Agenda

- Primal Network Simplex Method
- Dual Network Simplex Method
- Two-Phase Network Simplex Method
- One-Phase Primal-Dual Network Simplex Method
- Planar Graphs
- Integrality Theorem
Primal Network Simplex Method

Used when all primal flows are nonnegative (i.e., primal feasible).

Pivot Rules:

**Entering arc:** Pick a nontree arc having a negative (i.e. infeasible) dual slack.

**Leaving arc:** Add entering arc to make a cycle. Leaving arc is an arc on the cycle, pointing in the opposite direction to the entering arc, and of all such arcs, it is the one with the smallest primal flow.
**Explanation of leaving arc rule:**

- Increase flow on (d,e).
- Each unit increase produces a unit increase on arcs pointing in the same direction.
- Each unit increase produces a unit decrease on arcs pointing in the opposite direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.
Primal Method—Third Pivot

Entering arc: (c, g)
Leaving arc: (c, e)

Obj value = 335

Optimal!

Obj value = 316
Dual Network Simplex Method

Used when all dual slacks are nonnegative (i.e., dual feasible).

Pivot Rules:

*Leaving arc:* Pick a tree arc having a negative (i.e. infeasible) primal flow.

*Entering arc:* Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the opposite direction, and, of all such arcs, is the one with the smallest dual slack.
Dual Network Simplex Method—Second Pivot

Leaving arc: (d,a)
Entering arc: (b,c)

Obj value = 316

Optimal!
Recall initial tree solution:

Obj value = −62

-8  2  9
f  e  g  c
5  11  28  21
b  a  g  c
6  3  1  5
−4  38  1  −9
−8  0  2  5
−7  14  −2  0
−7  9

Leaving arc: (g,a)
Entering arc: (d,e)

- Remove leaving arc. Need to find a reconnecting arc.
- Consider some reconnecting arc. Add flow to it.
  - If it reconnects in the same direction as leaving arc, such as (f,d), then flow on leaving arc decreases.
  - Therefore, leaving arc’s flow can’t be raised to zero.
  - Therefore, leaving arc can’t leave. No good.

- Consider a potential arc reconnecting in the opposite direction, say (b,c).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.
Example.

- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.
Two-Phase Method—First Pivot

Use dual network simplex method.
Leaving arc: (d,e)  Entering arc: (e,f)

Dual Feasible!
Two-Phase Method—Phase II

- Turn off display of artificial dual slacks.
- Turn on display of dual slacks.
Two-Phase Method—Second Pivot

Entering arc: (g,b)
Leaving arc: (g,f)

Obj value = 500

Obj value = 290
Two-Phase Method—Third Pivot

Entering arc: (f,c)
Leaving arc: (a,f)

Optimal!
Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.
One-Phase Primal-Dual Method

- Artificial flows and slacks are multiplied by a parameter $\mu$.
- In the Figure, $6,1$ represents $6 + 1\mu$.
- **Question:** For which $\mu$ values is dictionary optimal?
- **Answer:**

\[
\begin{align*}
1 + \mu & \geq 0 \quad (a,b) \\
-2 + \mu & \geq 0 \quad (a,c) \\
\mu & \geq 0 \quad (a,d) \\
\mu & \geq 0 \quad (e,a) \\
-3 + \mu & \geq 0 \quad (a,g) \\
\mu & \geq 0 \quad (b,c) \\
3 + \mu & \geq 0 \quad (b,d) \\
\mu & \geq 0 \quad (c,e) \\
-2 + \mu & \geq 0 \quad (f,c) \\
\mu & \geq 0 \quad (a,d) \\
-2 + \mu & \geq 0 \quad (f,e) \\
\mu & \geq 0 \quad (f,b) \\
\mu & \geq 0 \quad (c,e) \\
\mu & \geq 0 \quad (g,d) \\
3 + \mu & \geq 0 \quad (g,e) \\
\end{align*}
\]

That is, $9 \leq \mu < \infty$.

- Lower bound on $\mu$ is generated by arc $(g,d)$.
- Therefore, $(g,d)$ enters.
- Arc $(a,d)$ leaves.
Second Iteration

- Range of $\mu$ values: $2 \leq \mu \leq 9$.
- Entering arc: (a,c)
- Leaving arc: (b,c)

New tree:
Third Iteration

- Range of $\mu$ values: $1.5 \leq \mu \leq 2$.
- Leaving arc: (a,g)
- Entering arc: (g,e)

New tree:
Fourth Iteration

- Range of $\mu$ values: $1 \leq \mu \leq 1.5$.
- A tie:
  - Arc (f,b) enters, or
  - Arc (f,c) leaves.
- Decide arbitrarily:
  - Leaving arc: (f,c)
  - Entering arc: (f,b)
Fifth Iteration

- Range of $\mu$ values: $1 \leq \mu \leq 1$.
- Leaving arc: (f,b)
- Nothing to Enter.

Primal Infeasible!
Online Network Simplex Pivot Tool

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**Planar Networks**

**Definition.** Network is called **planar** if can be drawn on a plane without intersecting arcs.

**Theorem.** Every planar network has a dual—dual nodes are faces of primal network.

Notes:
- Dual node $A$ is “node at infinity”.
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don’t forget node $A$).

**Theorem.** A dual pivot on the primal network is exactly a primal pivot on the dual network.
Theorem. Assuming integer data, every basic feasible solution assigns integer flow to every arc.

Corollary. Assuming integer data, every basic optimal solution assigns integer flow to every arc.