groups, not just those that are torsion-free. For let \widehat{G}^* be the divisible hull of G^* , and let \widehat{G} be the torsion-free proabelian dual of \widehat{G}^* . Since G^* is a subgroup of \widehat{G}^* , G is a quotient of \widehat{G} , and thus there is a surjective homomorphism $\widehat{G} \rightarrow \overline{G} \simeq G/H$. Let \widehat{H} be the kernel, then \widehat{H} is torsion-free, and \widehat{H} maps onto $H \leq G$. By inflation, the cohomology classes and alternating forms for H and G embed into the corresponding groups for the torsion-free \widehat{H} and \widehat{G} , and Theorem 2.10 computes the corestriction.

b. The corestriction map in Theorem 2.10 is not well defined on $\text{Bil}(H, \mathbb{Q}/\mathbb{Z})$. For example, suppose $G = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$ and $H = 2\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$, so $s = 2$. If $\mathbf{a} \in \text{Bil}(H, \mathbb{Q}/\mathbb{Z})$, then the choices for $\frac{1}{2}\mathbf{a}$ differ by elements of order 2. But if $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ is defined by $\mathbf{b}(\phi_i, \phi_j) = \delta_{i1}\delta_{j1}/8$. Then $\mathbf{c} = \text{res}(\mathbf{b})$ has order 2, but $j^{*\otimes 2}(\mathbf{c}) = j^{*\otimes 2} \iota^{*\otimes 2}(\mathbf{b}) = s^2\mathbf{b} = 4\mathbf{b} \neq 0$. Therefore $j^{*\otimes 2}(\frac{1}{2}\mathbf{a}) \neq j^{*\otimes 2}(\frac{1}{2}\mathbf{a} + \mathbf{c})$. This shows that in general the value of $j^{*\otimes 2}(\frac{1}{s}\mathbf{a})$ depends on the choice of $\frac{1}{s}\mathbf{a}$.

3. THE INDEX OF AN ALTERNATING FORM

The index of an alternating form is an invariant closely associated with the pfaffian of an alternating matrix that generalizes the (Schur) index of a Brauer class. More specifically, if G is the Galois group of a Galois field extension Ω/F , and μ is the torsion subgroup of F^\times there is a natural map $H^2(G, \mu) \rightarrow H^2(G, \Omega^\times) \simeq \text{Br}(\Omega/F)$. When $\mu \simeq \mathbb{Q}/\mathbb{Z}$ there is an isomorphism $H^2(G, \mu) \simeq H^2(G, \mathbb{Q}/\mathbb{Z})$, and by Theorem 2.4 an induced map $\text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Br}(\Omega/F)$. The index of an alternating form will be shown in Theorem 4.13 to be divisible by the index of its image in $\text{Br}(\Omega/F)$.

Definition 3.1. Suppose G is a proabelian group, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Define the *exponent*, *degree*, and *index* of \mathbf{a} to be the numbers

$$\exp(\mathbf{a}) = \exp(G/\text{rad}_{\mathbf{a}}(G)) \quad \deg(\mathbf{a}) = |G/\text{rad}_{\mathbf{a}}(G)| \quad \text{ind}(\mathbf{a}) = |G/\text{rad}_{\mathbf{a}}(G)|^{1/2}.$$

Since it is open in G , $\text{rad}_{\mathbf{a}}(G)$ has finite index in G , so $\exp(\mathbf{a})$, $\deg(\mathbf{a})$, and $\text{ind}(\mathbf{a})$ are well defined. Clearly $\exp(\mathbf{a}) = |\mathbf{a}|$, its order in $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. It will be shown presently that $\text{ind}(\mathbf{a}) \in \mathbb{N}$; it follows immediately that $\exp(\mathbf{a})$ and $\text{ind}(\mathbf{a})$ have the same prime divisors. Later, it will be shown that the index and exponent of an alternating form on an abelian Galois group over a field containing all roots of unity are closely related to the index and exponent of a corresponding Brauer class over the same field.

Definition 3.2. Suppose G is a proabelian group, and $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. Define a *matrix representing* \mathbf{b} to be the matrix $\mathcal{B} \in M_n(\mathbb{Q}/\mathbb{Z})$ determined by \mathbf{b} with respect to a basis $\{\phi_1, \dots, \phi_n\}$ of the finite group $G/\text{rad}_{\mathbf{b}}(G)$. Explicitly, if $b_{ij} = \mathbf{b}(\phi_i, \phi_j)$, then $\mathcal{B} = (b_{ij}) \in M_n(\mathbb{Q}/\mathbb{Z})$. Define the *row subgroup* $\text{row}(\mathbf{b})$ to be the subgroup of $(\mathbb{Q}/\mathbb{Z})^n$ generated by \mathcal{B} 's rows, up to isomorphism.

Note that if $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, then $(a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ is an alternating matrix, with $a_{ij} = \mathbf{a}(\phi_i, \phi_j) = -a_{ji}$, and $a_{ii} = 0$. Theorem 3.6 below shows how to compute the index of an alternating form \mathbf{a} directly from any alternating matrix representing \mathbf{a} . The key observation is given by the next result. The lemma that follows is the basis for later results on the standard form of alternating forms and Brauer classes (Theorem 4.12, Theorem 5.13, and Theorem 5.18).

Proposition 3.3. *Suppose G is a proabelian group, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Then $\text{row}(\mathbf{a})$ is well defined, $\text{row}(\mathbf{a}) \simeq G/\text{rad}_{\mathbf{a}}(G)$, and $|\text{row}(\mathbf{a})| = \deg(\mathbf{a})$.*

Proof. It suffices to show that $\text{row}(\mathcal{A}) \simeq G/\text{rad}_{\mathbf{a}}(G)$ for an arbitrary matrix representative \mathcal{A} .

Let $\bar{\phi} = \{\bar{\phi}_1, \dots, \bar{\phi}_n\}$ be a basis for $G/\text{rad}_{\mathbf{a}}(G)$, and let $\mathcal{A} \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ be the matrix for \mathbf{a} determined by $\bar{\phi}$. Let $r = \exp(\mathbf{a})$, then $G/\text{rad}_{\mathbf{a}}(G)$ is a quotient of $(\mathbb{Z}/r\mathbb{Z})^n$, such that the standard basis $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ maps to $\bar{\phi}$. By definition of inflation, $\mathbf{a}(\bar{\phi}_i, \bar{\phi}_j) = \text{inf}(\mathbf{a})(\phi_i, \phi_j)$, so \mathcal{A} is also the matrix for $\text{inf}(\mathbf{a})$ determined by $\underline{\phi}$. Clearly $G/\text{rad}_{\mathbf{a}}(G) \simeq (\mathbb{Z}/r\mathbb{Z})^n / \text{rad}_{\text{inf}(\mathbf{a})}((\mathbb{Z}/r\mathbb{Z})^n)$, so it now suffices to show $\text{row}(\mathcal{A}) \simeq G/\text{rad}_{\mathbf{a}}(G)$ with $G = (\mathbb{Z}/r\mathbb{Z})^n$ and $\mathbf{a} = \text{inf}(\mathbf{a})$. First, a lemma.

Lemma 3.4 “Symplectic Basis I”. *Suppose $G = (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Let $\{\phi_1, \dots, \phi_n\}$ be the standard basis for G . Then there exists a matrix $\mathcal{P} \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$ such that $\underline{\phi}\mathcal{P} = \underline{\psi}$, defined by $\psi_j = \prod_i \phi_i^{P_{ij}}$, is a symplectic basis for G with respect to \mathbf{a} , so that for some subset $\{\psi_1, \dots, \psi_{2m}\} \subset \underline{\psi}$, along with uniquely determined positive integers $d_i : d_i \mid d_{i-1}$ and units $w_i \pmod{d_i}$,*

$$\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}) = w_i/d_i, \quad 1 \leq i \leq m,$$

and $\mathbf{a}(\psi_k, \psi_l) = 0$ otherwise. The w_i may be arbitrarily designated. If $\mathcal{A} \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ is the matrix for \mathbf{a} with respect to $\underline{\phi}$, then $\mathcal{B} = \mathcal{P}^*\mathcal{A}\mathcal{P}$ is the alternating 2-block-diagonal matrix for \mathbf{a} determined by $\underline{\psi}$.

Proof. By [B, Lemma 1.10], there exists $\mathcal{P}' \in \text{GL}_n(\mathbb{Z})$ such that if $\underline{\psi}' = \underline{\phi}\mathcal{P}'$, then for some subset $\{\psi'_1, \dots, \psi'_{2m}\}$, $\mathbf{a}(\psi'_{2i-1}, \psi'_{2i}) = -\mathbf{a}(\psi'_{2i}, \psi'_{2i-1}) = u_i/d_i$ for $i : 1 \leq i \leq m$, where $d_i \mid d_{i-1}$, and u_i is a unit $\pmod{d_i}$, and $\mathbf{a}(\psi'_k, \psi'_l) = 0$ otherwise. The proof of this result is similar to the standard construction of a symplectic basis over a vector space. Since G is a free $\mathbb{Z}/r\mathbb{Z}$ -module, $\underline{\psi}'$ is a basis for G . Let $\mathcal{P}'_r \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$ be the reduction of $\mathcal{P}' \pmod{r}$.

Let w_i be any unit $\pmod{d_i}$. Then w_i/d_i and u_i/d_i generate the same subgroup of \mathbb{Q}/\mathbb{Z} , so $v'_i(u_i/d_i) = w_i/d_i$ for some $v'_i \in (\mathbb{Z}/d_i\mathbb{Z})^\times$. Choose $v_i \in (\mathbb{Z}/r\mathbb{Z})^\times$ such that $v_i \equiv v'_i \pmod{d_i}$, which is possible since $d_i \mid r$. Let $\mathcal{D} \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$ be the diagonal matrix with $(2i-1, 2i-1)$ -entry v_i , $i = 1, \dots, m$, and the other diagonal entries 1. Let $\underline{\psi} = \underline{\psi}'\mathcal{D}$. Then $\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = \mathbf{a}(v_i\psi'_{2i-1}, \psi'_{2i}) = v_i u_i/d_i = w_i/d_i$. Let $\mathcal{P} = \mathcal{P}'_r\mathcal{D}$, then the desired symplectic basis is $\underline{\psi} = \underline{\phi}\mathcal{P}$, with $\mathcal{P} \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$.

It is now obvious that $\text{rad}_{\mathbf{a}}(G) \simeq \prod_{i=1}^m d_i\mathbb{Z} \times d_i\mathbb{Z}$, hence $G/\text{rad}_{\mathbf{a}}(G) \simeq \prod_{i=1}^m (\mathbb{Z}/d_i\mathbb{Z} \times \mathbb{Z}/d_i\mathbb{Z})$. It follows that the d_i are uniquely determined, as the abelian group invariants of $G/\text{rad}_{\mathbf{a}}(G)$. Finally, applying the bilinear relations to $\mathbf{a}(\prod_k \phi^{P_{ki}}, \prod_l \phi^{P_{lj}})$ yields

$$\mathbf{a}(\psi_i, \psi_j) = \sum_{1 \leq k < l \leq n} (p_{ki}p_{lj} - p_{kj}p_{li})a_{kl},$$

the ij -th entry of $\mathcal{P}^*\mathcal{A}\mathcal{P}$. This proves the lemma. ■

To finish the proof of the proposition, let $\mathcal{B} = \mathcal{P}^*\mathcal{A}\mathcal{P}$ be the matrix for \mathbf{a} determined by $\underline{\psi}$ in Lemma 3.4, with $w_i = 1$. Then \mathcal{B} is a direct sum of m alternating 2×2 blocks, whose nonzero entries are $\pm 1/d_i$. Evidently $\text{row}(\mathcal{B}) \simeq G/\text{rad}_{\mathbf{a}}(G)$. Since \mathcal{P} is invertible, $\text{row}(\mathcal{A}) \simeq \text{row}(\mathcal{B})$. This completes the proof. ■

The determinant of an alternating matrix over a commutative ring is zero if the matrix has odd degree, and it is a square if the matrix has even degree. The positive square root is called the *pfaffian* of the matrix. The naïve computation of determinants over \mathbb{Q}/\mathbb{Z} is not well defined, but the definition can be adjusted to make sense, as follows.

A degree m submatrix $\mathcal{M} \subset \mathcal{A}$ of an alternating matrix $\mathcal{A} = (a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ is a matrix obtained by intersecting m rows i_1, \dots, i_m with m columns j_1, \dots, j_m . A degree m minor is the determinant of a degree m submatrix. A degree m alternating submatrix $\mathcal{S} \subset \mathcal{A}$ is obtained by intersecting m rows i_1, \dots, i_m with the corresponding set of m columns i_1, \dots, i_m .

Definition 3.5. Suppose G is a proabelian group, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. The *minor subgroup* $\text{mr}(\mathbf{a})$ is the subgroup of \mathbb{Q}/\mathbb{Z} generated by all of the minors of any matrix \mathcal{A} representing \mathbf{a} , computed naively. The *pfaffian subgroup* $\text{pf}(\mathbf{a})$ is the subgroup of \mathbb{Q}/\mathbb{Z} generated by the pfaffians of all even-degree alternating submatrices of \mathcal{A} , computed naively. Thus

$$\begin{aligned}\text{mr}(\mathbf{a}) &= \langle \text{mr}(\mathcal{M}) : \mathcal{M} \subset \mathcal{A} \text{ submatrix} \rangle \leq \mathbb{Q}/\mathbb{Z} \\ \text{pf}(\mathbf{a}) &= \langle \text{pf}(\mathcal{S}) : \mathcal{S} \subset \mathcal{A} \text{ alternating, even degree} \rangle \leq \mathbb{Q}/\mathbb{Z}.\end{aligned}$$

“Computed naively” means computed from an arbitrary representative of \mathbf{a} in $M_n(\mathbb{Q})$, then interpreted mod \mathbb{Z} . Both $\text{mr}(\mathbf{a})$ and $\text{pf}(\mathbf{a})$ are well defined. For by [B, Proposition 2.3, Proposition 2.8], $|\text{row}(\mathcal{A})| = |\text{mr}(\mathcal{A})| = |\text{pf}(\mathcal{A})|^2$ for any matrix \mathcal{A} representing \mathbf{a} . Therefore, since $\text{row}(\mathbf{a})$ is well defined by Proposition 3.3,

$$|\text{row}(\mathbf{a})| = |\text{mr}(\mathbf{a})| = |\text{pf}(\mathbf{a})|^2.$$

Since a finite subgroup of \mathbb{Q}/\mathbb{Z} is determined by its order, $\text{mr}(\mathbf{a})$ and $\text{pf}(\mathbf{a})$ are well defined. Note that if \mathcal{A} has degree n , there are $\sum_{i=1}^n \binom{n}{i}^2$ submatrices, but a mere $\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{2i}$ alternating, even degree submatrices.

The following theorem is immediate.

Theorem 3.6 “Index Formula”. *Suppose G is a proabelian group, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Then*

$$\text{ind}(\mathbf{a}) = |\text{pf}(\mathbf{a})|.$$

Proof. By Proposition 3.3, $|\text{row}(\mathbf{a})| = \text{deg}(\mathbf{a})$. Since $|\text{row}(\mathbf{a})| = |\text{pf}(\mathbf{a})|^2$, $|\text{pf}(\mathbf{a})| = \text{ind}(\mathbf{a})$ by Definition 3.1. ■

Example 3.7. Suppose $G = \widehat{\mathbb{Z}}^n$. Any matrix \mathcal{A} representing $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ may be viewed by inflation as an element of $\text{Alt}_n(\mathbb{Q}/\mathbb{Z})$. If $n = 2$, $\mathcal{A} = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}$, for some $a_{12} \in \mathbb{Q}/\mathbb{Z}$. Thus $\text{ind}(\mathbf{a}) = \text{exp}(\mathbf{a}) = |a_{12}|$ in this case. If $n = 3$, $\text{ind}(\mathbf{a}) = \text{exp}(\mathbf{a}) = \text{lcm}[|a_{12}|, |a_{13}|, |a_{23}|]$, since $\text{pf}(\mathcal{A})$ is defined with respect to \mathcal{A} 's *even* degree (alternating) submatrices. If $n = 4$,

$$\mathcal{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Compute $\text{exp}(\mathbf{a}) = \text{lcm}[|a_{12}|, |a_{13}|, |a_{14}|, |a_{23}|, |a_{24}|, |a_{34}|]$, and

$$\text{ind}(\mathbf{a}) = \text{lcm}[\text{exp}(\mathbf{a}), |a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}|].$$

This is the first case where index is not necessarily equal to exponent. The formula for a degree 6 pfaffian of a 6×6 alternating matrix (a_{ij}) is

$$\begin{aligned}& a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} \\ & + a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} - a_{15}a_{23}a_{46} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} \\ & + a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34}.\end{aligned}$$

Lemma 3.4 will be needed for the next section in a slightly more general form.

Lemma 3.8 “Symplectic Basis II”. *Suppose $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, where G is a finite abelian group that has homogeneous primary components. Then G has a basis $\underline{\psi} = \{\psi_i\}_I$ that is symplectic with respect to \mathbf{a} , so that for some subset $\{\psi_1, \dots, \psi_{2m}\} \subset \underline{\psi}$, along with uniquely determined numbers $d_i : d_i \mid d_{i-1}$,*

$$\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}) = 1/d_i, \quad 1 \leq i \leq m,$$

and $\mathbf{a}(\psi_k, \psi_l) = 0$ otherwise.

Proof. Let p_1, \dots, p_t denote the prime support of $r = \exp(\mathbf{a})$. Let G_h denote G 's (homogeneous) p_h -component, and let \mathbf{a}_h denote \mathbf{a} 's p_h -component, so $\mathbf{a} = \sum_{h=1}^t \mathbf{a}_h$. Since \mathbf{a}_h is supported on G_h , by Lemma 3.4 G_h has a symplectic basis $\underline{\phi}_h = \{\phi_{hi}\}_{I_h}$ for some index set I_h , such that for some subset $\{\phi_{h1}, \dots, \phi_{h(2m_h)}\} \subset \underline{\phi}_h$, along with uniquely determined numbers $e_{hi} : e_{hi} \leq e_{hi-1}$ and selectable units $w_{hi} \pmod{p_h^{e_{hi}}}$,

$$\mathbf{a}_h(\phi_{h2i-1}, \phi_{h2i}) = -\mathbf{a}_h(\phi_{h2i}, \phi_{h2i-1}) = w_{hi}/p_h^{e_{hi}}, \quad 1 \leq i \leq m_h,$$

and $\mathbf{a}_h(\phi_{hk}, \phi_{hl}) = 0$ otherwise.

Let $m = \max\{m_h\}$, let $H_i = \{h : i \leq m_h\}$, and for fixed $i \leq m$, let $d_i = \prod_{h \in H_i} p_h^{e_{hi}}$. Then since $e_{hi} \leq e_{hi-1}$ for each h , d_i divides d_{i-1} . Choose the w_{hi} so that $\sum_{h \in H_i} w_{hi}/p_h^{e_{hi}} = 1/d_i \pmod{\mathbb{Z}}$, using the standard “gcd identity”. For $i = 1, \dots, m$, let $\psi_{2i-1} = \prod_{h \in H_i} \phi_{h2i-1}$ and $\psi_{2i} = \prod_{h \in H_i} \phi_{h2i}$. Extend $\{\psi_1, \dots, \psi_{2m}\}$ to a basis $\underline{\psi}$ for G . Compute, for $i : 1 \leq i \leq m$,

$$\begin{aligned} \mathbf{a}(\psi_{2i-1}, \psi_{2i}) &= \sum_{h=1}^t \mathbf{a}_h(\psi_{2i-1}, \psi_{2i}) = \sum_{h \in H_i} \mathbf{a}_h(\phi_{h2i-1}, \phi_{h2i}) = \sum_{h \in H_i} w_{hi}/p_h^{e_{hi}} \\ &= 1/d_i = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}). \end{aligned}$$

Clearly $\mathbf{a}(\psi_k, \psi_l) = 0$ in all other cases, since all of \mathbf{a} 's support is spanned by the subset $\{\psi_1, \dots, \psi_{2m}\}$.

Any symplectic basis for G with respect to \mathbf{a} decomposes into primary components to yield symplectic bases for each \mathbf{a}_h , and its invariants decompose accordingly into invariants $p_h^{e_{hi}}$. Since by Lemma 3.4 these are uniquely determined for \mathbf{a}_h , the d_i are uniquely determined for \mathbf{a} . This completes the proof. ■

Remark 3.9. Lemma 3.8 shows that whenever the finite abelian group G has homogeneous primary components, G has a basis with respect to which \mathbf{a} is an alternating 2-block-diagonal matrix, with coefficients $1/d_i$ in \mathbb{Q}/\mathbb{Z} . This matrix will be called a *canonical form* for \mathbf{a} . If $\underline{\psi} = \{\psi_1, \dots, \psi_{2m}\}$ is a symplectic basis, let $\underline{\psi}^* = \{\psi_1^*, \dots, \psi_{2m}^*\}$ denote the *dual basis* for G^* , as in [L, Chapter 1, Section 1], defined by $\psi_i^*(\psi_j) = \delta_{ij}/|\psi_i|$. Then by (2.2),

$$\mathbf{a} = \sum_{i=1}^m \frac{n_i}{d_i} (\psi_{2i-1}^* \wedge \psi_{2i}^*),$$

where $n_i = \gcd(|\psi_{2i-1}^*|, |\psi_{2i}^*|)$. This expression will also be called a canonical form for \mathbf{a} .

4. TRANSPORT TO $\text{Br}(F)$

In this section all of the results on alternating forms, including index, standard form, and functoriality, are transported to the Brauer group.

Setup 4.1. Let F be a field of characteristic 0 that contains all roots of unity μ . Let F_{ab} denote F 's maximal abelian extension. Assume F_{ab}/F is nontrivial, then since $\mu \subset F$, $G_F^{\text{ab}} = G_{F_{\text{ab}}/F}$ is torsion-free. Let

$$\omega : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mu$$

denote a fixed isomorphism. Set $\omega_d^a = \omega(a/d)$, and $\omega_d = \omega_d^1$. Note any other choice of isomorphism $\omega' : \mathbb{Q}/\mathbb{Z} \rightarrow \mu$ differs from ω by a constant $u \in \widehat{\mathbb{Z}}^\times$, so that $\omega'_d = \omega_d^u$, where u is interpreted (mod d). Write $\omega' = \omega^u$ in this case. Let ω_* denote any of the isomorphisms $\mathcal{F}(-, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathcal{F}(-, \mu)$, where $\mathcal{F} = \text{Hom}, \text{Bil}, \text{H}^2$, or Alt , induced by ω . Let

$$1 \longrightarrow G_{\Omega/K} \xrightarrow{\iota} G_{\Omega/F} \xrightarrow{\pi} G_{K/F} \longrightarrow 1$$

be a short exact sequence of abelian Galois groups, where $\exp(G_{K/F}) = r \in \mathbb{N}$. View $G_{\Omega/K}$ as a subgroup of $G_{\Omega/F}$. For $d : r \mid d$, let

$$J_d : G_{\Omega/F} \rightarrow G_{\Omega/K}$$

denote exponentiation by d . If ι is open, set $s = |G_{K/F}|$, and let $j := j_s$ denote the transfer.

