Multiresolution on the Sphere

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Abstract. In this paper we study some basic tools for the construction of multiscale systems on the unit sphere. Particularly, we emphasize on properties of spherical harmonics and Legendre functions. Based on these orthogonal systems we discuss in some detail the decomposition of the classical Hilbert space on the sphere into subspaces of different level. To this end we explain different bases and frames. In our examples the building blocks consist of polynomials and spherical radial basis functions.

1 Introduction

The management of large sets of meteorological, geophysical or crystallographic data collected over the earth or other spheres is a challenging task. For synthesizing and analyzing such data a multiscale approach seems to be particularly adequate. The aim of these notes is to summarize background definitions and some known approximation methods useful for studying multiscale approximations on the sphere. In contrast to the situation for $L^2(\mathbb{R}^1)$ or $L^2(\mathbb{R}^m)$ the construction of suitable multiscale bases for the sphere is more complicated. Let us only mention here that apart from the five regular polytopes there are no equidistant point systems on the sphere. Hence, wavelet spaces cannot be simply generated by equidistant translates of one generating function. However, a lot of different approaches try to overcome these difficulties. Let us mention here only the work of Dahlke et al. [4], [30]. In [27] Schröder and Sweldens describe a simple technique for constructing biorthogonal wavelets on the sphere. Their construction is based on the lifting scheme. There one can find also further applications to computer graphics. Lyon and Schumaker [13] present a construction of spherical wavelets based on tensor products of polynomial splines and trigonometric splines. The main ingredient in our constructions turn out to be the well-known spherical harmonics. These homogeneous and harmonic polynomials restricted to the sphere play the same fundamental role as the sine and cosine frequencies for periodic functions or e.g. Legendre polynomials for the functions on the interval $[-1,1]$. In these univariate settings a lot of results concerning multiscale decompositions were published in the last years (see e.g. [21], [11], [19] and [5]). Here we follow these ideas and relate all our constructions to these basic blocks of the underlying orthonormal system of spherical harmonics.
In this direction a very useful source is the textbook of Freeden et al. [7], cf. also the many references therein. However, let us also stress the fact that polynomials cannot give optimal localization on the sphere. This behaviour can be studied in terms of uncertainty inequalities in a more quantitative way (see e.g. [16]).

In our paper we summarize some corresponding approximation procedures. Two different examples will be studied. A polynomial approach can be used very efficiently by applying the FFT and related fast algorithms. A second class of examples is based on spherical basis functions. In this case scattered data approximation is possible. Analogous questions are also handled by Iske in this Volume [10]. For more details related to our considerations compare e.g. [20], [22]. Different applications can also be found in [12], [16] or [24].

2 Preliminaries

2.1 Notations

Every point \( x \in \mathbb{R}^3 \backslash \{0\} \) in cartesian coordinates can be written in spherical coordinates \((r, \vartheta, \varphi)\) by \( x = (x_1, x_2, x_3)^T = (r \sin \vartheta \cos \varphi, r \sin \vartheta \sin \varphi, r \cos \vartheta)^T\), \( r \in \mathbb{R}^+, \vartheta \in [0, \pi] \) and \( \varphi \in [0, 2\pi) \). Here \( \vartheta \) is the longitudinal, \( \varphi \) the equatorial angle and \( r \) is the radius \( r = ||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2} \). Furthermore, \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), i.e.

\[
S^2 := \{ \xi \in \mathbb{R}^{m+1} : ||\xi||_2 = 1 \}.
\]

The inner product of \( \xi, \eta \in S^2 \) with spherical coordinates \((\vartheta, \varphi)\) and \((\vartheta', \varphi')\) can be written as

\[
\xi : \eta = \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cos (\varphi - \varphi').
\]

The Hilbert space \( L^2(S^2) \) of square-integrable functions on the sphere is defined as usual by \( f \in L^2(S^2) \) iff

\[
||f|| := ||f||_{L^2(S^2)} := \left( \int_{S^2} |f(\xi)|^2 d\omega(\xi) \right)^{1/2} < \infty.
\]

With the surface element \( d\omega(\xi) \) the inner product is given by

\[
\langle f, g \rangle := \int_{S^2} f(\xi) \overline{g(\xi)} d\omega(\xi) \quad \text{for } f, g \in L^2(S^2)
\]

and can be transformed to

\[
\langle f, g \rangle = \int_0^\pi \int_0^{2\pi} f(\vartheta, \varphi) \overline{g(\vartheta, \varphi)} \sin \vartheta d\vartheta d\varphi.
\]  

