Lemma 2:

Let $R_1, R_2$ be a decomposition of $R$.

If:

- $R_1 \cap R_2 \rightarrow R_1$
- or $R_1 \cap R_2 \rightarrow R_2$

Then:

$$\text{content}(R) = \text{content}(R_1) \times \text{content}(R_2).$$

Proof: we show only:

if $(R_1 \cap R_2 \rightarrow R_1)$ then:

$$\text{content}(R) = \text{content}(R_1) \times \text{content}(R_2).$$

Sketch:

Proof: $\text{content}(R) \leq \text{content}(R_1) \times \text{content}(R_2)$ \(\heartsuit\)

and $\text{content}(R_1) \times \text{content}(R_2) \leq \text{content}(R)$ \(\clubsuit\)

(A) $\text{content}(R) \leq \text{content}(R_1) \times \text{content}(R_2)$

Follows from Lemma 1.
sketch:

Let \( t_1 \) be an arb. tuple \( \in \text{content}(R_1) \)
\( t_2 \) be an arb. tuple \( \in \text{content}(R_2) \)

\( \implies \) \( t_1 \times t_2 \) is arb. tuple \( \in \text{content}(R_1 \times \text{content}(R_2)) \)

show: \( t_1 \times t_2 \in \text{content}(R) \).

Let \( R = (I, U_1, U_2) \)

\[ R_1 = (I, U_1) \quad (R_1 \cap R_2 = I) \]
\[ R_2 = (I, U_2) \]

Write tuples \( t_1 \) and \( t_2 \) as:

\[ t_1 = (i_1, u_1) \]
\[ t_2 = (i_2, u_2) \]

Consider 2 cases:

① \( i_1 \neq i_2 \).
② \( i_1 = i_2 \).

① \( i_1 \neq i_2 \):

\( t_1 \times t_2 = \text{no tuple} \).

Then: \( t_1 \times t_2 \in \text{content}(R) \)

is trivially satisfied!
(2) \( i = i_2 \)

\[
\begin{align*}
t_1 &= (i, u_1) = (i, u_1) \\
t_2 &= (i, u_2) = (i, u_2).
\end{align*}
\]

\( t_1 \times t_2 = (i, u_1) \times (i, u_2) = (i, u_1, u_2) \)

We must show that:

\( t = (i, u_1, u_2) \in \text{content}(R) \)

Because:

(A) \( t_1 = (i, u_1) \in \text{content}(R_1) \)

there must be a tuple like this in \( \text{content}(R) \)

\( (i, u_1, ???) \in \text{content}(R) \)

(B) \( t_2 = (i, u_2) \in \text{content}(R_2) \)

there must be a tuple like this in \( \text{content}(R) \)

\( (i, ???, u_2) \in \text{content}(R) \)

(C) Given that: \( R_1 \cap R_2 = \emptyset \Rightarrow R_1 \)

We have:

\[
\begin{array}{c}
\text{content}(R) \\
(i, u_1, ???) \\
(i, ???, u_2) \rightarrow (i, u_1, u_2) \in R
\end{array}
\]

must be \( u_1 \) !!!