For any $d \in \mathbb{N}$, let $\Omega^d \cap F^\times = \{x \in F^\times : x^{1/d} \in \Omega\} \leq F^\times$, and let

$$\langle -, - \rangle : G_{\Omega/F} \times (\Omega^d \cap F^\times) \longrightarrow \mu_d$$

denote the d -Kummer pairing, defined by $\langle \sigma, x \rangle_d = \sigma(x^{1/d})/x^{1/d}$, where $x^{1/d}$ is any root of the polynomial $t^d - x$. Since $\text{char}(F) = 0$ and F contains μ , the d -Kummer pairing makes sense for all $d \in \mathbb{N}$, and every $\phi^* \in {}_d G_{\Omega/F}^*$ is a *Kummer character*, which means there exists a *Kummer element* $x \in \Omega^d \cap F^\times$, uniquely defined (mod $F^{\times d}$), such that $\omega_* \cdot \phi^* = \langle -, x \rangle_d$. Let

$$\kappa_{\omega_d} : {}_d G_{\Omega/F}^* \xrightarrow{\sim} (\Omega^d \cap F^\times)/F^{\times d}$$

denote the resulting *Kummer isomorphism*. Its inverse is $x \mapsto \omega_*^{-1} \langle -, x \rangle_d$. By the bilinearity of the Kummer pairing, another choice $\omega' = \omega^u$, $u \in \widehat{\mathbb{Z}}^\times$, induces a new Kummer isomorphism $\kappa_{\omega'_d} = \kappa_{\omega_d}^u$, so that if $\kappa_{\omega_d}(\phi^*) = x$, then $\kappa_{\omega'_d}(\phi^*) = x^u$, where u is interpreted (mod d).

If $d \mid e$, the inclusion ${}_d G_{\Omega/F}^* \rightarrow {}_e G_{\Omega/F}^*$ induces the map

$$(4.2) \quad \begin{aligned} (\Omega^d \cap F^\times)/F^{\times d} &\longrightarrow (\Omega^e \cap F^\times)/F^{\times e} \\ x &\longmapsto x^{e/d} \end{aligned}$$

since, if $x \in \Omega^d \cap F^\times$, $\langle -, x \rangle_d = \langle -, x \rangle_e^{e/d} = \langle -, x^{e/d} \rangle_e$. Thus $\kappa_{\omega_e} \cdot \kappa_{\omega_d}^{-1}(x) = x^{e/d} \pmod{F^{\times e}}$.

If $\rho : G_{\Omega'/F'} \rightarrow G_{\Omega/F}$ is a map of abelian Galois groups, define for each $d \in \mathbb{N}$ the *Kummer transpose* of ρ by the right vertical arrow in the diagram

$$(4.3) \quad \begin{array}{ccc} {}_d G_{\Omega/F}^* & \xrightarrow{\kappa_{\omega_d}} & (\Omega^d \cap F^\times) / F^{\times d} \\ \rho^* \downarrow & & \downarrow \\ {}_d G_{\Omega'/F'}^* & \xrightarrow{\kappa_{\omega_d}} & (\Omega'^d \cap F'^{\times d}) / F'^{\times d}. \end{array}$$

When the meaning is clear, let ρ^* stand for both maps.

If $r \mid d$, then by Lemma 2.7, the sequence of Kummer transpose maps

$$(4.4) \quad 1 \longrightarrow \frac{K^d \cap F^\times}{F^{\times d}} \xrightarrow{\pi^*} \frac{\Omega^d \cap F^\times}{F^{\times d}} \xrightarrow{\iota^*} \frac{\Omega^d \cap K^\times}{K^{\times d}} \xrightarrow{j_d^*} \frac{K^d \cap F^\times}{F^{\times d}}$$

is exact, and if $G_{\Omega/F}$ is torsion-free, j_d^* is surjective. Thus if $G_{\Omega/F}$ is torsion-free, $\kappa_{\omega_d} \cdot j_d^*(G_{\Omega/K}^*)$ describes K/F as a Kummer extension. Explicitly, $K = F(Z^{1/d})$, where $Z \leq F^\times$ is a preimage of $\kappa_{\omega_d} \cdot j_d^*(G_{\Omega/K}^*)$ in F^\times .

Lemma 4.5. *The Kummer transpose maps π^* and ι^* in (4.4) are induced by the inclusions $K \subset \Omega$ and $F \subset K$, respectively. If ι is open, and $s = |G_{K/F}|$, the corestriction*

$$j_s^* = j_s^* : (\Omega^s \cap K^\times) / K^{\times s} \rightarrow (K^s \cap F^\times) / F^{\times s}$$

is induced by the norm $N_{K/F} : K^\times \rightarrow F^\times$.

Proof. For the first map, suppose $\phi^* \in {}_d G_{\Omega/F}^*$. Let $x = \kappa_{\omega_d}(\phi^*) \in F^\times / F^{\times d}$, then by definition $\omega_* \cdot \phi^*(\sigma) = \langle \sigma, x \rangle_d$ for all $\sigma \in G_{\Omega/F}$. Since ι^* is restriction, $\omega_* \cdot \iota^*(\phi^*)(\tau) = \langle \tau, x \rangle_d$ for $\tau \in G_{\Omega/K}$. Thus the Kummer element for $\iota^*(\phi^*)$ is x , viewed as an element of $K^\times / K^{\times d}$, hence $\iota^* : F^\times / F^{\times d} \rightarrow K^\times / K^{\times d}$ is the natural map. Similarly, π^* is the natural injection.

To show j_s^* is induced by the norm, let $\{\sigma_1, \dots, \sigma_s\}$ be a section $G_{K/F} \rightarrow G_{\Omega/F}$. Suppose $\phi^* \in {}_d G_{\Omega/K}^*$, and $\kappa_{\omega_d}(\phi^*) = y \in (\Omega^d \cap K^\times) / K^{\times d}$. Identify y with a preimage in $\Omega^d \cap K^\times$, then $N_{K/F}(y) \in \Omega^d \cap F^\times$. To prove the result it suffices to show $N_{K/F}(y) \equiv \kappa_{\omega_d} \cdot j_s^*(\phi^*)(\text{mod } F^{\times d})$.

By definition, $j_s^*(\phi^*)(\tau) = \phi^*(\tau^s) = \omega_*^{-1}(\langle \tau^s, y \rangle_d)$, for all $\tau \in G_{\Omega/F}$. Observe that $\langle \tau^s, y \rangle_d = \langle \tau, w \rangle_d$, where $w = \prod_{i=0}^{s-1} \tau^i(y) \in \Omega^{\langle \tau \rangle}$. Explicitly,

$$\frac{\tau^s(y^{1/d})}{y^{1/d}} = \prod_{i=0}^{s-1} \frac{\tau^{i+1}(y^{1/d})}{\tau^i(y^{1/d})} = \frac{\tau(w^{1/d})}{w^{1/d}}.$$

Let q be the order of $\tau(\text{mod } G_{\Omega/K})$. Reorder the σ_i if necessary so that $\sigma_1, \dots, \sigma_q$ represent the cosets of $G_{\Omega/K}$ in $\langle \tau \rangle G_{\Omega/K}$, and $\sigma_q, \sigma_{2q}, \dots, \sigma_s$ represent the cosets of $\langle \tau \rangle G_{\Omega/K}$ in $G_{\Omega/F}$. Since y is in K , it is fixed by τ^q , hence $w = z^{s/q}$, where

$$z = \prod_{i=0}^{q-1} \tau^i(y) = \prod_{i=1}^q \sigma_i(y).$$

Note since $y \in \Omega^{\times d}$, $z \in \Omega^{\times d}$. Since τ fixes z and $z \in K^{\times d}$, $\langle \tau, z \rangle_d = \langle \tau, \sigma_i(z) \rangle_d$, for all i . For σ_i is defined on any $z^{1/d}$, since $z^{1/d} \in \Omega$, and so $\sigma_i(z^{1/d})$ is a d -th root of $\sigma_i(z)$, hence by the definition of the Kummer pairing, $\langle \tau, \sigma_i(z) \rangle_d = \sigma_i(\langle \tau, z \rangle_d)$. Since $\langle \tau, z \rangle_d \in \mu \subset F$, $\sigma_i(\langle \tau, z \rangle_d) = \langle \tau, z \rangle_d$. It follows that

$$\langle \tau^s, y \rangle_d = \langle \tau, w \rangle_d = \langle \tau, z \rangle_d^{s/q} = \prod_{j=1}^{s/q} \langle \tau, \sigma_{jq}(z) \rangle_d = \langle \tau, \prod_{j=1}^{s/q} \sigma_{jq}(z) \rangle_d.$$

Since $\prod_{j=1}^{s/q} \sigma_{jq}(z) = \prod_{i=1}^s \sigma_i(y) = N_{K/F}(y)$, this shows $j^*(\phi^*)(\tau) = \omega_*^{-1} \langle \tau, N_{K/F}(y) \rangle_d$. Therefore $\kappa_{\omega_d} \cdot j^*(\phi^*) \equiv N_{K/F}(y) \pmod{F^{\times d}}$, as desired. ■

The Kummer transposes ι^* and π^* are so natural, the notation for them will be suppressed.

To apply this to the Brauer group, suppose $x \in F^{\times} / F^{\times m}$, $y \in F^{\times} / F^{\times n}$, and d divides $\gcd(m, n)$. Identify x and y with preimages in F^{\times} . Let

$$(x, y)_{\omega_d} = \sum F X^i Y^j : X^d = x; Y^d = y; [Y, X] = \omega_d$$

denote the *symbol algebra* of degree d . Note that a choice of ω is implicit in the definition. If $c \in \mathbb{N}$, let $(x, y)_{\omega_d}^c$ denote either of the (isomorphic) algebras $(x, y^c)_{\omega_d}$ or $(x^c, y)_{\omega_d}$. By the bilinearity of the bracket, if $(u, d) = 1$ then $(x, y)_{\omega_d} \simeq (x, y^u)_{\omega_d^u}$. In particular, if $\omega' = \omega^u$, $u \in \widehat{\mathbb{Z}}^{\times}$, is another choice of ω , then $(x, y)_{\omega_d} \simeq (x, y^u)_{\omega_d'}$, where u is interpreted (mod d). Let $[(x, y)_{\omega_d}]$ denote the class of $(x, y)_{\omega_d}$ in the relative Brauer group $\mathbf{H}^2(\mathbf{G}_{K/F}, K^{\times}) = \mathbf{Br}(K/F)$. Note $c[(x, y)_{\omega_d}] = [(x, y)_{\omega_d}^c] = [(x, y)_{\omega_d}^{\otimes c}]$. The *symbol subgroup* of $\mathbf{Br}(K/F)$ is the subgroup generated by all symbol classes that are split by K . By [D, Section 11, Lemma 6], if $d \mid e$, then $\frac{e}{d}[(x, y)_{\omega_e}] = [(x, y)_{\omega_d}]$.

If $\phi^* \in \mathbf{G}_F^*$ and $y \in F^{\times}$, let K/F be the cyclic field extension of degree $d = |\phi^*|$ fixed by $\ker(\phi^*)$, let $\phi \in \mathbf{G}_F$ be any element defined by $\phi^*(\phi) = 1/d$, and let

$$(\phi^*, y) = \sum K Y^j : Y^d = y; YX = \phi(X)Y, \forall X \in K$$

denote the *cyclic algebra* of degree d . If $\kappa_{\omega_d}(\phi^*) = x$, then $(\phi^*, y) \simeq (x, y)_{\omega_d}$.

Definition 4.6. Define the *interior subgroup* $\mathbf{Br}^{\circ}(K/F)$ to be the image of the natural map $\varepsilon_* : \mathbf{H}^2(\mathbf{G}_{K/F}, \mu) \rightarrow \mathbf{H}^2(\mathbf{G}_{K/F}, K^{\times})$, induced by the inclusion $\varepsilon : \mu \hookrightarrow K^{\times}$.

The next result is the main theorem of the paper. It computes a functorial map from $\mathbf{Alt}(G_{\Omega/F}, \mathbb{Q}/\mathbb{Z})$ to $\mathbf{Br}(\Omega/F)$, which is natural modulo the choice of ω . It shows that the image of the map is the interior subgroup $\mathbf{Br}^{\circ}(\Omega/F)$, and that the interior subgroup is asymptotically equal to the symbol subgroup.

Theorem 4.7. *Suppose F is a field containing μ , and Ω/F is a Galois extension with abelian Galois group $G_{\Omega/F}$. For $\phi^*, \psi^* \in G_{\Omega/F}^*$, let $m = |\phi^*|$, $n = |\psi^*|$, and $d = \gcd(m, n)$. Let $x = \kappa_{\omega_m}(\phi^*)$ and $y = \kappa_{\omega_n}(\psi^*)$. Let $\kappa^{\wedge 2}$ denote the functorial map $\varepsilon_* \cdot \omega_* \cdot \text{alt}^{-1}$. Then*

$$\begin{aligned} \kappa^{\wedge 2} : \text{Alt}(G_{\Omega/F}, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{Br}(\Omega/F) \\ \phi^* \wedge \psi^* &\longmapsto [(x, y)_{\omega_d}]. \end{aligned}$$

The image is $\text{Br}^\circ(\Omega/F)$. $\text{Br}^\circ(\Omega/F)$ is generated by all symbols $[(x, y)_{\omega_e}]$ with $x, y \in \Omega^e \cap F^\times$. In particular, $\text{Br}^\circ(F_{\text{ab}}/F) = \bigcup \text{Br}^\circ(\Omega/F)$, where the union is over all finite abelian extensions. $\text{Br}^\circ(F_{\text{ab}}/F)$ is the entire symbol subgroup of $\text{Br}(F)$, hence $\text{Br}^\circ(F_{\text{ab}}/F) = \text{Br}(F)$.

Suppose F' and F are fields containing μ , and $\rho : G_{\Omega'/F'} \rightarrow G_{\Omega/F}$ is a homomorphism of abelian Galois groups that induces a well defined homomorphism on Brauer groups. Then the induced map is given by

$$\begin{aligned} \rho^* \wedge^2 : \text{Br}^\circ(\Omega/F) &\longrightarrow \text{Br}^\circ(\Omega'/F') \\ [(x, y)_{\omega_e}] &\longmapsto [(\rho^*(x), \rho^*(y))_{\omega_e}] \end{aligned}$$

where $\rho^ : (\Omega^e \cap F^\times)/F^{\times e} \rightarrow (\Omega'^e \cap F'^{\times e})/F'^{\times e}$ is the Kummer transpose (4.3).*

Proof. The map $\varepsilon_* \cdot \omega_* \cdot \text{alt}^{-1}$ is functorial in $G_{\Omega/F}$ since it is the composite of functorial maps. Let $f = \varepsilon_* \cdot \omega_*(\phi^* \otimes \psi^*) = \varepsilon_* \cdot \omega_* \cdot \text{alt}^{-1}(\phi^* \wedge \psi^*) \in \text{Bil}(G_{\Omega/F}, \Omega^\times)$. By Theorem 2.4, to prove the first statement it suffices to show $[f] = [(x, y)_{\omega_d}]$. Let $I = F(x^{1/d})$ and $J = F(y^{1/d})$, and let $\langle \bar{\phi} \rangle = G_{I/F}$ and $\langle \bar{\psi} \rangle = G_{J/F}$, where $\langle \bar{\phi}, x \rangle_d = \langle \bar{\psi}, y \rangle_d = \omega_d$. Let π_1 and π_2 be the canonical maps from $G_{\Omega/F}$ to $G_{I/F}$ and $G_{J/F}$, respectively. Suppose $\sigma, \tau \in G_{\Omega/F}$, and

$$\begin{aligned} \pi_1(\sigma) &= \bar{\phi}^i & \pi_2(\sigma) &= \bar{\psi}^j \\ \pi_1(\tau) &= \bar{\phi}^k & \pi_2(\tau) &= \bar{\psi}^l \end{aligned}$$

where $0 \leq i, j, k, l < d$. Then by (2.2), $f(\sigma, \tau) = \omega_d^{il}$. For $a, b : 0 \leq a, b < d$, define

$$\begin{aligned} \bar{g} : G_{I/F} \times G_{I/F} &\longrightarrow I^\times \\ (\bar{\phi}^a, \bar{\phi}^b) &\longmapsto \begin{cases} y & \text{if } a + b \geq d \\ 1 & \text{else.} \end{cases} \end{aligned}$$

The resulting cyclic crossed product $(I/F, \bar{g}) = (\phi^*, y)$ is isomorphic to the symbol algebra $(x, y)_{\omega_d}$. Let $g = \text{inf}(\bar{g}) \in Z^2(G_{\Omega/F}, \Omega^\times)$. Define a 1-cochain $h \in C^1(G_{\Omega/F}, \Omega^\times)$ by $h(\sigma) = \omega_d^{ij} y^{i/d}$. Compute on one hand,

$$\frac{g}{f}(\sigma, \tau) = \begin{cases} \omega_d^{-il} y & \text{if } i + k \geq d \\ \omega_d^{-il} & \text{else.} \end{cases}$$

On the other hand, $h(\tau)\sigma = \omega_d^{kl}(y^{k/d})\sigma = \omega_d^{kl}(y^{k/d})\pi_2(\sigma) = \omega_d^{kl}\omega_d^{jk}y^{k/d}$, hence if $(i+k)_d$ is the smallest nonnegative integer congruent to $i+k \pmod{d}$,

$$\frac{h(\sigma)h(\tau)\sigma}{h(\sigma\tau)} = \frac{\omega_d^{ij+kl+jk}y^{(i+k)/d}}{\omega_d^{ij+kl+jk+il}y^{(i+k)_d/d}} = \begin{cases} \omega_d^{-il} y & \text{if } i + k \geq d \\ \omega_d^{-il} & \text{else.} \end{cases}$$

This shows g/f is a 2-cocycle, hence that $[g] = [f] \in H^2(G_{\Omega/F}, \Omega^\times) = \text{Br}(\Omega/F)$, and $[f] = [(x, y)_{\omega_d}]$, as desired.

Let $(x, y)_{\omega_e}$ be a symbol, where $x, y \in \Omega^e \cap F^\times$. Set $\phi^* = \kappa_{\omega_e}^{-1}(x)$ and $\psi^* = \kappa_{\omega_e}^{-1}(y)$, both elements of ${}_e(G_{\Omega/F}^*)$. Let $m = |\phi^*|$, $n = |\psi^*|$, $d = \gcd(m, n)$, and $c = \text{lcm}[m, n]$. Note $c \mid e$. Let $x' = \kappa_{\omega_m}(\phi^*)$ and $y' = \kappa_{\omega_n}(\psi^*)$. Identify x, y, x' , and y' with representatives in F^\times . By (4.2), $x'^{e/m} \equiv x \pmod{F^{\times e}}$ and $y'^{e/n} \equiv y \pmod{F^{\times e}}$. Now

$$[(x, y)_{\omega_e}] = [(x'^{e/m}, y'^{e/n})_{\omega_e}] = \frac{e^2}{mn} [(x', y')_{\omega_e}] = \frac{e^2}{cd} [(x', y')_{\omega_e}] = \frac{e}{c} [(x', y')_{\omega_d}] = \kappa^{\wedge 2} \left(\frac{e}{c} (\phi^* \wedge \psi^*) \right).$$

Since $\phi^*, \psi^* \in {}_e(G_{\Omega/F}^*)$, this shows $[(x, y)_{\omega_e}] \in \text{Br}^\circ(\Omega/F)$. Conversely, if $\xi \in \text{Br}^\circ(\Omega/F)$, then ξ is a sum of elements of the form $[(x, y)_{\omega_e}]$, where $x, y \in \Omega^e \cap F^\times$, by the computation of $\kappa^{\wedge 2}$. Therefore $\text{Br}^\circ(\Omega/F)$ is generated by all symbols $[(x, y)_{\omega_e}]$, $e \in \mathbb{N}$, where $x, y \in \Omega^e \cap F^\times$.

Any $x, y \in F^\times$ are in $F_{\text{ab}}^e \cap F^\times$ for some e , hence $\text{Br}^\circ(F_{\text{ab}}/F)$ is the entire symbol subgroup of $\text{Br}(F)$. By Theorem 2.8, $\text{Alt}(G_F^{\text{ab}}, \mathbb{Q}/\mathbb{Z}) = \bigcup \text{Alt}(G_{\Omega/F}, \mathbb{Q}/\mathbb{Z})$, where the union is over finite abelian extensions. Since inflation is also injective on the Brauer group, and since inflation is a morphism of cohomological functors, it follows that $\text{Br}^\circ(F_{\text{ab}}/F) = \bigcup \text{Br}^\circ(\Omega/F)$. Since $\mu \subset F$, $\text{Br}^\circ(F_{\text{ab}}/F) = \text{Br}(F)$ by the Merkurjev-Suslin Theorem.

It remains to verify the functorial map. Since both fields F' and F contain μ , $\kappa^{\wedge 2}$ is defined on $\text{Alt}(G_{\Omega/F}, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G_{\Omega'/F'}, \mathbb{Q}/\mathbb{Z})$. In addition to the above notation, let $m' = |\rho^*(\phi^*)|$, $n' = |\rho^*(\psi^*)|$, $d' = \gcd(m', n')$, and $c' = \text{lcm}[m', n']$. By the above, $[(x, y)_{\omega_e}] = \kappa^{\wedge 2} \left(\frac{e}{c} (\phi^* \wedge \psi^*) \right) = \frac{e}{c} [(x', y')_{\omega_d}]$, where $x'^{e/m} \equiv x$ and $y'^{e/n} \equiv y \pmod{F^{\times e}}$. By definition, $\kappa^{\wedge 2} \cdot \rho^{*\wedge 2} \left(\frac{e}{c} (\phi^* \wedge \psi^*) \right) = \frac{e}{c} \frac{e}{c'} [(\kappa_{\omega_{m'}} \cdot \rho^*(\phi^*), \kappa_{\omega_{n'}} \cdot \rho^*(\psi^*))_{\omega_{d'}}]$. Therefore

$$\begin{aligned} \kappa^{\wedge 2} \cdot \rho^{*\wedge 2} \left(\frac{e}{c} (\phi^* \wedge \psi^*) \right) &= \frac{e}{c} \frac{cd}{c'd'} [(\kappa_{\omega_{m'}} \cdot \rho^*(\phi^*), \kappa_{\omega_{n'}} \cdot \rho^*(\psi^*))_{\omega_{d'}}] \\ &= \frac{e}{c} \frac{mn}{m'n'} [(\kappa_{\omega_{m'}} \cdot \rho^*(\phi^*), \kappa_{\omega_{n'}} \cdot \rho^*(\psi^*))_{\omega_d}] \\ &= \frac{e}{c} [(\kappa_{\omega_{m'}} \cdot \rho^*(\phi^*)^{m/m'}, \kappa_{\omega_{n'}} \cdot \rho^*(\psi^*)^{n/n'})_{\omega_d}] \\ &= \frac{e}{c} [(\kappa_{\omega_m} \cdot \rho^*(\phi^*), \kappa_{\omega_n} \cdot \rho^*(\psi^*))_{\omega_d}] = \frac{e}{c} [(\rho^* \cdot \kappa_{\omega_m}(\phi^*), \rho^* \cdot \kappa_{\omega_n}(\psi^*))_{\omega_d}] \\ &= \frac{e}{c} \frac{e}{d} [(\rho^* \cdot \kappa_{\omega_m}(\phi^*), \rho^* \cdot \kappa_{\omega_n}(\psi^*))_{\omega_e}] = \frac{e^2}{mn} [(\rho^*(x'), \rho^*(y'))_{\omega_e}] \\ &= [(\rho^*(x')^{e/m}, \rho^*(y')^{e/n})_{\omega_e}] = [(\rho^*(x'^{e/m}), \rho^*(y'^{e/n}))_{\omega_e}] \\ &= [(\rho^*(x), \rho^*(y))_{\omega_e}]. \end{aligned}$$

To interpret some of the above elements, replace $\kappa_{\omega_{m'}} \cdot \rho^*(\phi^*)$ and $\kappa_{\omega_{n'}} \cdot \rho^*(\psi^*)$ with preimages in F'^{\times} . Since the induced map $\rho^{*\wedge 2} : \text{Br}(\Omega/F) \rightarrow \text{Br}(\Omega'/F')$ is well defined, $\rho^{*\wedge 2}([(x, y)_{\omega_e}]) = [(\rho^*(x), \rho^*(y))_{\omega_e}]$. This completes the proof. ■

Remarks 4.8. a. Since clearly $\omega_* \cdot \text{alt}^{-1} = \text{alt}^{-1} \cdot \omega_*$, the map

$$\begin{aligned} \varepsilon_* \cdot \text{alt}^{-1} : \text{Alt}(G_{\Omega/F}, \mu) &\longrightarrow \text{Br}(\Omega/F) \\ \omega_*(\phi^* \wedge \psi^*) &\longmapsto [(x, y)_{\omega_d}] \end{aligned}$$

is *natural*, that is, independent of the choice of ω . Explicitly, suppose $\omega' = \omega^u$ for some $u \in \widehat{\mathbb{Z}}^\times$, and $x' = \kappa_{\omega'_m}(\phi^*)$, $y' = \kappa_{\omega'_n}(\psi^*)$. Then $x' = x^u$ and $y' = y^u$, where u is interpreted $(\text{mod } d)$. Compute $\omega'_*(\phi^* \wedge \psi^*) = u\omega_*(\phi^* \wedge \psi^*)$, and $[(x', y')_{\omega'_d}] = [(x^u, y^u)_{\omega'_d}] = u[(x, y)_{\omega_d}]$, as desired.

b. By Theorem 4.7, $\kappa^{\wedge 2} : \text{Alt}(G_F^{\text{ab}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Br}(F)$ is surjective. However, it might not be injective. For example, if $F = \mathbb{C}(t)$, then G_F^{ab} is a large non-procyclic group, so $\text{Alt}(G_F^{\text{ab}}, \mu)$ is nontrivial. On the other hand, $\text{Br}(F) = \{0\}$ by Tsen's theorem.

The symplectic basis and index results for alternating forms may now be transported to the Brauer group.

Definition 4.9. a. Suppose $\mathbf{a} \in \text{Alt}(G_F^{\text{ab}}, \mathbb{Q}/\mathbb{Z})$, and $\kappa^{\wedge 2}(\mathbf{a}) = \boldsymbol{\xi} \in \text{Br}(F)$. Define a *matrix representing* $\boldsymbol{\xi}$ to be any (alternating) matrix (a_{ij}) representing \mathbf{a} , as in Definition 3.2.

b. Suppose $x_1, x_2 \in F^\times / F^{\times d}$ and $x_3, x_4 \in F^\times / F^{\times e}$ are nontrivial. Call a symbol $(x_1, x_2)_{\omega_d}$ *reduced* if the subgroup $\langle x_1^{1/d}, x_2^{1/d} \rangle \leq (F_{\text{ab}}^\times / F^\times)_{\text{tor}}$ has order d^2 . Call a collection of reduced symbols $(x_i, x_j)_{\omega_{d_{ij}}}$ *disjoint* if the subgroup of $(F_{\text{ab}}^\times / F^\times)_{\text{tor}}$ generated by all of the $x_i^{1/d_{ij}}, x_j^{1/d_{ij}}$, has order $\prod_{i,j} d_{ij}^2$.

c. Call a sum of classes of disjoint reduced symbols in $\text{Br}(F)$ a *disjoint reduced symbol presentation*.

Example 4.10. Suppose $\{x_1, \dots, x_n\}$ is a subset of F^\times that is pure independent in $F^\times / F^{\times d}$ for some $d \in \mathbb{N}$. For example, the x_i could be part of a transcendence basis for F over some subfield of F . Suppose $\boldsymbol{\xi} \in \text{Br}(F)$ has presentation

$$\boldsymbol{\xi} = \sum_{1 \leq i < j \leq n} [(x_i, x_j)_{\omega_{d_{ij}}}^{n_{ij}}].$$

Set $a_{ij} = n_{ij}/d_{ij} \in \mathbb{Q}/\mathbb{Z}$, and set $a_{ji} = -a_{ij}$ if $i < j$. Then $\mathcal{A} = (a_{ij})$ is an alternating matrix representing $\boldsymbol{\xi}$. This will be proved in Theorem 4.12. The symbols $(x_i, x_j)_{\omega_d}$ are reduced, and if $d_i \mid d$ for $i = 1, \dots, \lfloor n/2 \rfloor$, then the collection $\{(x_{2i-1}, x_{2i})_{\omega_{d_i}}\}$ is disjoint.

Remark 4.11. A reduced symbol algebra may be trivial, and two disjoint reduced symbol algebras may be isomorphic. For example, when $F = \mathbb{C}(t)$, any symbol algebra is trivial, hence any two symbol algebras of the same degree are isomorphic. On the other hand, Corollary 4.14 gives conditions under which a product of disjoint reduced symbol algebras is a division algebra.

The next theorem shows that it is always possible to express a Brauer class as a sum of disjoint reduced symbol classes. Moreover, it shows how to obtain this expression from a given presentation, provided the parameters of the given symbols are part of a homogeneous set, such as a transcendence base for F over some subfield. More generally, it suffices to have an m -independent set.

Theorem 4.12 “Disjoint Reduced Symbol Presentation I”. *Suppose $\boldsymbol{\xi} \in \text{Br}(F)$. For some $\underline{y} = \{y_1, \dots, y_{2m}\} \subset F^\times$ and $d_i \mid d_{i-1}$, $\boldsymbol{\xi}$ has a disjoint reduced symbol presentation*

$$\boldsymbol{\xi} = \sum_{i=1}^m [(y_{2i-1}, y_{2i})_{\omega_{d_i}}].$$

where, as in Definition 4.9, the subgroup of $F_{\text{ab}}^\times / F^\times$ generated by the elements $y_{2i-1}^{1/d_i}, y_{2i}^{1/d_i}$ has order $\prod_{i=1}^m d_i^2$.

Homogeneous Case. Suppose $\mathfrak{m} \subset \mathcal{P}$, $\underline{x} = \{x_1, \dots, x_n\} \subset F^\times$ is an \mathfrak{m} -independent set, and

$$\xi = \sum_{1 \leq i < j \leq n} [(x_i, x_j)_{\omega_{r_{ij}}^{s_{ij}}}],$$

where $r_{ij} \in \mathbb{N}(\mathfrak{m})$. Set $a_{ij} = -a_{ji} = s_{ij}/r_{ij}$, then $(a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ is a matrix that represents ξ . Let $r = \text{lcm}\{r_{ij}\}$. There exists a matrix $\mathcal{P} = (p_{ij}) \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$ that puts (a_{ij}) in standard form, so that $\mathcal{P}^*(a_{ij})\mathcal{P} = (b_{ij})$, where $b_{2i-1, 2i} = -b_{2i, 2i-1} = 1/d_i$ ($1 \leq i \leq 2m$, $d_i : d_i \mid d_{i-1} \mid r$), and $b_{ij} = 0$ otherwise. Let $\mathcal{Q}^* = (q_{ij}^*)$ be the inverse of \mathcal{P}^* , and set $\underline{y} = \underline{x}\mathcal{Q}^*$, so that $(\text{mod } F^{\times r})$,

$$y_j = \prod_{i=1}^n x_i^{q_{ij}^*}, \quad x_j = \prod_{i=1}^n y_i^{p_{ij}^*}.$$

Then

$$\xi = \sum_{i=1}^m [(y_{2i-1}, y_{2i})_{\omega_{d_i}}]$$

is a disjoint reduced symbol presentation.

Proof. Let $G = \mathbb{G}_F^{\text{ab}}$. Let $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ be any element such that $\kappa^{\wedge 2}(\mathbf{a}) = \xi$. Set $r = \exp(\mathbf{a})$. Since G is torsion-free, G/G^r has homogeneous primary components. Since G/G^r is bounded, i.e., of finite exponent, it is a direct product of cyclic groups. Since \mathbf{a} is continuous, it is supported on a finite direct summand H/H^r of G/G^r . Therefore by Lemma 3.8, there exists a symplectic basis for H/H^r with respect to \mathbf{a} , such that for some subset $\{\psi_1, \dots, \psi_{2m}\}$, and numbers $d_i : d_i \mid d_{i-1} \mid r$, $\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}) = 1/d_i$, and $\mathbf{a}(\psi_k, \psi_l) = 0$ otherwise. Extend this basis to a basis $\underline{\psi}$ for G/G^r . Define elements $\{\psi_1^*, \dots, \psi_{2m}^*\} \subset G^*$ by $\psi_i^*(\psi_j) = \delta_{ij}/|\psi|$, for all i . Set $m_i = |\psi_{2i-1}^*|$, $n_i = |\psi_{2i}^*|$, and $r_i = \text{gcd}(m_i, n_i)$. Then $|\psi_{2i-1}^* \wedge \psi_{2i}^*| = r_i$, and

$$\mathbf{a} = \sum_{i=1}^m \frac{r_i}{d_i} (\psi_{2i-1}^* \wedge \psi_{2i}^*).$$

For each $d \in \mathbb{N}$, let $\kappa_{\omega_d} : G^* \rightarrow F_{\text{ab}}^d \cap F^\times / F^{\times d}$ be the Kummer isomorphism, and set $y_{2i-1} = \kappa_{\omega_{m_i}}(\psi_{2i-1}^*)$, and $y_{2i} = \kappa_{\omega_{n_i}}(\psi_{2i}^*)$. For the statement of the theorem, identify these elements with preimages in F^\times . By Theorem 4.7,

$$\kappa^{\wedge 2}\left(\frac{r_i}{d_i}(\psi_{2i-1}^* \wedge \psi_{2i}^*)\right) = \frac{r_i}{d_i}[(y_{2i-1}, y_{2i})_{\omega_{r_i}}] = [(y_{2i-1}, y_{2i})_{\omega_{d_i}}].$$

Thus ξ has the desired form. Since the characters ψ_{2i-1}^* and ψ_{2i}^* are independent in G^* , this is a disjoint reduced symbol presentation. This proves the first statement.

Now for the homogeneous case. Assume $(s_{ij}, r_{ij}) = 1$. Let G be the Galois group of $F(x_1^{1/r}, \dots, x_n^{1/r})/F$. Since \underline{x} is \mathfrak{m} -independent, G is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^n$. Let $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ be the basis for G defined by $\langle \phi_i, x_j \rangle_r = \omega_r^{\delta_{ij}}$. Define $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ by $\mathbf{a}(\phi_i, \phi_j) = a_{ij} = s_{ij}/r_{ij}$. Since $(s_{ij}, r_{ij}) = 1$, then $r = \text{lcm}\{r_{ij}\} = \exp(\mathbf{a})$. Let $\{\phi_1^*, \dots, \phi_n^*\}$ be dual to $\underline{\phi}$. Then

$$\mathbf{a} = \sum_{1 \leq i < j \leq n} \frac{rs_{ij}}{r_{ij}} (\phi_i^* \wedge \phi_j^*).$$

By Theorem 4.7, $\kappa^{\wedge 2}(\phi_i^* \wedge \phi_j^*) = [(x_i, x_j)_{\omega_r}]$, therefore $\kappa^{\wedge 2}(\mathbf{a}) = \boldsymbol{\xi}$. Thus $\mathbf{a} = (a_{ij})$ is a matrix that represents $\boldsymbol{\xi}$. By Lemma 3.4, there exists a basis change $\underline{\psi} = \underline{\phi}\mathcal{P}$, where $\underline{\psi}$ is a symplectic basis for G with respect to \mathbf{a} , and $\mathcal{P} \in \mathrm{GL}_n(\mathbb{Z}/r\mathbb{Z})$. Thus for some subset $\{\psi_1, \dots, \psi_{2m}\} \subset \underline{\psi}$, $\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}) = b_{2i-1, 2i} = 1/d_i$, and $\mathbf{a}(\psi_k, \psi_l) = b_{kl} = 0$ otherwise, as desired. By Lemma 3.4, $(b_{ij}) = \mathcal{P}^*(a_{ij})\mathcal{P}$ is the resulting alternating 2-block-diagonal matrix for \mathbf{a} .

Let $\{\psi_1^*, \dots, \psi_{2m}^*\}$ be part of a basis for G^* dual to $\underline{\psi}$, defined by $\psi_i^*(\psi_j) = \delta_{ij}/r$, for all i , and set $y_i = \kappa_{\omega_r}(\psi_i^*) \in F^\times/F^{\times r}$. Then $\mathbf{a} = \sum_{i=1}^m \frac{r}{d_i} (\psi_{2i-1}^* \wedge \psi_{2i}^*)$. By Theorem 4.7, $\kappa^{\wedge 2}(\frac{r}{d_i} (\psi_{2i-1}^* \wedge \psi_{2i}^*)) = [(y_{2i-1}, y_{2i})_{\omega_{d_i}}]$. Thus $\boldsymbol{\xi}$ has the desired form. This is a disjoint reduced symbol presentation since the ψ_i^* are independent. Since $\underline{\psi} = \underline{\phi}\mathcal{P}$, the transpose is $\underline{\psi}^*\mathcal{P}^* = \underline{\phi}^*$, and by Kummer Theory, $\underline{y}\mathcal{P}^* = \underline{x}$. If $\mathcal{Q}^* = (\mathcal{P}^*)^{-1}$, then $\underline{y} = \underline{x}\mathcal{Q}^*$. This completes the proof. ■

The next theorem sets an upper bound for the index of a Brauer class $\boldsymbol{\xi}$ in terms of the index of any alternating form \mathbf{a} such that $\kappa^{\wedge 2}(\mathbf{a}) = \boldsymbol{\xi}$. The goal is to compute an index bound directly from an arbitrary symbol presentation of a given Brauer class. A mere bound is the best that can be expected for the general situation. For example, it follows by Remark 4.8 that the exponent and index of a Brauer class may be trivial, no matter what the exponent and index of the preimage in $\mathrm{Alt}(G_F, \mathbb{Q}/\mathbb{Z})$. On the other hand, Corollary 4.14 below shows the bound is realized over certain fields.

Theorem 4.13 “Index Bound for $\mathrm{Br}(F)$ ”. *Suppose $\boldsymbol{\xi} \in \mathrm{Br}(F)$. Let $\mathbf{a} \in \mathrm{Alt}(G_F^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$ be such that $\kappa^{\wedge 2}(\mathbf{a}) = \boldsymbol{\xi}$. Then*

$$\exp(\boldsymbol{\xi}) \mid \exp(\mathbf{a}) \quad \mathrm{ind}(\boldsymbol{\xi}) \mid \mathrm{ind}(\mathbf{a}) = \mathrm{pf}(\mathbf{a}).$$

The class $\boldsymbol{\xi}$ is represented by a central simple algebra of degree $\mathrm{pf}(\mathbf{a})$.

Proof. Since $\kappa^{\wedge 2}$ is a homomorphism, $\exp(\boldsymbol{\xi})$ divides $\exp(\mathbf{a})$. Assume the notation of Theorem 4.12 and its proof. By definition, $\mathrm{ind}(\mathbf{a}) = \prod d_i$, the degree of the algebra $\bigotimes_{i=1}^m (y_{2i-1}, y_{2i})_{\omega_{d_i}}$, which represents $\boldsymbol{\xi}$. Since the index of $\boldsymbol{\xi}$ divides the degree of any representative central simple algebra, $\mathrm{ind}(\boldsymbol{\xi})$ divides $\mathrm{ind}(\mathbf{a})$. By the index formula Theorem 3.6, $\mathrm{ind}(\mathbf{a}) = |\mathrm{pf}(\mathbf{a})|$. This completes the proof. ■

Corollary 4.14. *Let k be a field containing μ , and suppose $F = k(x_1, \dots, x_n)$, the rational function field, or $F = k((x_1)) \cdots ((x_n))$, the iterated power series field, in n variables. Let*

$$A = \bigotimes_{1 \leq i < j \leq n} (x_i, x_j)_{\omega_{r_{ij}}}^{s_{ij}}.$$

Then in the situation of Theorem 4.12 (homogeneous case), the division algebra underlying A is

$$D = \bigotimes_{i=1}^m (y_{2i-1}, y_{2i})_{\omega_{d_i}}.$$

Moreover, $\text{ind}(A) = |\text{pf}((a_{ij}))|$, where $(a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ is defined by $a_{ij} = s_{ij}/r_{ij} \in \mathbb{Q}/\mathbb{Z}$.

Proof. Since $k(x_1, \dots, x_n) \subset k((x_1)) \cdots ((x_n))$, it is enough to prove the result over $F = k((x_1)) \cdots ((x_n))$. The x_i form a pure independent subset of $F^\times / F^{\times r}$, where $r = \text{lcm}\{r_{ij}\}$. By Theorem 4.12, $A \sim D = \bigotimes_{i=1}^m (y_{2i-1}, y_{2i})_{\omega_{d_i}}$, for some numbers $m \leq \lfloor n/2 \rfloor$ and $d_i \mid r$, and some subset $\underline{y} = \{y_1, \dots, y_n\} \subset F^\times / F^{\times r}$, defined by $\underline{y} = \underline{x} \mathcal{Q}^*$, for some $\mathcal{Q}^* \in \text{GL}_n(\mathbb{Z}/r\mathbb{Z})$. Since \mathcal{Q}^* is invertible, \underline{y} is a pure independent subset of $F^\times / F^{\times r}$. Identify \underline{y} with any preimage in F^\times . Since the d_i divide r , by Kummer Theory $L = F(\{y_{2i-1}^{1/d_i}, y_{2i}^{1/d_i} : 1 \leq i \leq m\})$ has degree $\prod_{i=1}^m d_i^2$.

Let v denote the standard henselian valuation on F defined by the x_i , with lexicographically ordered value group $\Gamma_F \simeq \mathbb{Z}^n$. Since v is henselian, it has a unique extension v to L . By multiplicativity, $v(y_{2i-1}^{1/d_i}) = \frac{1}{d_i} v(y_{2i-1})$ and $v(y_{2i}^{1/d_i}) = \frac{1}{d_i} v(y_{2i})$. Since $v(\underline{x})$ gives a basis for $\Gamma_F / r\Gamma_F$, so does $v(\underline{y})$, and it follows easily that the $v(y_{2i-1}^{1/d_i}), v(y_{2i}^{1/d_i})$ are independent (mod $r\Gamma_F$), so they give a basis for $\Gamma_L / r\Gamma_F$. Since d_i divides r for all i , it then follows easily that $|\Gamma_L / \Gamma_F| = \prod d_i^2 = [L : F]$. Therefore L/F is totally ramified.

Claim. Each $(y_{2i-1}, y_{2i})_{\omega_{d_i}}$ is a division algebra. By the norm criterion [R, Corollary 30.7], it is enough to show that if y_{2i}^e is a norm from $K = F(y_{2i-1}^{1/d_i})$, then d_i divides e . Note that since K/F is totally ramified of degree d_i , $\Gamma_K = \langle \frac{1}{d_i} v(y_{2i-1}) \rangle + \Gamma_F$. Since v 's extension to K is uniquely defined, $v(y) = v(\sigma(y))$ for all $\sigma \in \text{G}_{K/F}$ and $y \in K$, hence $v(N(y)) = d_i v(y)$, and $v(N(K^\times)) = \langle v(y_{2i-1}) \rangle + d_i \Gamma_F$. Since $\langle v(y_{2i-1}) \rangle \cap \langle v(y_{2i}) \rangle \subset r\Gamma_F$, if $y_{2i}^e \in N(K^\times)$, then $v(y_{2i}^e) \in d_i \Gamma_F$, so $d_i \mid e$. This proves the claim.

Since v is henselian it has a unique extension from F to each $D_i = (y_{2i-1}, y_{2i})_{\omega_{d_i}}$; evidently $\Gamma_{D_i} = \langle \frac{1}{d_i} v(y_{2i-1}) \rangle + \langle \frac{1}{d_i} v(y_{2i}) \rangle + \Gamma_F \leq \Gamma_L$. By [TW, Extension Lemma 1.6], since $\Gamma_{D_i} \cap \Gamma_{D_j} = \Gamma_F$ for $i \neq j$, $D_i \otimes_F D_j$ is a division algebra, and $\Gamma_{D_i \otimes_F D_j} = \Gamma_{D_i} + \Gamma_{D_j}$. By induction, D is a division algebra.

The last statement is immediate by Theorem 4.13. ■

The induced map $\rho^{*\wedge 2}$ of Theorem 4.7 is computable in the most interesting cases, namely when ρ is a closed embedding, projection, or the transfer. In these cases, the induced maps on Brauer groups are well defined, and by Theorem 4.7, the computation of $\rho^{*\wedge 2}$ reduces to the computation of the Kummer transpose maps ρ^* , which was accomplished in Lemma 4.5.

Theorem 4.15. *Suppose K/F is a Galois extension with abelian Galois group $\text{G}_{K/F}$ of finite exponent. Then*

$$\begin{aligned} \text{res} : \text{Br}(F) &\twoheadrightarrow \text{Br}^\circ(F_{\text{ab}}/K) & \text{inf} : \text{Br}^\circ(K/F) &\hookrightarrow \text{Br}(F) \\ [(x, y)_{\omega_e}] &\longmapsto [(x, y)_{\omega_e}] & [(\bar{x}, \bar{y})_{\omega_e}] &\longmapsto [(\bar{x}, \bar{y})_{\omega_e}] \end{aligned}$$

Suppose $G_{K/F}$ is finite, of order s . For $e \in \mathbb{N}$, factor $s = ru$, where r is the divisor of s supported on $\text{supp}(e)$, and $(u, e) = 1$, and let v be a number such that $uv \equiv 1 \pmod{e}$. Then for all $x, y \in F_{\text{ab}}^e \cap K^\times$,

$$\begin{aligned} \text{cor} : \text{Br}^\circ(F_{\text{ab}}/K) &\longrightarrow \text{Br}(F) \\ [(x, y)_{\omega_e}] &\longmapsto v[(N_{K/F}(x), N_{K/F}(y))_{\omega_{er}}] \end{aligned}$$

Proof. The explicit descriptions of first two maps are well known. They follow immediately from the remarks preceding Theorem 2.8, which describe the inflation and restriction on alternating forms, then applying Theorem 4.7 and Lemma 4.5 to the case $\Omega = F_{\text{ab}}$. Since inflation and restriction are morphisms of cohomological functors, the induced maps on $\text{Br}^\circ(K/F)$ are indeed inflation and restriction. The latter is injective on Brauer groups. By Theorem 2.8, $\iota^{*\wedge 2} : \text{Alt}(G_{F_{\text{ab}}/F}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Alt}(G_{F_{\text{ab}}/K}, \mathbb{Q}/\mathbb{Z})$ is surjective, since $G_{F_{\text{ab}}/F}$ is torsion-free. Therefore the restriction $\text{Br}^\circ(F_{\text{ab}}/F) = \text{Br}(F) \rightarrow \text{Br}^\circ(F_{\text{ab}}/K)$ is surjective.

Suppose $x, y \in F_{\text{ab}}^e \cap K^\times$, so $[(x, y)_{\omega_e}] \in \text{Br}^\circ(F_{\text{ab}}/K)$ is a general element, by Theorem 4.7. By Theorem 2.10, the corestriction on $\text{Alt}(G_{F_{\text{ab}}/K}, \mathbb{Q}/\mathbb{Z})$ is obtained by applying $j^{*\wedge 2}$ to a choice of s -th root. Since the transfer j induces a well-defined map on Brauer groups, its effect on symbols can be computed directly using Theorem 4.7. By definition, the corestriction on alternating forms corresponds to corestriction on cohomology with coefficients in \mathbb{Q}/\mathbb{Z} , under the natural map. Since cor is a morphism of cohomological functors, the map $\varepsilon_* \cdot \omega_* : \mathbb{Q}/\mathbb{Z} \rightarrow F_{\text{ab}}^\times$ induces a commutative diagram

$$\begin{array}{ccc} \text{H}^2(G_{F_{\text{ab}}/K}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{cor}} & \text{H}^2(G_F^{\text{ab}}, \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \text{Br}(F_{\text{ab}}/K) & \xrightarrow{\text{cor}} & \text{Br}(F) \end{array}$$

Therefore, to compute the corestriction on $\text{Br}^\circ(F_{\text{ab}}/K)$, it suffices to find a preimage of $[(x, y)_{\omega_e}] \in \text{Br}(F_{\text{ab}}/K)$ in $\text{Alt}(G_{F_{\text{ab}}/K}, \mathbb{Q}/\mathbb{Z})$, take an s -th root, apply $\kappa^{\wedge 2}$, and use Lemma 4.5 to apply j^* to each parameter of the resulting symbol, as in Theorem 4.7.

Let $\phi^* = \kappa_{\omega_e}^{-1}(x)$, $\psi^* = \kappa_{\omega_e}^{-1}(y)$, $m = |\phi^*|$, $n = |\psi^*|$, $c = \text{lcm}[m, n]$, and $d = \text{gcd}(m, n)$. Note $c \mid e$. Factor $s = r'u'$, where r' is the divisor of s supported on the support of d , and $(u', d) = 1$. Let $v' : u'v' \equiv 1 \pmod{d}$. Find r' -th roots $\frac{1}{r'}\phi^*$ and $\frac{1}{r'}\psi^*$ in $G_{F_{\text{ab}}/K}^*$, which exist by the divisibility of $G_{F_{\text{ab}}/K}^*$. By (1.7), $v'(\frac{1}{r'}\phi^* \wedge \frac{1}{r'}\psi^*)$ is an s -th root of $\phi^* \wedge \psi^*$. If $x' = \kappa_{\omega_{mr'}}(\frac{1}{r'}\phi^*)$ and $y' = \kappa_{\omega_{nr'}}(\frac{1}{r'}\psi^*)$, then by definition, $\kappa^{\wedge 2}(v'(\frac{1}{r'}\phi^* \wedge \frac{1}{r'}\psi^*)) = v'[(x', y')_{\omega_{dr'}}]$, and $\kappa^{\wedge 2}(\phi^* \wedge \psi^*) = [(x', y')_{\omega_d}]$.

Replace x and $y \pmod{K^{\times er'}}$ by $x = \kappa_{\omega_{er'}}(\frac{1}{r'}\phi^*)$ and $y = \kappa_{\omega_{er'}}(\frac{1}{r'}\psi^*)$. Since they are congruent to the original x and $y \pmod{K^{\times e}}$ these new elements still determine the original element $[(x, y)_{\omega_e}]$. By (4.2), $x \equiv x'^{e/m} \pmod{F^{\times er'}}$ and $y \equiv y'^{e/n} \pmod{F^{\times er'}}$ by (4.2). Since $mn = cd$,

$$\kappa^{\wedge 2}(\frac{e}{c}(\phi^* \wedge \psi^*)) = \frac{e}{c}[(x', y')_{\omega_d}] = \frac{e^2}{cd}[(x', y')_{\omega_e}] = [(x, y)_{\omega_e}].$$

Thus $\frac{e}{c}(\phi^* \wedge \psi^*)$ is a preimage of $[(x, y)_{\omega_e}]$ in $\text{Alt}(G_{F_{\text{ab}}/K}, \mathbb{Q}/\mathbb{Z})$, and $\frac{e}{c}v'(\frac{1}{r'}\phi^* \wedge \frac{1}{r'}\psi^*)$ is an s -th root of $\frac{e}{c}(\phi^* \wedge \psi^*)$. Now compute $\kappa^{\wedge 2}(\frac{e}{c}v'(\frac{1}{r'}\phi^* \wedge \frac{1}{r'}\psi^*)) = \frac{e}{c}v'[(x', y')_{\omega_{dr'}}]$, and rewrite

$$\frac{e}{c}v'[(x', y')_{\omega_{dr'}}] = \frac{e}{c}\frac{e}{d}v'[(x', y')_{\omega_{er'}}] = v'[(x'^{e/m}, y'^{e/n})_{\omega_{er'}}] = v'[(x, y)_{\omega_{er'}}] = \frac{u'v'}{u}[(x, y)_{\omega_{er}}].$$

Note in the last equality $\frac{u'}{u} = \frac{r}{r'}$ is an integer. By Theorem 4.7, $\text{cor}([(x, y)_{\omega_e}])$ is computed by applying j^* to the parameters in $\frac{u'v'}{u}[(x, y)_{\omega_{er}}]$. Thus by Lemma 4.5,

$$\text{cor}([(x, y)_{\omega_e}]) = \frac{u'v'}{u}[(N_{K/F}(x), N_{K/F}(y))_{\omega_{er}}].$$

As $|\frac{e}{c}(\phi^* \wedge \psi^*)|$ divides d , and $u'v'/u$ is relatively prime to d , the order of $[(N_{K/F}(x), N_{K/F}(y))_{\omega_{er}}]$ divides d . Therefore $\frac{u'v'}{u}[(N_{K/F}(x), N_{K/F}(y))_{\omega_{er}}] = v[(N_{K/F}(x), N_{K/F}(y))_{\omega_{er}}]$. This completes the proof. ■

Remarks 4.16. a. In the situation of Theorem 4.15, suppose $y \in F^\times$ and $x \in F_{\text{ab}}^d \cap K^\times$. Then $[(x, y)_{\omega_d}] \in \text{Br}^\circ(F_{\text{ab}}/K)$, and

$$\begin{aligned} \text{cor}([(x, y)_{\omega_d}]) &= v[(N_{K/F}(x), y^s)_{\omega_{dr}}] = ruv[(N_{K/F}(x), y)_{\omega_{dr}}] = uv[(N_{K/F}(x), y)_{\omega_d}] \\ &= [(N_{K/F}(x), y)_{\omega_d}], \end{aligned}$$

which agrees with the well known theorem [D, Section 11, Theorem 7] (which applies whether or not K/F is abelian).

b. If G_F is abelian, then $F_{\text{ab}} = K_{\text{ab}}$, hence $\text{Br}^\circ(F_{\text{ab}}/K) = \text{Br}(K)$. This situation occurs, for example, when $F = \mathbb{C}((x_1)) \cdots ((x_n))$, the field of iterated power series. Then the restriction is surjective, and the corestriction formula applies to all of $\text{Br}(K)$.

5. BASES

Let G be a proabelian group. One of the main goals of this paper is to effectively compute the theory of the group $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ of continuous alternating forms on G , and then, when G is the Galois group over a field F containing μ , to effectively compute the homomorphism $\text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Br}(F)$ resulting from the natural isomorphism $\text{Alt}(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, \mathbb{Q}/\mathbb{Z})$.

The next two sections show how to use bases to achieve this goal when G is torsion-free. First, bases are defined for torsion-free proabelian groups G , dualized, and then extended to the groups $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Compatible bases are then defined in connection with the ground field F , where they produce a set of generators for $\text{Br}(F)$.

All of this would be greatly simplified if it could be assumed that G was a finite product of procyclic groups of some fixed type, like \mathbb{Z}_p , $\widehat{\mathbb{Z}}$, or $\mathbb{Z}/d\mathbb{Z}$. However, in order for the map to $\text{Br}(F)$ to be surjective, it must be assumed that G is the Galois group of the maximal abelian extension of F , a generally inhomogeneous, generally infinite product of various \mathbb{Z}_p 's. For this reason, a completely general proabelian G will be treated, even though this introduces a variety of technical problems that can obscure the relative simplicity of some of the concepts.

*Bases for G and G^**

Let G be a proabelian group that is either torsion-free or torsion. By the structure theory ([RZ, Theorem 4.3.3, Theorem 4.3.9]), there is a topological isomorphism $G \simeq \prod_I G_i$, where I is a set, G_i is procyclic, and $\prod_I G_i$ has the product topology. In the torsion-free case, $G_i = \mathbb{Z}_{\mathfrak{m}_i}$, for some $\mathfrak{m}_i \subset \mathcal{P}$, and in the torsion case $G_i = \mathbb{Z}/d_i\mathbb{Z}$, for some $d_i \in \mathbb{N}$ that divides the exponent. By [RZ, Lemma 4.3.7], a torsion proabelian group has bounded exponent. If $G_i \simeq \mathbb{Z}_{\mathfrak{m}_i}$, $\phi \in G_i$, and $c \in \mathbb{Z}_{\mathfrak{m}_i}$, then ϕ^c is the element whose image in each finite quotient G_i/G_i^d is $\phi_d^{c_d}$, where ϕ_d and c_d are elements of G_i/G_i^d and $\mathbb{Z}_{\mathfrak{m}_i}/d\mathbb{Z}_{\mathfrak{m}_i}$, respectively.

By Pontrjagin Duality [RZ, Lemma 2.9.4], $G \simeq \prod_I G_i$ if and only if $G^* \simeq \prod_I G_i^*$, where in the torsion-free case $G_i = \mathbb{Z}_{\mathfrak{m}_i}$ if and only if $G_i^* = \mathbb{Z}[1/\mathfrak{m}_i]/\mathbb{Z}$, and in the torsion-case, $G_i = \mathbb{Z}/d_i\mathbb{Z}$ if and only if $G_i^* = (1/d_i)\mathbb{Z}/\mathbb{Z}$. Thus a proabelian group is torsion-free if and only if its dual is divisible, and it is torsion if and only if its dual has bounded exponent.

The guiding example for the definition below arises when G is isomorphic to a finite homogeneous product of torsion-free procyclic groups, all of the same type, such as $\widehat{\mathbb{Z}}^n$. By a basis for such a group is meant a topological basis, given by the images of the canonical generators under a topological isomorphism $\widehat{\mathbb{Z}}^n \xrightarrow{\sim} G$. It is then easy to define *dual* bases for the d -torsion subgroups ${}_dG^*$, using the transpose ${}_dG^* \xrightarrow{\sim} (\frac{1}{d}\mathbb{Z}/\mathbb{Z})^n$. In fact, for practical purposes this is the only case that will be needed. For if \mathbf{a} is any alternating form on an arbitrary torsion-free proabelian G then by continuity \mathbf{a} 's radical contains all but finitely many factors in any product representation of G . Moreover, \mathbf{a} 's p -primary components are all supported on the p -primary part of G , which is a product of \mathbb{Z}_p 's. Thus the computation of matrix forms for any particular \mathbf{a} can always be reduced to the case where G is a finite homogeneous product of procyclic groups of the same type. Here is the general definition.

Definitions 5.1. a. Let G be a torsion or torsion-free proabelian group. A *topological basis* for G is the image of the standard generators $\{\mathbf{e}_i\}$ under a topological isomorphism

$$\Phi : \prod_I G_i \xrightarrow{\sim} G$$

where each G_i is procyclic, with canonical generator \mathbf{e}_i . Let $\phi_i = \Phi(\mathbf{e}_i)$, and write $\underline{\phi} = \{\phi_i\}_I$. Partition I into a disjoint union $\bigcup_J I_j$, defined for each $j \in J$ by $I_j = \{i \in I : G_i \simeq G_j\}$. Then

$$\prod_I G_i = \prod_J \prod_{I_j} G_i = \prod_J G_j^{|I_j|}.$$

Call $G_j^{|I_j|}$ a *homogeneous component*. Call a decomposition Φ and the corresponding basis $\underline{\phi}$ *homogeneous* if the G_i are all isomorphic. Call Φ and $\underline{\phi}$ *primary* if the G_i are all p -groups, for various p .

A *topological basis change* is given by a commutative diagram

$$\begin{array}{ccc} G & = & G \\ \Phi' \uparrow & & \uparrow \Phi \\ \prod_I G_i & \xrightarrow{\iota} & \prod_I G_i \end{array}$$

where Φ, Φ' , and ι are topological isomorphisms, and ι 's restriction to each homogeneous component $G_j^{|I_j|}$ is an isomorphism onto $G_j^{|I_j|}$. A basis change on G is given explicitly by $\phi'_j = \prod_i \phi_i^{p_{ij}^*}$, where in the torsion-free case, $p_{ij} \in \mathbb{Z}_{\mathfrak{m}_i}$, and in the torsion case, $p_{ij} \in \mathbb{Z}/d_i\mathbb{Z}$, where $d_i = |G_i|$. If $\theta \in G$, let $[\theta]_{\underline{\phi}}$ denote θ 's coordinate expression with respect to $\underline{\phi}$, and let $\mathcal{P} = [\iota]_{\underline{\phi}'}$ be ι 's coordinate expression with respect to $\underline{\phi}'$ and $\underline{\phi}$. Then write $\mathcal{P} : [\theta]_{\underline{\phi}'} \mapsto [\theta]_{\underline{\phi}}$ on G , and

$$\underline{\phi}' = \underline{\phi}\mathcal{P}.$$

b. Let G^* be a discrete torsion abelian group of bounded exponent dividing d . A *basis* for G^* is the preimage of the set of canonical generators $\mathbf{e}_i^* \in G_i^* = \frac{1}{d_i}\mathbb{Z}/\mathbb{Z}$, $d_i \mid d$, under an isomorphism

$$\Phi^* : G^* \xrightarrow{\sim} \prod_I G_i^*$$

Let $\phi_i^* = (\Phi^*)^{-1}(\mathbf{e}_i^*)$, and write $\underline{\phi}^* = \{\phi_i^*\}$. Decompose I into a disjoint union $I = \bigcup_J I_J$ as in (a), with $I_J = \{i \in I : G_i^* \simeq G_J^*\}$. Then $\prod_I G_i^* = \prod_J (\prod_{I_J} G_i^*)$. Say G^* is *homogeneous* if the G_i^* are isomorphic for all i . A *basis change* from $\underline{\phi}^{*'} to $\underline{\phi}^*$ is given by a commutative diagram$

$$\begin{array}{ccc} G^* & = & G^* \\ \Phi'^* \downarrow & & \downarrow \Phi^* \\ \prod_I G_i^* & \xleftarrow{\iota^*} & \prod_I G_i^* \end{array}$$

where Φ^*, Φ'^* , and ι^* are isomorphisms, and ι^* maps each $\prod_{I_J} G_i^*$ onto $\prod_{I_J} G_i^*$.

A basis change is given explicitly by $\phi_j^{*'} = \sum_i p_{ij}^* \phi_i^*$ for elements $p_{ij}^* \in \mathbb{Z}/d_i\mathbb{Z}$, finitely many of which are nonzero. If $d_i \in \mathbb{N}(\mathfrak{m}_i)$, then $\mathbb{Z}/d_i\mathbb{Z}$ is a quotient of $\mathbb{Z}_{\mathfrak{m}_i}$, and the p_{ij}^* may be represented in $\mathbb{Z}_{\mathfrak{m}_i}$. Let $\mathcal{P}^* = [\iota^*]_{\underline{\phi}^{*'}}$ be ι^* 's coordinate expression with respect to $\underline{\phi}^*$ and $\underline{\phi}^{*'}$, then for $\theta^* \in G^*$, $\mathcal{P}^* : [\theta^*]_{\underline{\phi}^{*'}} \mapsto [\theta^*]_{\underline{\phi}^*}$, and

$$\underline{\phi}^{*'}\mathcal{P}^* = \underline{\phi}^*.$$

c. Let G be a torsion or torsion-free proabelian group. Suppose $d \in \mathbb{N}$ divides $|G|$. If $\underline{\phi}$ is a basis for G , then the *d-dual basis* ${}_d\underline{\phi}^*$ for the d -torsion subgroup ${}_dG^*$ is defined by

$${}_d\phi_i^*(\phi_j) = \delta_{ij}/d_i \in \mathbb{Q}/\mathbb{Z},$$

where $d_i = (d, |\phi_i|) = |{}_d\phi_i^*|$. A basis change ${}_d\underline{\phi}^{*'}\mathcal{P}^* = {}_d\underline{\phi}^*$ on ${}_dG^*$ is the *transpose* of a topological basis change $\underline{\phi}' = \underline{\phi}\mathcal{P}$ on G if ${}_d\underline{\phi}^{*'}$ and ${}_d\underline{\phi}^*$ are dual to $\underline{\phi}'$ and $\underline{\phi}$, respectively. Then ${}_d\phi_j^* = \sum_i p_{ij}^*({}_d\phi_i^*)$, and $p_{ij}^* \equiv p_{ji} \pmod{d_i}$. For compute

$$p_{ji}/d_i = {}_d\phi_j^*\left(\prod_{k \in I} \phi_k^{p_{ki}^*}\right) = \left(\sum_{k \in I} p_{kj}^*({}_d\phi_k^*)\right)(\phi_i) = p_{ij}^*/d_i.$$

Since this holds for all d dividing $|G|$, in the torsion-free case the coefficients p_{ij}^* can be taken to lie in $\mathbb{Z}_{\mathfrak{m}_i}$, and then $p_{ij}^* = p_{ji} \in \mathbb{Z}_{\mathfrak{m}_i}$. In this case, write

$$\underline{\phi}^{*'}\mathcal{P}^* = \underline{\phi}^*$$

as shorthand for the family of basis changes $\{\underline{d}\phi'^{*}\mathcal{P}^* = \underline{d}\phi^*\}$.

Example 5.2. If $G = \widehat{\mathbb{Z}}^n$, the free proabelian group of rank n , then the standard basis element is $\underline{\phi} = \{1_i\}_{i=1}^n$, where 1_i is the i -th summand's canonical generator. A basis change is represented by a matrix $\mathcal{P} = (p_{ij}) \in \text{GL}_n(\widehat{\mathbb{Z}})$. The new basis $\underline{\phi}' = \underline{\phi}\mathcal{P}$ is given by $\phi'_j = \prod_{i=1}^n \phi_i^{p_{ij}}$. The dual of G is $G^* = (\mathbb{Q}/\mathbb{Z})^n$. The basis for ${}_dG^*$ dual to $\underline{\phi}$ is $\underline{d}\phi^* = \{(1/d)_i\}_{i=1}^n$, where $(1/d)_i$ is the i -th summand's canonical generator. Set $\underline{\phi}^* = \{d\underline{\phi}^*\}$. If $\mathcal{P} \in \text{GL}_n(\widehat{\mathbb{Z}})$ is a basis change for G , $\mathcal{P}^* \in \text{GL}_n(\widehat{\mathbb{Z}})$ is the matrix transpose of \mathcal{P} , and the transpose basis change is $\underline{\phi}'^*\mathcal{P}^* = \underline{\phi}^*$, so ${}_d\phi_j^* = \sum_{i=1}^n p_{ij}^* d\phi_i^*$.

By the above definitions, a given topological basis $\underline{\phi}$ for a torsion-free proabelian group G determines a system $\underline{\phi}^* = \{d\underline{\phi}^*\}$ of dual bases for the various torsion groups ${}_dG^*$. The dual bases for each homogeneous component are coherent in the following sense.

Lemma 5.3. *Suppose $G = \prod_I G_i$ is homogeneous, with each $G_i \simeq \mathbb{Z}_{\mathfrak{m}}$, for some $\mathfrak{m} \subset \mathcal{P}$. Let $\underline{\phi}$ be a homogeneous topological basis. Then for all $d, e \in \mathbb{N}$ such that $d \mid e$ and $e/d \in \mathbb{N}(\mathfrak{m})$, ${}_d\underline{\phi}^* = (e/d)_e\underline{\phi}^*$. That is, ${}_d\phi_i^* = (e/d)_e\phi_i^*$ for all $i \in I$. Conversely, suppose $G^* = \prod_I G_i^*$ is homogeneous, with each $G_i^* \simeq \mathbb{Z}[1/\mathfrak{m}]/\mathbb{Z}$. Let $\underline{\phi}^* = \{d\underline{\phi}^* : d \in \mathbb{N}(\mathfrak{m})\}$ be a family of d -bases for G^* such that ${}_d\underline{\phi}^* = (e/d)_e\underline{\phi}^*$ whenever $d \mid e$ and $e \in \mathbb{N}(\mathfrak{m})$. Then there exists a topological basis $\underline{\phi}$ for $G = \prod_I G_i$ that is dual to $\{d\underline{\phi}^*\}$, meaning each ${}_d\underline{\phi}^*$ is dual to $\underline{\phi}$.*

Proof. For the first statement, let $e_i = (e, |\phi_i|)$ and $d_i = (d, |\phi_i|)$, then $e/d = e_i/d_i$ since $e/d \in \mathbb{N}(\mathfrak{m})$, so ${}_d\phi_i^*(\phi_j) = \delta_{ij}/d_i = (e/d)(\delta_{ij}/e_i) = (e/d)_e\phi_i^*(\phi_j)$.

For the converse, let G be the (proabelian) dual of G^* , then $G = \prod_I G_i$, where G_i is the procyclic dual of G_i^* . Let ${}_d\underline{\phi}$ be the dual of ${}_d\underline{\phi}^*$ in G/G^d , for all d . If $d \mid e$ and $e \in \mathbb{N}(\mathfrak{m})$, the dual of ${}_e\underline{\phi}$ in ${}_dG^*$ is ${}_d\underline{\phi}^*$, since

$${}_d\phi_i^*({}_e\phi_j) = ((e/d)_e\phi_i^*)({}_e\phi_j) = (e/d)(\delta_{ij}/e_i) = \delta_{ij}/d_i.$$

Therefore the image of ${}_e\underline{\phi}$ in G/G^d is ${}_d\underline{\phi}$, and $\{d\underline{\phi} : d \in \mathbb{N}(\mathfrak{m})\}$ is a projective system of sets. Since G is \mathfrak{m} -homogeneous, $G \simeq \varprojlim_d G/G^d$, where the limit is taken over $d \in \mathbb{N}(\mathfrak{m})$. Let

$$\underline{\phi} = \varprojlim_{d \in \mathbb{N}(\mathfrak{m})} ({}_d\underline{\phi}) \subset G,$$

so that $\phi_i = \varprojlim_d ({}_d\phi_i)$. Each $\phi_i \in \underline{\phi}$ generates G_i topologically, so it is clear that $\underline{\phi}$ is a topological basis for G . Moreover, ${}_d\phi_i^*(\phi_j) = \delta_{ij}/d_i$, so $\underline{\phi}$ is dual to $\underline{d}\phi^*$. ■

Bases for $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$, and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$

By Theorem 2.3 and Theorem 2.4, a (topological) basis $\underline{\phi} = \{\phi_i\}_I$ for a torsion or torsion-free proabelian group G determines compatible dual d -bases $\{d\underline{\phi}^*\}$, and from them elements ${}_m\phi_i^* \otimes_n \phi_j^*$ and ${}_m\phi_i^* \wedge_n \phi_j^*$, for $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, respectively. If d is such that $d_{ij} := \gcd(d, |\phi_i|, |\phi_j|) = \gcd(m, n, |\phi_i|, |\phi_j|)$, then by (1.3), ${}_m\phi_i^* \otimes_n \phi_j^* = {}_d\phi_i^* \otimes_d \phi_j^*$, and ${}_m\phi_i^* \wedge_n \phi_j^* =$

${}_d\phi_i^* \wedge {}_d\phi_j^*$. These elements have order d_{ij} , except for ${}_d\phi_i^* \wedge {}_d\phi_i^*$, which is trivial. By (2.2), their action is described explicitly by

$${}_d\phi_i^* \otimes {}_d\phi_j^* \left(\prod_{k \in I} \phi_k^{a_k}, \prod_{k \in I} \phi_k^{b_k} \right) = \frac{a_i b_j}{d_{ij}} \in \mathbb{Q}/\mathbb{Z},$$

and

$${}_d\phi_i^* \wedge {}_d\phi_j^* \left(\prod_{k \in I} \phi_k^{a_k}, \prod_{k \in I} \phi_k^{b_k} \right) = \frac{a_i b_j - a_j b_i}{d_{ij}} \in \mathbb{Q}/\mathbb{Z}.$$

To control notation, set

$$\begin{aligned} (\phi_i^* \otimes \phi_j^*)_d &:= m \phi_i^* \otimes_n \phi_j^* \\ (\phi_i^* \wedge \phi_j^*)_d &:= m \phi_i^* \wedge_n \phi_j^*, \end{aligned}$$

where for any m and n , d is such that $\gcd(d, |\phi_i|, |\phi_j|) = \gcd(m, n, |\phi_i|, |\phi_j|)$. Thus $|(\phi_i^* \wedge \phi_j^*)_d| = \gcd(d, |\phi_i|, |\phi_j|)$.

Proposition 5.4. *Suppose G is a torsion-free proabelian group, with topological basis $\underline{\phi} = \{\phi_i\}_{i \in I}$, indexed by a linearly ordered set I . Then*

$$\begin{aligned} \underline{d}\phi^{*\otimes 2} &= \{(\phi_i^* \otimes \phi_j^*)_d : i, j \in I \text{ and } (\phi_i^* \otimes \phi_j^*)_d \neq 0\} \\ [\underline{d}\phi^{*\otimes 2}] &= \{[(\phi_i^* \otimes \phi_j^*)_d] : i < j \in I \text{ and } [(\phi_i^* \otimes \phi_j^*)_d] \neq 0\} \\ \underline{d}\phi^{*\wedge 2} &= \{(\phi_i^* \wedge \phi_j^*)_d : i < j \in I \text{ and } (\phi_i^* \wedge \phi_j^*)_d \neq 0\} \end{aligned}$$

are bases for ${}_d\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, ${}_d\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$ and ${}_d\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, respectively.

Proof. By Theorem 2.3, $\underline{d}\phi^{*\otimes 2}$ generates $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, and since $(\phi_i^* \wedge \phi_j^*)_d = -(\phi_j^* \wedge \phi_i^*)_d$, $\underline{d}\phi^{*\wedge 2}$ generates $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. By Theorem 2.4, $[\underline{d}\phi^{*\otimes 2}]$ generates $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$. In fact, check directly that $(\phi_i^* \otimes \phi_j^*)_d + (\phi_j^* \otimes \phi_i^*)_d$ equals the coboundary ∂h of the 1-cochain $h \in C^1(G, \mathbb{Q}/\mathbb{Z})$ given by $h(\prod_k \phi_k^{a_k}) = -a_i a_j / d_{ij}$. Therefore $[(\phi_i^* \otimes \phi_j^*)_d] = -[(\phi_j^* \otimes \phi_i^*)_d]$ in $\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$.

If $\mathbf{a} = \sum n_{ij} (\phi_i^* \otimes \phi_j^*)_d = 0$, then $\mathbf{a}(\phi_i, \phi_j) = n_{ij} / d_{ij} = 0$, where $d_{ij} = \gcd(d, |\phi_i|, |\phi_j|)$, for all $i, j \in I$. Therefore $\underline{d}\phi^{*\otimes 2}$ is an independent set, hence a basis for ${}_d\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$. Similarly, $\underline{d}\phi^{*\wedge 2}$ is a basis for ${}_d\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, and by Theorem 2.4, $[\underline{d}\phi^{*\otimes 2}]$ is a basis for ${}_d\text{H}^2(G, \mathbb{Q}/\mathbb{Z})$. ■

Definition 5.5. Suppose $\underline{\phi} = \{\phi_i\}_I$ is a basis for G , indexed by a linearly ordered set I , $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Set $b_{ij} = \mathbf{b}(\phi_i, \phi_j)$ and $a_{ij} = \mathbf{a}(\phi_i, \phi_j)$. The *coordinate* (or *matrix*) *expressions* $[\mathbf{b}]_{\underline{\phi}^{*\otimes 2}}$ and $[\mathbf{a}]_{\underline{\phi}^{*\wedge 2}}$, or just $[\mathbf{b}]_{\underline{\phi}}$ and $[\mathbf{a}]_{\underline{\phi}}$ if the meaning is clear, are the $I \times I$ matrices (b_{ij}) and (a_{ij}) . Note almost all of the b_{ij} and a_{ij} are zero, and if $b_{ij} = n_{ij} / d_{ij}$ and $a_{ij} = m_{ij} / d_{ij}$ in \mathbb{Q}/\mathbb{Z} , then $\mathbf{b} = \sum_{i,j} n_{ij} (\phi_i^* \otimes \phi_j^*)_d$ and $\mathbf{a} = \sum_{i < j} m_{ij} (\phi_i^* \wedge \phi_j^*)_d$, where $d_{ij} = \gcd(d, |\phi_i|, |\phi_j|)$. Let

$$\text{M}_{|\underline{\phi}|}(\mathbb{Q}/\mathbb{Z}), \quad \text{Alt}_{|\underline{\phi}|}(\mathbb{Q}/\mathbb{Z})$$

denote the group of all such matrices, with pointwise addition.

Remark 5.6. The coordinate (matrix) expression is a matrix representative, in the sense of Definition 3.2. Moreover, every matrix representative is a matrix expression, for some basis. For suppose (b_{ij}) is the matrix for \mathbf{b} , determined by a basis $\bar{\underline{\phi}}$ for $G/\text{rad}_{\mathbf{b}}(G)$. It will be shown in Lemma 5.12, below, that $\bar{\underline{\phi}}$ extends to a basis $\underline{\phi}$ for G , hence $(b_{ij}) = [\mathbf{b}]_{\underline{\phi}}$.

Proposition 5.7 “Basis Change on Bilinear and Alternating Forms”. *Suppose G is a torsion-free proabelian group with topological basis $\underline{\phi}$. Suppose $\underline{\phi}' = \underline{\phi}\mathcal{P}$ is a topological basis change on G , defined explicitly by $\phi'_j = \prod_I \phi_i^{p_{ij}}$, $p_{ij} \in \mathbb{Z}_{m_i}$. Then the transpose basis changes*

$$({}_d\underline{\phi}'^{*\otimes 2})\mathcal{P}^{*\otimes 2} = {}_d\underline{\phi}^{*\otimes 2}, \quad ({}_d\underline{\phi}'^{*\wedge 2})\mathcal{P}^{*\wedge 2} = {}_d\underline{\phi}^{*\wedge 2}$$

on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ are defined explicitly by the finite sums

$$(\phi_i^* \otimes \phi_j^*)_d = \sum_{k,l \in I} p_{ki}^* p_{lj}^* (\phi_k'^* \otimes \phi_l'^*)_d, \quad (\phi_i^* \wedge \phi_j^*)_d = \sum_{k < l} p_{kl,ij}^* (\phi_k'^* \wedge \phi_l'^*)_d,$$

where $p_{kl,ij}^* = p_{ki}^* p_{lj}^* - p_{kj}^* p_{li}^*$, and $p_{ij}^* = p_{ji} \in \mathbb{Z}_{m_i}$. Thus if $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ have matrix expressions $[\mathbf{b}]_{\underline{\phi}} = (b_{ij})$ and $[\mathbf{a}]_{\underline{\phi}} = (a_{ij})$ with respect to $\underline{\phi}$ and $[\mathbf{b}]_{\underline{\phi}'} = (b'_{ij})$ and $[\mathbf{a}]_{\underline{\phi}'} = (a'_{ij})$ with respect to $\underline{\phi}'$, then $[\mathbf{b}]_{\underline{\phi}'} = \mathcal{P}^{*\otimes 2}([\mathbf{b}]_{\underline{\phi}})$ and $[\mathbf{a}]_{\underline{\phi}'} = \mathcal{P}^{*\wedge 2}([\mathbf{a}]_{\underline{\phi}})$, and $b'_{ij} = \sum_{k,l} p_{ik}^* p_{jl}^* b_{kl}$ and $a'_{ij} = \sum_{k < l} p_{ij,kl}^* a_{kl}$.

Proof. Compute $(\phi_i^* \otimes \phi_j^*)_d(\phi'_k, \phi'_l) = (\phi_i^* \otimes \phi_j^*)_d(\prod_i \phi_i^{p_{ik}}, \prod_j \phi_j^{p_{jl}}) = p_{ik} p_{jl} / d_{ij}$. Since a basis change stabilizes G 's homogeneous components, $d_{ij} = d_{kl}$ whenever p_{ik} and p_{jl} are nonzero, therefore $(\phi_i^* \otimes \phi_j^*)_d(\phi'_k, \phi'_l) = p_{ki}^* p_{lj}^* (\phi_k'^* \otimes \phi_l'^*)_d(\phi'_k, \phi'_l)$. It follows that

$$(\phi_i^* \otimes \phi_j^*)_d = \sum_{k,l} p_{ki}^* p_{lj}^* (\phi_k'^* \otimes \phi_l'^*)_d,$$

as desired. Similarly, in $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$,

$$(\phi_i^* \wedge \phi_j^*)_d(\phi'_k, \phi'_l) = (p_{ik} p_{jl} - p_{jk} p_{il})(\phi_k'^* \wedge \phi_l'^*)_d(\phi'_k, \phi'_l).$$

The explicit computation of matrix expressions on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ is immediately obtained as usual by setting $b_{ij} = n_{ij}/d_{ij}$ for all i, j , substituting the basis change formula into the expression $\mathbf{b} = \sum_{k,l} n_{kl} (\phi_k^* \otimes \phi_l^*)_d$, and then retrieving the coefficients of each $(\phi_i^* \otimes \phi_j^*)_d$. A similar computation yields the result for $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. ■

Remark 5.8. Here is how to compute the basis changes directly, using (1.3). On $\tilde{\otimes}^2 G^*$, $(\phi_i^* \otimes \phi_j^*)_d = {}_d\phi_i^* \otimes (\sum_l p_{lj}^* ({}_d\phi_l'^*))$. Since $|p_{lj}^* ({}_d\phi_l'^*)|$ and $|{}_d\phi_l'^*|$ divide $|{}_d\phi_i^*|$, by (1.3) this is $\sum_l {}_d\phi_i^* \otimes p_{lj}^* ({}_d\phi_l'^*) = \sum_l p_{lj}^* (\phi_i^* \otimes \phi_l'^*)_d$. Expanding ${}_d\phi_i^*$ in the same way yields

$$\sum_l p_{lj}^* \left(\left(\sum_k p_{ki}^* ({}_d\phi_k'^*) \right) \otimes {}_d\phi_l'^* \right) = \sum_{k,l} p_{ki}^* p_{lj}^* (\phi_k'^* \otimes \phi_l'^*)_d,$$

as desired. A similar computation establishes the result for $\tilde{\wedge}^2 G^*$. By Theorem 2.3, this produces the basis changes on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$.

Example 5.9. Suppose $G = \widehat{\mathbb{Z}}^n$, with the standard basis $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$. The coordinate expressions of $\mathbf{b} \in \text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ are matrices $(b_{ij}) \in M_n(\mathbb{Q}/\mathbb{Z})$ and $(a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$, where $b_{ij} = \mathbf{b}(\phi_i, \phi_j)$ and $a_{ij} = \mathbf{a}(\phi_i, \phi_j)$. These matrices behave as bilinear forms on the coordinate expressions of elements of G in the usual way. By Example 5.2, a basis change $\underline{\phi}' = \underline{\phi}\mathcal{P}$ on G is represented by a matrix $\mathcal{P} = (p_{ij}) \in \text{GL}_n(\widehat{\mathbb{Z}})$, so that $\phi'_j = \prod_i \phi_i^{p_{ij}}$. Let

$\mathcal{P}^* = (p_{ij}^*)$ be the transpose of \mathcal{P} , so $p_{ij}^* = p_{ji}$. By Proposition 5.7, $(b'_{ij}) = [\mathbf{b}]_{\underline{\phi}'} = \mathcal{P}^{*\otimes 2}([\mathbf{b}]_{\underline{\phi}})$, and $(a'_{ij}) = [\mathbf{a}]_{\underline{\phi}'} = \mathcal{P}^{*\wedge 2}([\mathbf{a}]_{\underline{\phi}})$, with

$$b'_{ij} = \sum_{1 \leq k, l \leq n} p_{ki}^* p_{lj}^* b_{kl}, \quad a'_{ij} = \sum_{1 \leq k < l \leq n} p_{ij,kl}^* a_{kl},$$

where $p_{ij,kl}^* = p_{ik}^* p_{jl}^* - p_{il}^* p_{jk}^*$. Thus the matrix for \mathbf{b} and for \mathbf{a} in the new basis are computed by congruence transformation:

$$[\mathbf{b}]_{\underline{\phi}'} = \mathcal{P}^*([\mathbf{b}]_{\underline{\phi}})\mathcal{P}, \quad [\mathbf{a}]_{\underline{\phi}'} = \mathcal{P}^*([\mathbf{a}]_{\underline{\phi}})\mathcal{P}.$$

Alternatively, represent the matrix $(b_{ij}) = [\mathbf{b}]_{\underline{\phi}}$ by the n^2 -tuple $(b_{11}, b_{12}, \dots, b_{nn}) \in (\mathbb{Q}/\mathbb{Z})^{n^2}$, ordered lexicographically. Then $\mathcal{P}^{*\otimes 2} \in \mathrm{GL}_{n^2}(\widehat{\mathbb{Z}})$ is the matrix whose (ij, kl) -entry is $p_{ik}^* p_{jl}^*$, and whose action on n^2 -tuples is the usual one. Similarly, represent the matrix for $\mathbf{a} \in \mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z})$ by the $\binom{n}{2}$ -tuple $(a_{12}, a_{13}, \dots, a_{n-1n}) \in (\mathbb{Q}/\mathbb{Z})^{\binom{n}{2}}$, and $\mathcal{P}^{*\wedge 2} \in \mathrm{GL}_{\binom{n}{2}}(\widehat{\mathbb{Z}})$ is then the matrix whose (ij, kl) -entry is $p_{ij,kl}^*$, and whose action on $\binom{n}{2}$ -tuples is the usual one.

Proposition 5.10. *Suppose G is a torsion-free proabelian group, with topological basis $\underline{\phi} = \{\phi_i\}_{i \in I}$, indexed by a linearly ordered set I . Then*

$$\mathrm{Bil}(G, \mathbb{Q}/\mathbb{Z}) \simeq \prod_{i,j \in I} G_{ij}^*, \quad \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z}) \simeq \prod_{i < j} G_{ij}^*$$

where, if $\mathfrak{m}_k = \mathrm{supp}(|\phi_k|)$ for $k \in I$, G_{ij}^* is the divisible group $\mathbb{Z}[\frac{1}{\mathfrak{m}_i \cap \mathfrak{m}_j}]/\mathbb{Z}$.

Proof. By Theorem 2.4, $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Since $\underline{\phi}$ is a basis, ${}_d\underline{\phi}^{*\otimes 2}$ and ${}_d\underline{\phi}^{*\wedge 2}$ are bases for ${}_d\mathrm{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and ${}_d\mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z})$ by Proposition 5.4, and $|(\phi_i^* \otimes \phi_j^*)_d| = |(\phi_i^* \wedge \phi_j^*)_d| = d_{ij}$, where $d_{ij} = \mathrm{gcd}(d, |\phi_i|, |\phi_j|)$. Thus if

$${}_dG_{ij}^* = \frac{1}{d_{ij}}\mathbb{Z}/\mathbb{Z} = {}_d(\mathbb{Z}[\frac{1}{\mathfrak{m}_i \cap \mathfrak{m}_j}]/\mathbb{Z}),$$

then ${}_d\mathrm{Bil}(G, \mathbb{Q}/\mathbb{Z}) \simeq \prod_{i,j} {}_dG_{ij}^*$, and ${}_d\mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z}) \simeq \prod_{i < j} {}_dG_{ij}^*$. Since G_{ij}^* , $\mathrm{Bil}(G, \mathbb{Q}/\mathbb{Z})$, and $\mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z})$ are the union of their torsion subgroups, the result follows. ■

Example 5.11. If $G = \widehat{\mathbb{Z}}^n$, then $G_{ij}^* \simeq \mathbb{Q}/\mathbb{Z}$, so

$$\mathrm{Bil}(G, \mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^{n^2} \simeq \mathrm{M}_n(\mathbb{Q}/\mathbb{Z}), \quad \mathrm{Alt}(G, \mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^{\binom{n}{2}} \simeq \mathrm{Alt}_n(\mathbb{Q}/\mathbb{Z}).$$

Lemma 5.12. *Suppose G is a torsion-free proabelian group, and $\pi : G \rightarrow \overline{G}$ is a projection onto a torsion proabelian group \overline{G} . Then a topological basis $\overline{\underline{\phi}}$ for \overline{G} lifts to a topological basis $\underline{\phi}$ for G .*

Proof. Since the proabelian group \overline{G} is torsion, it is a product of finite cyclic groups, and $\exp(\overline{G}) = r$ is finite. Let $\overline{\underline{\phi}} = \{\overline{\phi}_i\}_I$ be a topological basis, and let $\overline{\underline{\phi}}^*$ be its dual, a basis for \overline{G}^* . View \overline{G}^* as a subgroup of G^* via inflation. Let H^* be a divisible hull of \overline{G}^* in G^* . This exists by [F, Theorem 24.4]. *Claim:* there exists a system of bases $\underline{\phi}^* = \{{}_d\underline{\phi}^*\}$ for H^* extending $\overline{\underline{\phi}}^*$, that

is coherent in the sense of Lemma 5.3. Since H^* is a direct sum of its homogeneous components, it suffices to assume H^* is homogeneous. If d is such that $r \mid d$ and $\text{supp}(d) = \text{supp}(r)$, then since H^* is divisible there exist elements ${}_d\phi_i^*$, each of order d , extending the basis elements $\bar{\phi}_i^*$, for each $i \in I$. These elements are easily seen to constitute a basis for ${}_dH^*$, and the claim follows by induction on d . Since H^* is divisible, it is a summand of G^* by [K, Theorem 2]; let $\underline{\phi}^*$ be an extension to all of G^* . Its dual $\underline{\phi}$ is a topological basis for G extending $\bar{\phi}$. This completes the proof. ■

The next theorem show that there is a standard form for any alternating form \mathbf{a} on a torsion-free proabelian group G . This standard form is obtained by constructing a symplectic basis for G , essentially using the standard algorithm. When G is homogeneous, for example when $G \simeq \widehat{\mathbb{Z}}^n$, or $G \simeq \mathbb{Z}_p^n$, this symplectic basis may be obtained from any given basis via basis change. The resulting 2-block-diagonal matrix representing \mathbf{a} is then obtained from the original alternating matrix by congruence transformation. This is the crucial case since, as mentioned at the beginning of Section 5, any alternating form \mathbf{a} over any torsion-free proabelian G may be decomposed into a sum of its p -primary components, and each p -primary component is supported on the p -primary part of G , a (homogeneous) product of \mathbb{Z}_p 's. Moreover, since \mathbf{a} is continuous, its p -primary part is supported on a finite product of \mathbb{Z}_p 's.

Theorem 5.13 “Symplectic Basis III”. *Suppose G is a torsion-free proabelian group, and $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$. Then G has a basis $\underline{\psi} = \{\psi_i\}_I$ that is symplectic with respect to \mathbf{a} , so that for some subset $\{\psi_1, \dots, \psi_{2m}\} \subset \underline{\psi}$, along with uniquely determined numbers $d_i : d_i \mid d_{i-1}$,*

$$\mathbf{a}(\psi_{2i-1}, \psi_{2i}) = -\mathbf{a}(\psi_{2i}, \psi_{2i-1}) = 1/d_i, \quad 1 \leq i \leq m,$$

and $\mathbf{a}(\psi_k, \psi_l) = 0$ otherwise. Thus $[\mathbf{a}]_{\underline{\psi}}$ is an alternating 2-block-diagonal matrix. In basis notation,

$$\mathbf{a} = \sum_{i=1}^m (\psi_{2i-1}^* \wedge \psi_{2i}^*)_{d_i}.$$

If G has a homogeneous basis $\underline{\phi}$, then $\underline{\psi}$ is obtainable in the standard way by a basis change $\underline{\psi} = \underline{\phi}\mathcal{P}$, and $[\mathbf{a}]_{\underline{\psi}} = \mathcal{P}^{*\wedge 2}[\mathbf{a}]_{\underline{\phi}}$ is the resulting alternating 2-block-diagonal matrix for \mathbf{a} with respect to $\underline{\psi}$.

Proof. Let $r = \exp(\mathbf{a})$, then $\mathbf{a} \in \text{Alt}(G/G^r, \mathbb{Q}/\mathbb{Z})$. Since G is torsion-free, G/G^r has homogeneous primary components. By continuity, \mathbf{a} is supported on a finite summand of G/G^r . Therefore it may be assumed that G/G^r is finite, and Lemma 3.8 applies. By Lemma 5.12, the resulting symplectic basis on G/G^r lifts to G . This proves the first part.

Suppose G is homogeneous, with $G \simeq \mathbb{Z}_{\mathfrak{m}}^c$ for some $\mathfrak{m} \subset \mathcal{P}$ and cardinal number c . Since \mathbf{a} is continuous, it may be assumed that c is finite. Set $r = \exp(\mathbf{a})$. Since $r \in \mathbb{N}(\mathfrak{m})$, $G/G^r \simeq (\mathbb{Z}/r\mathbb{Z})^c$, and Lemma 3.4 applies. Let $\bar{\phi}$ be the image of $\underline{\phi}$ in G/G^r , and let $\bar{\psi} = \bar{\phi}\mathcal{P}_r$, where $\mathcal{P}_r \in \text{GL}_c(\mathbb{Z}/r\mathbb{Z})$, be a symplectic basis for G/G^r with respect to \mathbf{a} . Extend $\bar{\psi}$ to the $\text{supp}(r)$ -primary part of G using Lemma 5.12. Let $\mathfrak{m}' = \text{supp}(r)$. By the proof of Lemma 5.12, the extended set is a homogeneous basis for the \mathfrak{m}' -primary summand of G . Let $\underline{\psi}$ be the homogeneous basis for G

obtained by multiplying this basis “diagonally” with a homogeneous basis for the $\mathfrak{m} \setminus \mathfrak{m}'$ -primary part of G . That is, if G_i is a procyclic component of the homogeneous decomposition of G , ϕ_i is a basis element for the $G_i(\mathfrak{m}')$, and ϕ'_i is a basis element for $G_i(\mathfrak{m} \setminus \mathfrak{m}')$, then let $\psi_i = \phi_i \phi'_i$. Then $\underline{\psi} = \underline{\phi} \mathcal{P}$ for some basis change \mathcal{P} , extending the basis change $\bar{\psi} = \bar{\phi} \mathcal{P}_r$ in G/G' , and $\underline{\psi}$ is symplectic with respect to \mathbf{a} . The last statement follows by Proposition 5.7. ■

Bases in F^\times

To transport the basis computations from $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ to the Brauer group $\text{Br}(F)$, some notion of basis must be defined for F^\times which is dual to the notion of basis for the torsion-free group G . These “bases” for F^\times will be constructed from the Kummer map ${}_d\kappa$ of (4.1), between d -torsion subgroups of G^* and exponent- d quotients $F^\times / F^{\times d}$.

As usual, assume F is a field of characteristic 0 that contains the roots of unity μ . Let $\omega : \mu \rightarrow \mathbb{Q}/\mathbb{Z}$ be a fixed identification. Let $\underline{\phi} = \{\phi_i\}_I$ be a (topological) basis for a *torsion-free* abelian Galois group $G_{\Omega/F}$, and let $\underline{\phi}^*$ stand for the family of dual bases ${}_d\underline{\phi}^* = \{{}_d\phi_i^*\}_I$ for ${}_dG_{\Omega/F}^*$, for each $d \in \mathbb{N}$. The choice of ω determines a corresponding family of Kummer isomorphisms $\kappa_{\omega_d} : {}_dG_{\Omega/F}^* \rightarrow \Omega^d \cap F^\times / F^{\times d}$.

Definition 5.14. For all $d \in \mathbb{N}$, define the *Kummer dual* of the basis $\underline{\phi}$ to be the basis ${}_d\underline{x} = \kappa_{\omega_d}({}_d\underline{\phi}^*) \subset (\Omega^d \cap F^\times) / F^{\times d}$ dual to $\underline{\phi}$ via Kummer Theory and the choice ω . By Definition 5.1c, $\langle \phi_i, {}_d x_j \rangle_d = \omega_{d_j}^{\delta_{ij}}$, for all $i, j \in I$, where $d_j = (d, |\phi_j|) = |{}_d\phi_j^*|$.

By Definition 5.1c, the Kummer transpose of a topological basis change $\underline{\phi}' = \underline{\phi} \mathcal{P}$, where $\mathcal{P} = (p_{ij})$, is given by ${}_d\underline{x}' \mathcal{P}^* = {}_d\underline{x}$. Explicitly,

$${}_d x_j = \prod_i ({}_d x'_i)^{p_{ij}^*},$$

where $p_{ij}^* = p_{ji}$ is interpreted (mod $|{}_d x_i|$). Note that for fixed j , almost all of the p_{ij}^* are zero, so that the product is finite. Moreover, if $p_{ij}^* \neq 0$, then in the notation of Definition 5.1a, i and j are in the same homogeneous component $G_j^{|\phi_j|}$ of G , so that by Definition 5.1c, p_{ij}^* may be taken to lie in $\mathbb{Z}_{\mathfrak{m}_i}$.

The following lemma is a Kummer Theory version of Lemma 5.3.

Lemma 5.15. *Suppose $G_{\Omega/F}$ is homogeneous, with $G_{\Omega/F} \simeq \mathbb{Z}_{\mathfrak{m}}^c$ for some $\mathfrak{m} \subset \mathcal{P}$ and some cardinal number c . Let $\underline{\phi}$ be a homogeneous basis. Then for all $d, e \in \mathbb{N} : d \mid e$ and $e/d \in \mathbb{N}(\mathfrak{m})$, ${}_d\underline{x}$ is the image of ${}_e\underline{x}$ under the natural projection map $F^\times / F^{\times e} \rightarrow F^\times / F^{\times d}$. Conversely, suppose $\mathfrak{m} \subset \mathcal{P}$, and for all $d \in \mathbb{N}(\mathfrak{m})$, ${}_d\underline{x}$ is a homogeneous pure independent set of exponent d in $F^\times / F^{\times d}$, such that $\{{}_d\underline{x}\}$ is coherent under the natural projections. Then there exists a Galois extension Ω/F with torsion-free homogeneous abelian group $G_{\Omega/F} \simeq \mathbb{Z}_{\mathfrak{m}}^c$ and topological*

basis $\underline{\phi}$, with the property that $\langle \phi_i, {}_d x_j \rangle_d = \omega_d^{\delta_{ij}}$ for all $d \in \mathbb{N}(\mathfrak{m})$. Explicitly, Ω is the composite of the Kummer extensions defined by the subgroups $\langle {}_d \underline{x} \rangle \leq F^\times / F^{\times d}$.

Proof. Let d', e' be the parts of d and e supported on \mathfrak{m} , then since $e/d \in \mathbb{N}(\mathfrak{m})$, $e/d = e'/d'$. By Lemma 5.3, ${}_d \underline{\phi}^* = (e/d) {}_e \underline{\phi}^*$, therefore by the definition of κ_{ω_d} and κ_{ω_e} , $\langle -, {}_e x_j \rangle_e^{e/d} = \langle -, {}_d x_j \rangle_d$. On the other hand, since $e/d = e'/d'$, $\langle -, {}_e x_j \rangle_e^{e/d} = \langle -, {}_e x_j \rangle_e$. Therefore ${}_e x_j \equiv {}_d x_j \pmod{F^{\times d}}$. This proves the first statement.

Conversely, suppose the elements in the set $\{{}_d \underline{x} \subset F^\times / F^{\times d} : d \in \mathbb{N}(\mathfrak{m})\}$ are coherent under the natural projections. Let ${}_d \underline{\phi}^*$ be the corresponding characters of G_F . Since ${}_d \underline{x}$ is a homogeneous pure independent set of exponent d , so is ${}_d \underline{\phi}^*$, hence each ${}_d \underline{\phi}^*$ is a basis for a summand of ${}_d G_F^*$. Since $e/d \in \mathbb{N}(\mathfrak{m})$, $\langle -, {}_e x_j \rangle_d = \langle -, {}_e x_j \rangle_e^{e/d}$, and so ${}_d \underline{\phi}^* = (e/d) {}_e \underline{\phi}^*$. That is, the ${}_d \underline{\phi}^*$ form a coherent set. By Lemma 5.3, there exists a Galois group $G_{\Omega/F} \simeq \mathbb{Z}_{\mathfrak{m}}^c$ with a topological basis $\underline{\phi} = \{\phi_i\}_I$ whose dual basis is the ${}_d \underline{\phi}^*$. Thus $\langle \phi_i, {}_d x_j \rangle_d = \omega_d^{\delta_{ij}}$, for all $i, j \in I$, and $d \in \mathbb{N}(\mathfrak{m})$. Clearly Ω is the composite of the Kummer extensions defined by the ${}_d \underline{x}$, so the last statement follows. ■

Definition 5.16. In the situation of Lemma 5.15, let $x_i = \varprojlim_d ({}_d x_i)$ for each i , where $d \in \mathbb{N}(\mathfrak{m})$, and set $\underline{x} = \{x_i\}_I$. This is a subset of the closed subgroup $\varprojlim_d (\Omega^d \cap F^\times / F^{\times d})$ of the \mathfrak{m} -adic completion $F_{\mathfrak{m}}^\times$ of F^\times . Write

$$F(\underline{x}^{1/\mathfrak{m}^\infty}) = \varprojlim_{d \in \mathbb{N}(\mathfrak{m})} F(\{x_i^{1/d}\}_I)$$

for this field extension. If $\underline{\psi} = \underline{\phi} \mathcal{P}$ is a topological basis change, and $\underline{\psi}$ and $\underline{\phi}$ correspond to \underline{y} and \underline{x} , respectively, write

$$\underline{y}^{\mathcal{P}^*} = \underline{x}$$

for the induced basis change.

Without the machinery of inverse limits, the basis change expression can be viewed as shorthand for the family of basis changes $\{{}_d \underline{y}^{\mathcal{P}^*} = {}_d \underline{x}\}$.

Remark 5.17. The following special case is particularly interesting. Suppose $\underline{x} \subset F^\times$ is an \mathfrak{m} -independent subset of F^\times . For example, \underline{x} could be part of a transcendence basis for F over some subfield of F , or a set of nonassociate prime elements in a ring with quotient field F . Then $\Omega = F(\underline{x}^{1/\mathfrak{m}^\infty})$ is the composite of all radical extensions of degree $d \in \mathbb{N}(\mathfrak{m})$ defined by \underline{x} . By the above, the Galois group $G_{\Omega/F} \simeq \mathbb{Z}_{\mathfrak{m}}^c$ has a homogeneous basis $\underline{\phi}$, such that $\langle \phi_i, x_j \rangle_d = \omega_d^{\delta_{ij}}$.

In this construction, the element $\phi_i \in G_{\Omega/F}$ is the ‘‘Frobenius’’ automorphism corresponding to the ‘‘prime’’ x_i . In this sense, the basis $\underline{\phi}$ of any homogeneous torsion-free abelian $G_{\Omega/F}$ is a set of Frobenius automorphisms for the prime elements \underline{x} in $F_{\mathfrak{m}}^\times$.

Generators in $\text{Br}(F)$

The basis elements ${}_d\underline{x} = \{{}_dx_i\}$ for the various $F^\times / F^{\times d}$ suggest an obvious generating set for part of $\text{Br}(F)$, namely the set of symbol classes that are expressible in terms of the ${}_dx_i$. These elements will be used to import the explicit basis calculations on $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$ into $\text{Br}(F)$.

Assume F contains μ , and $G_{\Omega/F}$ is a torsion-free abelian Galois group, with topological basis $\underline{\phi} = \{\phi_i\}_I$, for some linearly ordered set I . By Proposition 5.4, the set ${}_d\underline{\phi}^{*\wedge 2} = \{(\phi_i^* \wedge \phi_j^*)_d : i < j \in I\}$ is a basis for $\text{Alt}(G_{\Omega/F}, \mathbb{Q}/\mathbb{Z})$. Let ${}_d\underline{x} = \kappa_{\omega_d}({}_d\underline{\phi}^*)$ be the Kummer dual, as in Definition 5.14. Assume for simplicity that $G_{\Omega/F}$ is homogeneous, so $G_{\Omega/F} \simeq \mathbb{Z}_m^c$ for some $m \subset \mathcal{P}$ and some cardinal number c . The general case can be reduced to this situation using the primary decomposition. By Theorem 4.7, the elements $[(x_i, x_j)_{\omega_d}]$, where $d \in \mathbb{N}(m)$, generate ${}_d\text{Br}^\circ(\Omega/F)$. If $\kappa^{\wedge 2}$ is injective, then they form a *basis* for ${}_d\text{Br}^\circ(\Omega/F)$.

Theorem 5.18 “Disjoint Reduced Symbol Presentation II”. *Assume $\mu \subset F$. Suppose Ω/F is a Galois extension with torsion-free abelian group $G_{\Omega/F}$, and $\underline{\xi} \in \text{Br}^\circ(\Omega/F)$. Then $G_{\Omega/F}$ has a basis $\underline{\psi}$ such that for some subset $\{\psi_1, \dots, \psi_{2m}\} \subset \underline{\psi}$, along with numbers $d_i : d_i \mid d_{i-1}$, $\underline{\xi}$ has a disjoint reduced symbol presentation*

$$\underline{\xi} = \sum_{i=1}^m [(y_{2i-1}, y_{2i})_{\omega_{d_i}}]$$

where $y_{2i-1} = \kappa_{\omega_{d_i}}(d_i \psi_{2i-1}^*)$ and $y_{2i} = \kappa_{\omega_{d_i}}(d_i \psi_{2i}^*)$. If $\kappa^{\wedge 2}$ is injective, the d_i are uniquely determined.

If $G_{\Omega/F}$ is homogeneous, with homogeneous topological basis $\underline{\phi} = \{\phi_i\}_I$, then the basis $\underline{\psi}$ is obtainable through a basis change $\underline{\psi} = \underline{\phi}\mathcal{P}$. More precisely, suppose the Kummer dual of $\underline{\phi}$ is $\underline{x} = \{x_i\}_I$, and

$$\underline{\xi} = \sum_{1 \leq i < j \leq n} [(x_i, x_j)_{\omega_{r_{ij}}}^{s_{ij}}].$$

Set $a_{ij} = -a_{ji} = s_{ij}/r_{ij}$, then $[\mathbf{a}]_{\underline{\phi}} = (a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ represents $\underline{\xi}$. Let $\underline{\psi} = \underline{\phi}\mathcal{P}$ be a symplectic basis for $G_{\Omega/F}$ with respect to the form \mathbf{a} , and let $\underline{y} = \{y_i\}_I$ be the Kummer dual of $\underline{\psi}$, so that $\underline{y} = \underline{x}\Omega^*$, where $\Omega = \mathcal{P}^{-1}$. Then

$$\underline{\xi} = \sum_{i=1}^m [(y_{2i-1}, y_{2i})_{\omega_{d_i}}]$$

is a disjoint reduced symbol presentation.

Proof. The proof follows immediately by Theorem 4.12, Theorem 5.13, and Lemma 5.15. ■

Corollary 5.19. *Let $F = \mathbb{C}((x_1)) \cdots ((x_n))$. Then for each $d \in \mathbb{N}$, $\{[(x_i, x_j)_{\omega_d}] : 1 \leq i < j \leq n\}$ is a basis for ${}_d\text{Br}(F)$, and $\text{Br}(F) \simeq \mathbb{Q}/\mathbb{Z}^{\binom{n}{2}}$. Let $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ be the topological basis for G_F^{ab} Kummer dual to $\underline{x} = \{x_1, \dots, x_n\}$ via ω . Then the map*

$$\begin{aligned} \kappa^{\wedge 2} : \text{Alt}(G_F, \mu) &\xrightarrow{\sim} \text{Br}(F) \\ \omega_*(\phi_i^* \wedge \phi_j^*)_d &\longmapsto [(x_i, x_j)_{\omega_d}]. \end{aligned}$$

is a natural isomorphism.

Proof. The first statement is immediate from the third by Proposition 5.4. The iterated henselian valuation on F defines a decomposition $F^\times \simeq \mathbb{Z}^n \times U$, where U is divisible. Clearly the image of \underline{x} in $F^\times / F^{\times d}$ is a basis for all d , $\underline{\phi}$ is a basis for G_F^{ab} , and $G_F^{\text{ab}} \simeq \widehat{\mathbb{Z}}^n$. Therefore the second statement follows from the third by Example 5.11.

It can be shown with a little valuation theory that every finite extension of F is radical, hence G_F is abelian. Therefore $\text{Alt}(G_F^{\text{ab}}, \mu) \simeq H^2(G_F^{\text{ab}}, \mu) = H^2(G_F, \mu)$. By Hilbert's Theorem 90 and the long exact cohomology sequence associated with the Kummer sequence

$$1 \rightarrow \mu_d \rightarrow F_{\text{alg}}^\times \rightarrow F_{\text{alg}}^\times \rightarrow 1,$$

$H^2(G_F, \mu_d) \simeq {}_d\text{Br}(F)$. It follows that $H^2(G_F, \mu) = \varinjlim_d H^2(G_F, \mu_d) \simeq \varinjlim_d ({}_d\text{Br}(F)) = \text{Br}(F)$. By Theorem 4.7, the composite of these maps yields the natural map $\text{Alt}(G_F^{\text{ab}}, \mu) \rightarrow \text{Br}(F)$ given by $\kappa^{\wedge 2} \cdot \omega_*^{-1}$, an isomorphism that sends $\omega_*(\phi_i^* \wedge \phi_j^*)_d$ to $[(x_i, x_j)_{\omega_d}]$. This completes the proof. \blacksquare

Example 5.20. The inverse of $\kappa^{\wedge 2}$ in Corollary 5.19 is analogous to the Hasse invariant in local class field theory. Suppose F is a *local field*, which is by definition a complete discretely valued field with finite residue field. The *Hasse invariant map* in local class field theory is an isomorphism

$$\text{inv}_F : \text{Br}(F) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Explicitly, every Brauer class ξ is represented by a cyclic algebra $[({}_d\phi^*, \pi^c)]$, where ${}_d\phi^*$ is the *Frobenius character* of order d , defined on a Frobenius generator ϕ in G_F by ${}_d\phi^*(\phi) = 1/d$, and π is any uniformizer. Then by definition $\text{inv}_F(\xi) = c/d$. The map's naturality relies on the naturality of both the Frobenius map and the valuation defined by π .

Let $F = \mathbb{C}((x_1)) \cdots ((x_n))$. By Corollary 5.19, a choice of ω , together with the choice of the ordered set $\{\underline{x}\}$, determines an isomorphism

$$\text{inv}_F : \text{Br}(F) \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})^{\binom{n}{2}} \xrightarrow{\sim} \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$$

sending $[(x_i, x_j)_{\omega_d}^c]$ to the ij -th standard generator $(0, \dots, 0, c/d, 0, \dots, 0)$ corresponding to the element $c(\phi_i^* \wedge \phi_j^*)_d$. In particular, when $n = 2$ there is an isomorphism $\text{Br}(F) \simeq \mathbb{Q}/\mathbb{Z}$, analogous to the Hasse invariant map. In Example 6.5 below it is shown how $\text{Alt}_n(\mathbb{Q}/\mathbb{Z})$, hence $(\mathbb{Q}/\mathbb{Z})^{\binom{n}{2}}$, transforms under restriction and corestriction. In the $n = 2$ case, the usual formulas of local class field theory are obtained.

6. COMPUTING FUNCTORIAL MAPS

This section contains computations of inflation, restriction, and corestriction on bilinear forms, cohomology, and alternating forms with coefficients in \mathbb{Q}/\mathbb{Z} , and finally on the Brauer group of a field that contains μ .

Setup 6.1. Suppose there is an exact sequence of proabelian groups

$$1 \longrightarrow H \xrightarrow{\iota} G \xrightarrow{\pi} \overline{G} \longrightarrow 1$$

where G is torsion-free and \overline{G} has finite exponent r . For each d divisible by r , let j_d denote the composite of exponentiation by d and ι^{-1} ,

$$j_d : G \longrightarrow H.$$

Recall ι^* is restriction, π^* is inflation, and the composition $j_d^* \cdot \iota^* : G^* \rightarrow G^*$ is multiplication by d . If ι is open and $s = |\overline{G}|$, then $j = j_s$ is the transfer, and j^* is corestriction.

Definitions 6.2. Assume Setup 6.1.

a. Decompositions $\Psi : \prod_J \mathbb{Z}_{\mathfrak{m}_j}^{b_j} \xrightarrow{\sim} H$ and $\Phi : \prod_J \mathbb{Z}_{\mathfrak{m}_j}^{c_j} \xrightarrow{\sim} G$ and the corresponding bases $\underline{\psi}$ and $\underline{\phi}$ are *compatible* if $\iota(\Psi(\mathbb{Z}_{\mathfrak{m}_j}^{b_j})) \leq \Phi(\mathbb{Z}_{\mathfrak{m}_j}^{c_j})$, for all $j \in J$.

b. If $\underline{\psi}$ and $\underline{\phi}$ are compatible, write $\mathcal{P} = (p_{ij}) = [\iota]_{\underline{\psi}}^{\underline{\phi}}$, meaning that for each $\psi_j \in \Psi(\mathbb{Z}_{\mathfrak{m}_k}^{b_k})$, $\psi_j = \prod_i \phi_i^{p_{ij}}$, where $\phi_i \in \Phi(\mathbb{Z}_{\mathfrak{m}_k}^{c_k})$ and $p_{ij} \in \mathbb{Z}_{\mathfrak{m}_k}$. Let $\mathcal{P}^* = [\iota^*]_{\underline{\phi}^*}^{\underline{\psi}^*}$ stand for the family of coordinate representations $[\iota^*]_{\underline{\phi}^*}^{d\underline{\psi}^*} : {}_dG^* \rightarrow {}_dH^*$. Thus $\iota^*(d\phi_j^*) = \sum_{i \in I} p_{ij}^*(d\psi_i^*)$. If \underline{y} and \underline{x} are the respective Kummer bases as in Definition 5.16, let $\mathcal{P}^* = [\iota^*]_{\underline{x}}^{\underline{y}}$ denote the corresponding family of coordinate representations of the Kummer transpose of ι^* . Apply similar notation to maps on bilinear forms and alternating forms.

Compatible decompositions always exist. For example, the primary decompositions for H and G are obviously compatible.

Lemma 6.3. *Assume Setup 6.1.*

a. $H \simeq G$. In particular, if G has a homogeneous decomposition, then H has a compatible homogeneous decomposition.

b. ι is diagonalizable. Specifically, let $\underline{\psi} = \{\psi_i\}_I$ and $\underline{\phi} = \{\phi_i\}_I$ be compatible bases. Then there exist basis changes $\underline{\psi}' = \underline{\psi}\mathcal{C}$ and $\underline{\phi}' = \underline{\phi}\mathcal{B}^{-1}$, such that for some numbers $d_i \in \mathbb{N}$, $\iota(\psi'_i) = \phi_i^{d_i}$ for each $i \in I$. Thus if $[\iota]_{\underline{\psi}}^{\underline{\phi}} = \mathcal{P}$ then $[\iota]_{\underline{\psi}'}^{\underline{\phi}'} = \mathcal{B}\mathcal{P}\mathcal{C} = \mathcal{D} = \text{diag}\{d_i\}_I$.

Proof. View H as a closed subgroup of G . To prove $H \simeq G$ it suffices to show H and G have isomorphic p -primary components for all p . Therefore assume $H \simeq \mathbb{Z}_p^{b_p}$ and $G \simeq \mathbb{Z}_p^{c_p}$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{G}^* & \longrightarrow & G^* & \longrightarrow & H^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{G}^* & \longrightarrow & G^* & \longrightarrow & H^* \longrightarrow 0 \end{array}$$

where the vertical arrows are multiplication by p . By the Snake Lemma, the sequence of $\mathbb{Z}/p\mathbb{Z}$ -vector spaces

$$0 \longrightarrow {}_p\overline{G}^* \longrightarrow {}_pG^* \xrightarrow{\pi^*} {}_pH^* \longrightarrow \overline{G}^*/p\overline{G}^* \longrightarrow G^*/pG^* \longrightarrow 0$$

is exact. Since G^* is divisible, $G^*/pG^* = \{0\}$. Since these are all vector spaces, there are direct sum decompositions

$$\begin{aligned} {}_pG^* &\simeq {}_p\bar{G}^* \oplus \pi^*({}_pG^*) \\ {}_pH^* &\simeq \bar{G}^*/p\bar{G}^* \oplus \pi^*({}_pG^*) \end{aligned}$$

Now compare $\mathbb{Z}/p\mathbb{Z}$ -dimensions. Since \bar{G}^* has bounded order, it is a direct sum of (finite) cyclic p -groups, and so $\dim(\bar{G}^*/p\bar{G}^*) \leq \text{rank}(\bar{G}^*) = \dim({}_p\bar{G}^*)$. Therefore

$$\dim({}_pH^*) = \dim(\bar{G}^*/p\bar{G}^*) + \dim(\pi^*({}_pG^*)) \leq \dim({}_p\bar{G}^*) + \dim(\pi^*({}_pG^*)) = \dim({}_pG^*).$$

Thus $b_p \leq c_p$. On the other hand, since ι 's cokernel has exponent r , $G^r \leq H$. Since $r\mathbb{Z}_p \simeq \mathbb{Z}_p$ as \mathbb{Z}_p -modules, $G^r \simeq \mathbb{Z}_p^{c_p}$, therefore $c_p \leq b_p$. Thus $b_p = c_p$. Since a map of proabelian groups takes p -primary components into p -primary components, the primary decompositions of H and G are compatible. This proves (a).

To prove (b) it may be assumed that $H \simeq G \simeq \mathbb{Z}_m^c$ for some $m \in \mathcal{P}$ and some cardinal number $c = |I|$. Since any two bases of G are then related by a basis change, it suffices to show there exists a topological basis $\underline{\phi} = \{\phi_i\}_I$ for G such that for some numbers d_i , $\underline{\psi} = \{\phi_i^{d_i}\}_I$ is a topological basis for H .

Let $\underline{\phi}$ be the topological basis for G extending a basis $\bar{\underline{\phi}}$ for \bar{G} , constructed in Lemma 5.12. Set $d_i = |\bar{\phi}_i|$. *Claim:* $\underline{\psi} = \{\phi_i^{d_i}\}_I$ is a topological basis for H . Since “ $*$ ” is an exact functor, the sequence

$$0 \longrightarrow \bar{G}^* \longrightarrow G^* \longrightarrow H^* \longrightarrow 0$$

is exact. Suppose $\tau \in G$. Since $\underline{\phi}$ is a basis, $\tau = \prod_I \phi_i^{g_i}$, for some $g_i \in \mathbb{Z}_m$. Since $\bar{\underline{\phi}}$ is a basis for \bar{G} , by exactness $\tau \in H$ if and only if $\bar{\phi}_j^*(\tau) = 0$ for all j , i.e., if and only if $d_i \mid g_i$ for all $i \in I$. Thus $\underline{\psi}$ is a topological basis for H . This proves the claim, and completes the proof. ■

Computing Functorial Maps in $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$

Theorem 6.4. *Assume Setup 6.1. Let $\underline{\phi} = \{\phi_i\}_I$ and $\underline{\psi} = \{\psi_i\}_I$ be compatible bases for G and H with respect to ι , indexed by a linearly ordered set I , and let $\mathcal{P} = (p_{ij}) = [\iota]_{\underline{\psi}}^{\underline{\phi}}$.*

a. Let $\mathcal{BPC} = \mathcal{D} = \text{diag}\{d_i\}_I$ be a diagonalization of \mathcal{P} as in Lemma 6.3, and set $\mathcal{B} = (b_{ij})$. Then the kernel of the restriction map on $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$ and $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, respectively, have bases

$$\left\{ \sum_{k,l} b_{ki}^* b_{lj}^* (\phi_k^* \otimes \phi_l^*)_{d_{ij}} : i, j \in I \right\}, \quad \left\{ \sum_{k < l} b_{kl,ij}^* (\phi_k^* \wedge \phi_l^*)_{d_{ij}} : i < j \in I \right\},$$

where $d_{ij} = d_i d_j$, $\mathcal{B}^* = (b_{ij}^*)$ and $b_{kl,ij}^* = b_{ki}^* b_{lj}^* - b_{li}^* b_{kj}^*$.

b. The coordinate expressions of the restriction maps are

$$\begin{aligned} \mathcal{P}^{*\otimes 2} : M_{|\underline{\phi}|}(\mathbb{Q}/\mathbb{Z}) &\longrightarrow M_{|\underline{\psi}|}(\mathbb{Q}/\mathbb{Z}), & \mathcal{P}^{*\wedge 2} : \text{Alt}_{|\underline{\phi}|}(\mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{Alt}_{|\underline{\psi}|}(\mathbb{Q}/\mathbb{Z}) \\ (b_{ij}) &\longmapsto (b'_{ij}), & (a_{ij}) &\longmapsto (a'_{ij}), \end{aligned}$$

where $b'_{ij} = \sum_{k,l} p_{ik}^* p_{jl}^* b_{kl}$ and $a'_{ij} = \sum_{k<l} p_{ij,kl}^* a_{kl}$, where $p_{ij}^* = p_{ji}$, and $p_{kl,ij}^* = p_{ki}^* p_{lj}^* - p_{li}^* p_{kj}^*$.

c. Suppose ι is open. Let $s = [G : H]$ and $\mathcal{N} = (n_{ij}) = [j]_{\underline{\phi}}^{\underline{\psi}}$. The coordinate expression of the corestriction map is

$$\begin{aligned} \mathcal{N}^{*\wedge 2} \cdot \frac{1}{s} : \text{Alt}_{|\underline{\psi}|}(\mathbb{Q}/\mathbb{Z}) &\longrightarrow \text{Alt}_{|\underline{\phi}|}(\mathbb{Q}/\mathbb{Z}) \\ (a_{ij}) &\longmapsto (a'_{ij}) \end{aligned}$$

defined by $a'_{ij} = \sum_{k<l} n_{ij,kl}^* \frac{1}{s} a_{kl}$, where $n_{ij}^* = n_{ji}$, $n_{ij,kl}^* = n_{ik}^* n_{jl}^* - n_{il}^* n_{jk}^*$, and $\frac{1}{s} a_{ij}$ is any element in \mathbb{Q}/\mathbb{Z} such that $s \frac{1}{s} a_{ij} = a_{ij}$.

Proof. Let $\mathcal{BPC} = \mathcal{D} = \text{diag}\{d_i\}_I$ be the diagonalization of Lemma 6.3. Then $\mathcal{C}^* \mathcal{P}^* \mathcal{B}^* = \mathcal{D}^* = \text{diag}\{d_i\}_I$, and both $\underline{\phi}^{*'} = \underline{\phi}^* \mathcal{B}^*$ and $\underline{\psi}^{*'} = \underline{\psi}^* (\mathcal{C}^*)^{-1}$ are (families of) basis changes for G^* and H^* , respectively, and $\mathcal{D}^* = [\iota^*]_{\underline{\phi}^{*'}}^{\underline{\psi}^{*'}}$. By Proposition 5.4, $\underline{\phi}^{*'\otimes 2}$, $\underline{\psi}^{*'\otimes 2}$, $\underline{\phi}^{*'\wedge 2}$, and $\underline{\psi}^{*'\wedge 2}$ are (families of) bases for $\text{Bil}(G, \mathbb{Q}/\mathbb{Z})$, $\text{Bil}(H, \mathbb{Q}/\mathbb{Z})$, $\text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, and $\text{Alt}(H, \mathbb{Q}/\mathbb{Z})$, respectively. Compute by the formulas in Theorem 2.3, $[\iota^{*\otimes 2}]_{\underline{\phi}^{*'}}^{\underline{\psi}^{*'}} = \mathcal{D}^{*\otimes 2} = \text{diag}\{d_{ij}\}_{i,j}$ and $[\iota^{*\wedge 2}]_{\underline{\phi}^{*'}}^{\underline{\psi}^{*'}} = \mathcal{D}^{*\wedge 2} = \text{diag}\{d_{ij}\}_{i<j}$. Thus $\text{res}(\phi_i^{*'} \otimes \phi_j^{*'})_d = d_{ij}(\psi_i^{*'} \otimes \psi_j^{*'})_d$ and $\text{res}(\phi_i^{*'} \wedge \phi_j^{*'})_d = d_{ij}(\psi_i^{*'} \wedge \psi_j^{*'})_d$. Since $\underline{\phi}^{*'\otimes 2}$ and $\underline{\phi}^{*'\wedge 2}$ are bases, it follows that the kernels are the subgroups with bases $\{(\phi_i^{*'} \otimes \phi_j^{*'})_{d_{ij}}, i, j \in I\}$, and $\{(\phi_i^{*'} \wedge \phi_j^{*'})_{d_{ij}}, i < j \in I\}$. By Proposition 5.7, $(\phi_i^{*'} \otimes \phi_j^{*'})_{d_{ij}} = \sum_{k,l} b_{ki}^* b_{lj}^* (\phi_k^* \otimes \phi_l^*)_{d_{ij}}$, and $(\phi_i^{*'} \wedge \phi_j^{*'})_{d_{ij}} = \sum_{k<l} b_{kl,ij}^* (\phi_k^* \wedge \phi_l^*)_{d_{ij}}$, so the kernels of the restrictions are as stated. This proves (a).

For (b), view H as a subgroup of G , which is allowed since ι is closed. Since $\mathcal{P} = [\iota]_{\underline{\psi}}^{\underline{\phi}}$, $\mathcal{P}^* = [\iota^*]_{\underline{\phi}^*}^{\underline{\psi}^*}$. A computation exactly as in Proposition 5.7 for basis change gives the desired formulas for the coordinate expressions $\mathcal{P}^{*\otimes 2} = [\iota^{*\otimes 2}]_{\underline{\phi}^*}^{\underline{\psi}^*}$ and $\mathcal{P}^{*\wedge 2} = [\iota^{*\wedge 2}]_{\underline{\phi}^*}^{\underline{\psi}^*}$.

Suppose $H \leq G$ is an open subgroup, and $[G : H] = s$. By Theorem 2.10, $\text{cor}(\mathbf{a}) = j^{*\wedge 2}(\frac{1}{s}\mathbf{a})$ on $\text{Alt}(H, \mathbb{Q}/\mathbb{Z})$, where $\frac{1}{s}\mathbf{a}$ is an s -th root of \mathbf{a} . Clearly if $[\mathbf{a}]_{\underline{\psi}} = (a_{ij})$, then $[\frac{1}{s}\mathbf{a}]_{\underline{\psi}} = (\frac{1}{s}a_{ij})$, where $\frac{1}{s}a_{ij}$ is any s -th root of a_{ij} in $\mathbb{Z}[1/m]/\mathbb{Z}$. If $\mathcal{N} = [j]_{\underline{\phi}}^{\underline{\psi}} = (n_{ij})$, the transpose is $\mathcal{N}^* = [j^*]_{\underline{\psi}^*}^{\underline{\phi}^*} = (n_{ij}^*)$, and

$$\mathcal{N}^{*\wedge 2} = [j^{*\wedge 2}]_{\underline{\psi}^*}^{\underline{\phi}^*} : [\frac{1}{s}\mathbf{a}]_{\underline{\psi}} \longmapsto [\text{cor}(\mathbf{a})]_{\underline{\phi}}.$$

If $[\text{cor}(\mathbf{a})]_{\underline{\phi}} = (a'_{ij})$, then a routine computation shows $a'_{ij} = \sum_{k<l} n_{ij,kl}^* \frac{1}{s} a_{kl}$. ■

Example 6.5. Suppose $G = \widehat{\mathbb{Z}}^n$. Let $H \leq G$ be a subgroup of index s . Let $\iota : H \hookrightarrow G$ be the natural inclusion. Suppose $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ and $\underline{\psi} = \{\psi_1, \dots, \psi_n\}$ are bases. The restriction and corestriction are computed on $\text{Alt}_n(\mathbb{Q}/\mathbb{Z})$ from Theorem 6.4 as follows.

Restriction. Let $\mathcal{P}^* = [\iota^*]_{\underline{\phi}^*}^{\underline{\psi}^*}$, then $\mathcal{P}^* = (p_{ij}^*) \in \text{M}_n(\widehat{\mathbb{Z}})$ has rank n . If $\mathbf{a} \in \text{Alt}(G, \mathbb{Q}/\mathbb{Z})$, then $[\mathbf{a}]_{\underline{\phi}} = (a_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$, where $a_{ij} = \mathbf{a}(\phi_i, \phi_j)$. Set $\text{res}(\mathbf{a}) = \mathbf{a}'$, and $[\mathbf{a}']_{\underline{\psi}} = (a'_{ij}) \in \text{Alt}_n(\mathbb{Q}/\mathbb{Z})$. As in Example 5.9, order the pairs ij lexicographically, and view (a_{ij}) as the $\binom{n}{2}$ -tuple $(a_{ij}) = \{a_{12}, a_{13}, \dots, a_{n-1n}\}$. By Theorem 6.4, $[\text{res}(\mathbf{a})]_{\underline{\psi}^*}^{\underline{\phi}^*} = \mathcal{P}^{*\wedge 2} = (p_{kl,ij}^*) \in \text{M}_{\binom{n}{2}}(\widehat{\mathbb{Z}})$, where $p_{kl,ij}^* = p_{ki}^* p_{lj}^* - p_{li}^* p_{kj}^*$ is the entry in the kl -th row and ij -th column. Compute $\mathcal{P}^{*\wedge 2}(a_{ij}) = (a'_{ij})$,

where $a'_{ij} = \sum_{k<l} p_{ij,kl}^* a_{kl}$. Equivalently, $[\text{res}]_{\underline{\phi}^*}^{\underline{\psi}^*}$ is the congruence transformation $(a_{ij}) \mapsto (a'_{ij}) = \mathcal{P}^*(a_{ij})\mathcal{P}$:

$$\begin{array}{ccc} \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Psi^{*\wedge 2}} & \text{Alt}_n(\mathbb{Q}/\mathbb{Z}) \\ \text{res} \uparrow & & \uparrow \mathcal{P}^*(-)\mathcal{P} \\ \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Phi^{*\wedge 2}} & \text{Alt}_n(\mathbb{Q}/\mathbb{Z}) \end{array}$$

Here $\Psi^{*\wedge 2}$ and $\Phi^{*\wedge 2}$ are the coordinate representation maps with respect to $\underline{\psi}$ and $\underline{\phi}$, respectively. Let $\mathcal{D}^* = \text{diag}\{d_1, \dots, d_n\}$ be a diagonalization of \mathcal{P}^* , as in Lemma 6.3, with $d_i \in \mathbb{N}$. Clearly $\prod d_i = [G : H]$, consequently $\det(\mathcal{P}^*) = u[G : H]$ where $u \in \widehat{\mathbb{Z}}^\times$ and $\det(\mathcal{P}^*)$ is computed in the ring $\widehat{\mathbb{Z}}$. Thus

$$|\det(\mathcal{P}^*)| = [G : H],$$

where $|\cdot|$ is the canonical norm on $\widehat{\mathbb{Z}}$.

In the special case $n = 2$, $\mathcal{P}^{*\wedge 2} = p_{12,12}^* = p_{11}^*p_{22}^* - p_{12}^*p_{21}^* = \det(\mathcal{P}^*)$, so restriction is multiplication by $u[G : H]$:

$$\begin{array}{ccc} \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}_2(\mathbb{Q}/\mathbb{Z}) \\ \text{res} \uparrow & & \uparrow u[G:H] \\ \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}_2(\mathbb{Q}/\mathbb{Z}) \end{array}$$

Corestriction. Since $\text{rank}(\mathcal{P}^*) = n$, $\mathcal{P}^* \in \text{GL}_n(\widehat{\mathbb{Q}})$. Since $\mathcal{P}^*\mathcal{N}^*$ is multiplication by s , $\mathcal{N}^* = s(\mathcal{P}^*)^{-1}$. Let $\mathcal{Q}^* = (\mathcal{P}^*)^{-1} \in \text{GL}_n(\widehat{\mathbb{Q}})$, and define the action of $\mathcal{Q}^* = \frac{1}{s}\mathcal{N}^*$ on \mathbb{Q}/\mathbb{Z} by letting $\frac{1}{s}$ produce an s -th root of an element of \mathbb{Q}/\mathbb{Z} . By Theorem 6.4

$$[\text{cor}]_{\underline{\psi}^*}^{\underline{\phi}^*} = s\mathcal{Q}^{*\wedge 2} = (sq_{kl,ij}^*) \in \text{GL}_n(\widehat{\mathbb{Q}})$$

where $q_{kl,ij}^* = q_{ki}^*q_{lj}^* - q_{li}^*q_{kj}^*$. Write $[\mathbf{b}]_{\underline{\psi}} = (b_{ij})$, then $[\text{cor}(\mathbf{b})]_{\underline{\phi}} = s\mathcal{Q}^{*\wedge 2}(b_{ij}) = (b'_{ij})$, where $b'_{ij} = \sum_{k<l} sq_{ij,kl}^* b_{kl} = \sum_{k<l} n_{ij,kl}^* \frac{1}{s} b_{kl}$. By notation abuse, $[\text{cor}]_{\underline{\psi}^*}^{\underline{\phi}^*}$ is congruence transformation:

$$\begin{array}{ccc} \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Psi^{*\wedge 2}} & \text{Alt}_n(\mathbb{Q}/\mathbb{Z}) \\ \text{cor} \downarrow & & \downarrow \sqrt{s}\mathcal{Q}^*(-)\sqrt{s}\mathcal{Q} \\ \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Phi^{*\wedge 2}} & \text{Alt}_n(\mathbb{Q}/\mathbb{Z}) \end{array}$$

In the special case $n = 2$, the corestriction is multiplication by v , where $v = u^{-1} \in \widehat{\mathbb{Z}}^\times$, as above, which is an *isomorphism*. Thus

$$\begin{array}{ccc} \text{Alt}(H, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}_2(\mathbb{Q}/\mathbb{Z}) \\ \text{cor} \downarrow & & \downarrow v \\ \text{Alt}(G, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Alt}_2(\mathbb{Q}/\mathbb{Z}) \end{array}$$

Computing Functorial Maps in $\text{Br}(F)$

Setup 6.6. The following situation holds for the rest of the section. Let F be a field that contains the roots of unity μ . Let $\omega : \mathbb{Q}/\mathbb{Z} \rightarrow \mu$ denote a fixed isomorphism. Let

$$(6.7) \quad 1 \longrightarrow G_{\Omega/K} \xrightarrow{\iota} G_{\Omega/F} \xrightarrow{\pi} G_{K/F} \longrightarrow 1$$

be a short exact sequence of abelian groups, where $G_{\Omega/F}$ is torsion-free, and $\exp(G_{K/F}) = r \in \mathbb{N}$. Assume $r > 1$. For $d : r \mid d$, let

$$(6.8) \quad J_d : G_{\Omega/F} \rightarrow G_{\Omega/K}$$

denote exponentiation by d . If ι is open, set $s = |G_{K/F}|$, and let $J := J_s$ denote the transfer. Let

$$\kappa_{\omega_d} : G_{\Omega/F}^* \xrightarrow{\sim} (\Omega^d \cap F^\times) / F^{\times d}$$

denote the Kummer isomorphism of Setup 4.1.

Lemma 6.9. *Suppose $\underline{\psi}$ and $\underline{\phi}$ are compatible topological bases for $G_{\Omega/K}$ and $G_{\Omega/F}$, respectively, and $\exp(G_{K/F}) = r$. Then for all $d : r \mid d$, $d\underline{z} := \kappa_{\omega_d} \cdot J_d^*(d\underline{\psi}^*)$ generates the Kummer subgroup $K^d \cap F^\times / F^{\times d}$ of $F^\times / F^{\times d}$ determined by K/F . If $\underline{z} \subset F^\times$ is any preimage of $d\underline{z}$, and $\underline{y} = \underline{z}^{1/d} \subset K^\times$, then $K = F(\underline{y})$.*

Proof. Since $G_{\Omega/F}$ is torsion-free, this is immediate by Lemma 2.7, and the comments preceding Lemma 4.5. ■

Suppose $F = \mathbb{C}(x_1, \dots, x_n)$ is the rational function field in n -variables over \mathbb{C} . Every abelian extension of F is Kummer. A conspicuous class of Kummer extensions are those extensions K/F that can be defined by choosing n radical elements $y_j := \prod x_i^{q_{ji}}$, where each q_{ji} is a rational number between 0 and 1 (inclusive), and adjoining these elements to F . For this purpose, the x_i could easily be replaced by a pure independent subset of F^\times , such as a set of primes in $\mathbb{C}[x_1, \dots, x_n]$. Let X be the subgroup of F^\times generated by the x_i . Each Kummer extension defined in this way may be represented by an $n \times n$ matrix \mathcal{Q} with entries q_{ji} in \mathbb{Q} , whose j -th row consists of the exponents that define y_j in terms of the x_i . For example, the extension $F((x_1x_2)^{1/4}, x_2^{1/2})$ of $F = \mathbb{C}(x_1, x_2, x_3)$ is represented by the matrix

$$\mathcal{Q} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $y_1 = (x_1x_2)^{1/4}$, $y_2 = x_2^{1/2}$, and $y_3 = x_3$. This matrix is invertible, and the rows of the inverse \mathcal{P} express the x_i in terms of the y_j . In this case,

$$\mathcal{P} = \mathcal{Q}^{-1} = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $x_1 = y_1^4 y_2^{-2}$, $x_2 = y_2^2$, and $x_3 = y_3$. The y_j generate a subgroup Y of K^\times , and the image of Y in $F_{\text{alg}}^\times / F^\times$ is canonically isomorphic to the dual $G_{K/F}^*$ of the Galois group $G_{K/F}$. The group in the above example has exponent 4, and so Y^4 is a subgroup Z of X , with generators $z_i := y_i^4$. This subgroup contains X^4 ; more generally $X^d \leq Z = Y^d \leq X$, where $d = \exp(G_{K/F})$. Obviously $z_j = \prod x_i^{n_{ji}}$, where $n_{ji} = dq_{ji} = 4q_{ji}$, and the rows of the matrix

$$\mathcal{N}_4 = 4\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

express the z_j in terms of the x_i . Thus the elements $z_1 = x_1 x_2$, $z_2 = x_2^2$, and $z_3 = x_3^4$ determine the subgroup Z . By Kummer Theory, the image of Z in $F^\times / F^{\times 4}$ is also isomorphic to $G_{K/F}^*$.

Since the machinery developed in Theorem 6.4 to compute restriction and corestriction is phrased in terms of bases of Galois groups and their character groups, it isn't obvious how it would apply to such a Kummer extension, given not by a subgroup of the Galois group, but purely in terms of field elements. The next result shows how the subgroups X, Y , and Z can be used to reconcile the two points of view. It will allow restriction and corestriction on the Brauer group to be computed (in Theorem 6.13) in the likely case when the starting data is a Kummer extension that is defined explicitly in terms of field elements.

Lemma 6.10. *Fix $\mathfrak{m} \subset \mathcal{P}$ and a number $d \in \mathbb{N}(\mathfrak{m})$. Let $\underline{x} = \{x_1, \dots, x_n\}$ be an \mathfrak{m} -independent subset of F^\times , let $\underline{y} = \{y_1, \dots, y_n\}$ be a subset of F_{alg}^\times , and let $\underline{z} = \{z_1, \dots, z_n\}$ be defined by $z_i = y_i^d$. Let $X = \langle \underline{x} \rangle$, $Y = \langle \underline{y} \rangle$, and $Z = \langle \underline{z} \rangle$. Assume $X^d \leq Z \leq X$. Set $K = F(Y) \subset \Omega = F(X^{1/\mathfrak{m}^\infty})$, so that (6.7) and (6.8) hold. Then*

a. *The subset \underline{y} is \mathfrak{m} -independent in K^\times , and there are isomorphisms ${}_d G_{\Omega/F}^* \simeq X/X^d$, ${}_d G_{\Omega/K}^* \simeq Y/Y^d$, and ${}_d G_{K/F}^* \simeq Z/X^d$, given by κ_{ω_d} over the appropriate field.*

b. *The natural maps $\pi^* : Z \hookrightarrow X$, $\iota^* : X \hookrightarrow Y$, and $j_d^* : Y \hookrightarrow \pi^*(Z)$, induce their counterparts for the groups ${}_d G_{K/F}^*$, ${}_d G_{\Omega/F}^*$, and ${}_d G_{\Omega/K}^*$, transpose to π , ι , and j_d in (6.7) and (6.8).*

c. *There exist topological bases $\underline{\psi} = \{\psi_1, \dots, \psi_n\}$ and $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ for $G_{\Omega/K}$ and $G_{\Omega/F}$, respectively, whose Kummer duals in $K^\times / K^{\times e}$ and $F^\times / F^{\times e}$ are the images of \underline{x} and \underline{y} , for all $e \in \mathbb{N}(\mathfrak{m})$. Thus $\langle \phi_i, x_j \rangle_e = \omega_e^{\delta_{ij}}$ and $\langle \psi_i, y_j \rangle_e = \omega_e^{\delta_{ij}}$. In particular, the maps $[l]_{\underline{\psi}}^{\underline{\phi}} = \mathcal{P} = (p_{ij})$ and $[j_d]_{\underline{\phi}}^{\underline{\psi}} = \mathcal{N}_d = (n_{ij})$ are the transposes of the maps $[l^*]_{\underline{x}}^{\underline{y}} = \mathcal{P}^* = (p_{ij}^*)$ and $[j_d^*]_{\underline{y}}^{\underline{x}} = \mathcal{N}_d^* = (n_{ij}^*)$, respectively, where $p_{ij}^* = p_{ji}$ and $n_{ij}^* = n_{ji}$ are integers. In other words, if*

$$x_j = \prod_{i=1}^n y_i^{p_{ij}^*}, \quad z_j = y_j^d = \prod_{i=1}^n x_i^{n_{ij}^*},$$

then

$$\psi_j = \prod_{i=1}^n \phi_i^{p_{ij}}, \quad \phi_j^d = \prod_{i=1}^n \psi_i^{n_{ij}}.$$

d. $[K : F] = |\det(\mathcal{P})|$, where $\mathcal{P} = [l]_{\underline{\psi}}^{\underline{\phi}}$.

Proof. Since \underline{x} is \mathfrak{m} -independent in F^\times , \underline{x} is independent in $F^\times / F^{\times d}$. By [F, Lemma 32.1], X is \mathfrak{m} -pure in F^\times , hence $F^d \cap X = X^d$. By [F, Proposition 27.1], $X/X^d \hookrightarrow F^\times / F^{\times d}$ is a summand. Therefore by the Definition 5.16 of Ω as a Kummer extension, $\kappa_{\omega_d} : {}_d G_{\Omega/F}^* \xrightarrow{\sim} X/X^d$.

By definition, the extension K/F is determined by the image $Z/Z \cap F^d$ of Z in $F^\times / F^{\times d}$. Since $X \cap F^d = X^d \leq Z \leq X$, $Z/Z \cap F^d = Z/X^d$. Thus by Kummer Theory, $\kappa_{\omega_d} : G_{K/F}^* \xrightarrow{\sim} Z/X^d$.

Since \underline{x} is \mathfrak{m} -independent in F^\times and μ is divisible, X is torsion-free. Since $Z \leq X$, Z is torsion-free. Since $X^d \leq Z$, $\text{rank}(X) = \text{rank}(Z)$, so $X \simeq Z$, and \underline{z} is a basis for Z . The exponentiation-by- d homomorphism J_d^* on K^\times maps Y onto Z . The kernel $\mu_d \cap Y$ is trivial, for if $\prod y_i^{u_i} \in \mu_d$, then $\prod z_i^{u_i} = 1$, which implies $u_i = 0$ since the z_i are independent. Therefore $Y \simeq Z$, and Y is torsion-free, with basis \underline{y} . To show \underline{y} is \mathfrak{m} -independent in K^\times , it suffices by [F, Lemma 32.1] to show $Y \cap K^p = Y^p$ for all $p \in \mathfrak{m}$. The inclusion $Y \cap K^p \geq Y^p$ is obvious, so it suffices to show $Y \cap K^p \leq Y^p$.

Since $Y^d = Z \leq X \leq F^\times$, by Kummer Theory, $(K^\times / F^\times)_{\text{tor}} = YF^\times / F^\times$. Suppose $p \in \mathfrak{m}$, and $a \in Y \cap K^p$. Then $a = y^p$ for some $y \in K^\times$. Since a is in Y , its image in K^\times / F^\times has finite order, and since $a = y^p$, the image of y in K^\times / F^\times also has finite order, so it lies in $(K^\times / F^\times)_{\text{tor}} = YF^\times / F^\times$. Thus $y = y'x$ for some $y' \in Y$ and $x \in F^\times$, hence $a \in Y^p F^{\times p}$. Therefore $Y \cap K^p \leq Y^p F^{\times p} \cap Y$. Since $Y \cap K^p \geq Y^p F^{\times p} \cap Y$ is clear, $Y \cap K^p = Y^p F^{\times p}$. A quick calculation shows $Y^p F^{\times p} = (Y \cap F^p)Y^p$, so $Y \cap K^p = (Y \cap F^p)Y^p$. It now suffices to show $Y \cap F^p \leq Y^p$.

Let “ $-$ ” denote the map $K^\times \rightarrow K^\times / \mu$. If $\bar{Y} \cap \bar{F}^p \leq \bar{Y}^p$, then $Y \cap F^p \leq Y^p \mu \cap Y = Y^p(\mu \cap Y) = Y^p$, as Y is torsion-free. Therefore it suffices to show $\bar{Y} \cap \bar{F}^p \leq \bar{Y}^p$. Note that since $\bar{X}^d \leq \bar{Y}^d$ and \bar{K} is torsion-free, $\bar{X} \leq \bar{Y}$. Also $\bar{X} \cap \bar{F}^d = \bar{X}^d$, since $X \cap F^d = X^d$, and there is a commutative diagram

$$\begin{array}{ccc} X/X^d & \xrightarrow{\sim} & \bar{X}/\bar{X}^d \\ \downarrow & & \downarrow \\ F^\times / F^{\times d} & \xrightarrow{\sim} & \bar{F}^\times / \bar{F}^{\times d} \end{array}$$

Thus

$$(\bar{Y} \cap \bar{F})^d \leq \bar{Y}^d \cap \bar{F}^d \leq \bar{X} \cap \bar{F}^d = \bar{X}^d.$$

Therefore $\bar{Y} \cap \bar{F} \leq \bar{X}$, as \bar{K} is torsion-free. Therefore $\bar{Y} \cap \bar{F}^p \leq \bar{X} \cap \bar{F}^p = \bar{X}^p \leq \bar{Y}^p$, as desired. Thus \underline{y} is \mathfrak{m} -independent in K^\times .

Since \underline{y} is \mathfrak{m} -independent in K^\times , a proof similar to the one for \underline{x} in F^\times in the first paragraph shows $\kappa_{\omega_d} : {}_d G_{\Omega/K}^* \xrightarrow{\sim} Y/Y^d$. This completes the proof of (a).

The inclusions $\pi^* : Z \hookrightarrow X$ and $\iota^* : X \hookrightarrow Y$ clearly induce the inflation ${}_d G_{K/F}^* \hookrightarrow {}_d G_{\Omega/F}^*$ and the restriction ${}_d G_{\Omega/F}^* \rightarrow {}_d G_{\Omega/K}^*$. If $\sigma \in G_{\Omega/F}$ and $y \in Y$, then $\langle \sigma, J_d^*(y) \rangle_d = \langle \sigma, y^d \rangle_d = \langle \sigma^d, y \rangle_d = \langle J_d(\sigma), y \rangle_d = J_d^*(\langle -, y \rangle_d)(\sigma)$, using the bilinearity of the Kummer symbol. Therefore the exponentiation-by- d map $J_d^* : Y \hookrightarrow \pi^*(Z)$ induces $J_d^* : {}_d G_{\Omega/K}^* \rightarrow G_{\Omega/F}^*$. This proves (b).

Suppose $e \in \mathbb{N}(\mathfrak{m})$. The images of \underline{x} and \underline{y} in X/X^e and Y/Y^e are clearly coherent under the natural projections, since, after all, $\underline{x} \subset \bar{F}^\times$ and $\underline{y} \subset K^\times$. Therefore by Lemma 5.15, the desired bases $\underline{\phi}$ and $\underline{\psi}$ for $G_{\Omega/F}$ and $G_{\Omega/K}$ exist, Kummer dual to the images of \underline{x} and \underline{y} in X/X^e and Y/Y^e , respectively. By (b), the maps on X/X^e and Y/Y^e induced by $\iota^* : X \rightarrow Y$ and $J_d^* : Y \rightarrow X$ are transpose to ι and J_d . Thus if $[\iota]_{\underline{\psi}}^{\underline{\phi}} = \mathcal{P} = (p_{ij})$ and $[J_d]_{\underline{\phi}}^{\underline{\psi}} = \mathcal{N}_d = (n_{ij})$,

then $[\iota^*]_{\underline{x}}^{\underline{y}} = \mathcal{P}^* = (p_{ij}^*)$ and $[j_d^*]_{\underline{y}}^{\underline{x}} = \mathcal{N}_d^* = (n_{ij}^*)$, where $p_{ij}^* = p_{ji}$ and $n_{ij}^* = n_{ji}$ are integers. This proves (c).

Finally, the matrix $[j_d^*]_{\underline{y}}^{\underline{x}}$ expresses each $y_j^d = z_j$ as a product of x_i 's. Since X and Z are free abelian groups, it follows by [Ro, Chapter 9, Corollary 9.63] that $|\det([j_d^*]_{\underline{y}}^{\underline{x}})| = |\det(\mathcal{N}_d^*)| = [X : Z]$. Similarly, since $j_d^* \cdot \iota^*$ is exponentiation by d on X ,

$$|\det([j_d^* \cdot \iota^*]_{\underline{x}}^{\underline{x}})| = |\det(\mathcal{N}_d^* \cdot \mathcal{P}^*)| = [X : Z] |\det(\mathcal{P}^*)| = [X : X^d] = d^n.$$

Therefore $[Z : X^d] = |\det(\mathcal{P}^*)|$. Since K/F is defined by the image of Z in X/X^d , $[Z : X^d] = [K : F]$, hence $[K : F] = |\det(\mathcal{P}^*)|$. This proves (d). ■

Remark 6.11. It is easy to construct situations that satisfy the hypotheses of Lemma 6.10. For example, F could be a rational function field in n variables, or the quotient field of a ring containing n nonassociate prime elements. Let $\underline{x} = \{x_1, \dots, x_n\}$ be either of these sets. Let $\mathcal{P} = (p_{ij}) \in M_n(\mathbb{Z})$ be any matrix with nonzero determinant d . Set $\mathcal{Q} = (q_{ij}) = \mathcal{P}^{-1} \in \text{GL}_n(\mathbb{Q})$, and use this to choose $\underline{y} = \{y_1, \dots, y_n\}$ by the rule $y_j = \prod x_i^{q_{ij}}$. Since the adjoint of \mathcal{P} is defined over \mathbb{Z} , $dq_{ij} \in \mathbb{Z}$, so $z_j = y_j^d \in X$. Since \mathcal{P} is defined over \mathbb{Z} , $X \leq Y$, therefore $X^d \leq Z$. Thus all of the hypotheses of Lemma 6.10 are satisfied.

Example 6.12. Revisit the example immediately preceding Lemma 6.10, with $F = \mathbb{C}(x_1, x_2, x_3)$. Let $\Omega = F(\underline{x}^{1/\infty})$, so that $G_{\Omega/F} \simeq \widehat{\mathbb{Z}}^3$, and let K be the Kummer extension

$$K = F((x_1 x_2)^{1/4}, x_2^{1/2}).$$

The exponent is 4, which divides $d = 8$. In the notation of Lemma 6.10, $y_1 = (x_1 x_2)^{1/4}$, $y_2 = x_2^{1/2}$, and $y_3 = x_3$. The subgroup $Z = \langle z_1, z_2, z_3 \rangle$ is given by $\underline{z} = \underline{y}^8$, so that $z_1 = y_1^8 = (x_1 x_2)^2$, $z_2 = y_2^8 = x_2^4$, and $z_3 = y_3^8 = x_3^8$. In matrix notation, $\underline{z} = \underline{x} \mathcal{N}^*$, where

$$\mathcal{N}^* = \mathcal{N}_8^* = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Let $\mathcal{Q}^* := \frac{1}{8} \mathcal{N}_8^*$. Since $\mathcal{P}^* \mathcal{N}_8^* = 8\mathcal{J}$, $\mathcal{P}^* = \mathcal{Q}^{*-1}$. Compute

$$\mathcal{Q}^* = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{P}^* = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $K = \mathbb{C}(y_1, y_2, y_3)$, where $\underline{y} = \underline{x} \mathcal{Q}^*$, i.e., $y_1 = (x_1 x_2)^{1/4}$, $y_2 = x_2^{1/2}$, and $y_3 = x_3$. Inversely, $\underline{x} = \underline{y} \mathcal{P}^*$, i.e., $x_1 = y_1^4 y_2^{-2}$, $x_2 = y_2^2$, and $x_3 = y_3$.

By Lemma 6.10d, the degree of K/F is

$$[K : F] = |\det(\mathcal{P}^*)| = 8.$$

By Lemma 6.10c, $\mathcal{P} = [\iota]_{\underline{\psi}}^{\underline{\phi}}$, the transpose of \mathcal{P}^* , is the coordinate expression for the embedding $G_{\Omega/K} \hookrightarrow G_{\Omega/F}$ with respect to bases $\underline{\psi} = \{\psi_1, \psi_2, \psi_3\}$ and $\underline{\phi} = \{\phi_1, \phi_2, \phi_3\}$, whose elements are the Frobenius generators for the y_j and x_i , respectively, as in Lemma 5.15. Thus $\underline{\psi} = \underline{\phi} \mathcal{P}$, i.e., $\psi_1 = \phi_1^4$, $\psi_2 = \phi_1^{-2} \phi_2^2$, and $\psi_3 = \phi_3$.

Theorem 6.13 “Functoriality in $\text{Br}(F)$ ”. Assume Setup 6.6. Suppose $G_{\Omega/F} \simeq \mathbb{Z}_m^c$ is homogeneous, for some $\mathfrak{m} \subset \mathcal{P}$ and cardinal number c . Let $\underline{\psi} = \{\psi_i\}_I$ and $\underline{\phi} = \{\phi_i\}_I$ be compatible homogeneous topological bases for $G_{\Omega/F}$ and $G_{\Omega/K}$, indexed by a linearly ordered set I . Let $\underline{x} = \{x_i\}_I$ and $\underline{y} = \{y_i\}_I$ be respective Kummer dual bases, as in Definition 5.16. Set $\mathcal{P} = (p_{ij}) = [l]_{\underline{\psi}}^{\underline{\phi}}$, and Let $\mathcal{BPC} = \mathcal{D} = \text{diag}\{d_i\}_I$ be a diagonalization of \mathcal{P} as in Lemma 6.3, where $\mathcal{B} = (b_{ij})$ and \mathcal{C} are basis changes. Then

- $\text{Br}^\circ(\Omega/F) = \langle [(x_i, x_j)_{\omega_e}] : i < j, e \in \mathbb{N}(\mathfrak{m}) \rangle$.
- $\text{Br}^\circ(K/F) = \langle \sum_{k < l} b_{kl,ij}^* [(x_k, x_l)_{\omega_{d_i d_j}}] : i < j \rangle$, where $b_{kl,ij}^* = b_{ik}b_{jl} - b_{il}b_{jk}$.
- The restriction homomorphism’s coordinate expression

$$\begin{aligned} \mathcal{P}^{*\wedge 2} : \text{Br}^\circ(\Omega/F) &\longrightarrow \text{Br}^\circ(\Omega/K) \\ [\boldsymbol{\xi}]_{\underline{x}} &\longmapsto [\text{res}(\boldsymbol{\xi})]_{\underline{y}} \end{aligned}$$

is defined explicitly by $\text{res}([(x_i, x_j)_{\omega_d}]) = \sum_{k < l} p_{kl,ij}^* [(y_k, y_l)_{\omega_d}]$, where $p_{kl,ij}^* = p_{ik}p_{jl} - p_{il}p_{jk}$.

d. Suppose $G_{K/F}$ is finite, of order s . Let $\mathcal{N} = (n_{ij}) = [j]_{\underline{\phi}}^{\underline{\psi}}$, the coordinate expression of the transfer. Then the corestriction’s coordinate expression

$$\begin{aligned} \mathcal{N}^{*\wedge 2} \cdot \frac{1}{s} : \text{Br}^\circ(\Omega/K) &\longrightarrow \text{Br}^\circ(\Omega/F) \\ [\boldsymbol{\eta}]_{\underline{y}} &\longmapsto [\text{cor}(\boldsymbol{\eta})]_{\underline{x}} \end{aligned}$$

is defined explicitly by $\text{cor}([(y_i, y_j)_{\omega_d}]) = \sum_{k < l} n_{kl,ij}^* [(x_k, x_l)_{\omega_{ds}}]$, where $n_{kl,ij}^* = n_{ik}n_{jl} - n_{il}n_{jk}$.

Proof. Suppose $e \in \mathbb{N}(\mathfrak{m})$. By the definition of Ω/F , $\Omega^e \cap F^\times / F^{\times e}$ has Kummer basis elements $e\underline{x}$, so (a) follows from Theorem 4.7. Parts (b), (c), and (d) are immediate from Theorem 6.4a,b,c. ■

Remark 6.14. The hypothesis on $G_{\Omega/F}$ holds for Ω equal to the maximal abelian p -extension of F , for then $G_{\Omega/F} \simeq \mathbb{Z}_p^c$. In this case $\text{Br}^\circ(\Omega/F) \simeq \text{Br}(\Omega/F)$. For suppose $\boldsymbol{\xi} = [(x, y)_{\omega_d}]$ for some p -power d and elements $x, y \in F^\times$. Then $F(x^{1/d})$ and $F(y^{1/d})$ are subfields of Ω , so $x, y \in \Omega^d \cap F^\times$, hence $\boldsymbol{\xi} \in \text{Br}^\circ(\Omega/F)$ by Theorem 4.7. Since $\text{Br}(\Omega/F)$ is generated by symbols of p -primary order, it follows that $\text{Br}^\circ(\Omega/F) = \text{Br}(\Omega/F)$.

Example 6.15. Continue Example 6.12. It is easy to diagonalize \mathcal{P} :

$$\mathcal{D} = \mathcal{B}\mathcal{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To compute $\text{Br}^\circ(K/F)$ and the coordinate expressions of res and cor , compute

$$\mathcal{B}^{*\wedge 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathcal{P}^{*\wedge 2} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & 2 \end{pmatrix} \quad \frac{1}{8}\mathcal{N}^{*\wedge 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 4 \end{pmatrix} \quad \mathcal{D}^{*\wedge 2} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

By Theorem 6.13b, $\text{Br}^\circ(K/F)$ is generated by the elements $[(x_i, x_j)_{\omega_{d_i d_j}}] \mathcal{B}^{*\wedge 2}$, so

$$\text{Br}^\circ(K/F) = \langle [(x_1, x_2)_{\omega_8}], [(x_1 x_2, x_3)_{\omega_4}], [(x_2, x_3)_{\omega_2}] \rangle.$$

The restriction and corestriction on basic classes are given by multiplication by $\mathcal{P}^{*\wedge 2}$ and $\frac{1}{8}\mathcal{N}^{*\wedge 2}$,

$$\begin{aligned} \text{res}([(x_1, x_2)_{\omega_d}]) &= 8[(y_1, y_2)_{\omega_d}] & \text{cor}([(y_1, y_2)_{\omega_d}]) &= [(x_1, x_2)_{\omega_d}] \\ \text{res}([(x_1, x_3)_{\omega_d}]) &= 4[(y_1, y_3)_{\omega_d}] - 2[(y_2, y_3)_{\omega_d}] & \text{cor}([(y_1, y_3)_{\omega_d}]) &= 2[(x_1, x_3)_{\omega_d}] + 2[(x_2, x_3)_{\omega_d}] \\ \text{res}([(x_2, x_3)_{\omega_d}]) &= 2[(y_2, y_3)_{\omega_d}] & \text{cor}([(y_2, y_3)_{\omega_d}]) &= 4[(x_2, x_3)_{\omega_d}]. \end{aligned}$$

Here d is any number. It is easy to check that $\text{cor} \cdot \text{res}$ is multiplication by $8 = [K : F]$. For example,

$$\begin{aligned} \text{cor}(\text{res}([(x_1, x_3)_{\omega_d}])) &= \text{cor}(4[(y_1, y_3)_{\omega_d}] - 2[(y_2, y_3)_{\omega_d}]) \\ &= 4(2[(x_1, x_3)_{\omega_d}] + 2[(x_2, x_3)_{\omega_d}]) - 2(4[(x_2, x_3)_{\omega_d}]) \\ &= 8[(x_1, x_3)_{\omega_d}]. \end{aligned}$$

Here, $[(x_1, x_3)_{\omega_d}]$ corresponds to the 3-tuple $(0, 1/d, 0)$. Apply $\mathcal{P}^{*\wedge 2}$ to obtain $(0, 4/d, -2/d)$, which corresponds to $4[(y_1, y_3)_{\omega_d}] - 2[(y_2, y_3)_{\omega_d}]$. Apply $8\mathcal{Q}^{*\wedge 2}$ to $(0, 4/d, -2/d)$ to obtain $(0, 8/d, 8/d - 8/d) = (0, 8/d, 0)$, which corresponds to $8[(x_1, x_3)_{\omega_d}]$, as desired.

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