Our next aim is to define particular polynomial spaces. Let \( \text{Hom}_n(\mathbb{R}^3) \) be the space of homogeneous polynomials of degree \( n \), i.e. polynomials \( T_n \), which
satisfy $T_n(\lambda x) = \lambda^n T_n(x)$ for arbitrary $\lambda \in \mathbb{C}$. Moreover, $\text{Harm}_n(\mathbb{R}^3)$ is the space of homogeneous harmonic polynomials, i.e.

$$\text{Harm}_n(\mathbb{R}^3) := \{ T_n \in \text{Hom}_n(\mathbb{R}^3) : \Delta_x T_n(x) = 0, x \in \mathbb{R}^3 \}.$$  \hspace{1cm} (2)

Here $\Delta_x$ is the Laplace operator, given by

$$\Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$  

The Laplace equation from (2) can be also written as

$$\Delta_{(n, \theta, \varphi)} T_n := \frac{\partial^2 T_n}{\partial r^2} + \frac{2}{r} \frac{\partial T_n}{\partial r} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial T_n}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 T_n}{\partial \varphi^2} = 0.$$  

The dimension of these spaces can be computed as

$$\dim(\text{Hom}_n(\mathbb{R}^3)) = \frac{(n + 1)(n + 2)}{2} \quad \text{and} \quad \dim(\text{Harm}_n(\mathbb{R}^3)) = 2n + 1.$$  

(For the proof cf. [7, Chap. 2.2 and 2.3]). Finally we mention the orthogonal group $SO(3)$, i.e.

$$SO(3) := \{ T \in \mathbb{R}^{3 \times 3} : T^T = T^{-1}, \det(T) = 1 \},$$  \hspace{1cm} (3)

which describes the set of rotations on the sphere $S^2$. 

\textbf{Fig. 1.} Spherical coordinates $(r, \theta, \varphi)$
2.2 Approximation on the Sphere

Let $\mathcal{V}$ be a finite dimensional subspace of $L^2(\mathbb{S}^2)$ with $\dim \mathcal{V} = K$. With respect to a basis $\{\varphi_k\}_{k=1}^K$ of $\mathcal{V}$ any $f \in \mathcal{V}$ has a unique representation

$$f = \sum_{k=1}^K a_k \varphi_k \quad \text{with} \quad a_k \in \mathbb{C}.$$ 

Now we assume to have a set of $L$ data on the sphere. More precisely, for $\xi_l \in \mathbb{S}^2, l = 1, \ldots, L$ we have data $f_l \in \mathbb{C}$ for $l = 1, \ldots, L$. Denoting

$$f := (f_1, \ldots, f_L)^T \in \mathbb{C}^L,$$
$$a := (a_1, \ldots, a_K)^T \in \mathbb{C}^K$$

and

$$\Phi := \begin{pmatrix} \varphi_1(\xi_1) & \cdots & \varphi_K(\xi_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(\xi_L) & \cdots & \varphi_K(\xi_L) \end{pmatrix} \in \mathbb{C}^{L \times K},$$

we can formulate our approximation problem in the following way:

Find $\tilde{a} \in \mathbb{C}^K$, with $||f - \Phi \tilde{a}||_2 \leq ||f - \Phi a||_2$ for all $a \in \mathbb{C}^K$ \hspace{1cm} (4)

or, equivalently,

$$\min_{a \in \mathbb{C}^K} ||f - \Phi a||_2.$$

Here $||\cdot||_2$ denotes the usual euclidean norm. Let us now distinguish the cases $K \leq L$ and $K > L$.

The Least Squares Problem For $K \leq L$, the solution of the approximation problem (4) is given by the normal equations

$$\Phi^H \Phi a = \Phi^H f.$$

Assuming that the matrix $\Phi$ has full rank, i.e. $\text{rank}(\Phi) = K$, we obtain

$$a = (\Phi^H \Phi)^{-1} \Phi^H f.$$ 

By writing $\varphi(\xi) := (\varphi_1(\xi), \ldots, \varphi_K(\xi))^T$ it follows

$$f(\xi) = a^H \varphi(\xi) = \Phi^H \Phi (\Phi^H \Phi)^{-1} \varphi(\xi). \quad \hspace{1cm} (5)$$

Moreover, defining the function $G(c, \xi_l)$ independent of the data $f_l$, $l = 1, \ldots, L$, as

$$G(\xi, \xi_l) := \varphi(\xi_l)^H (\Phi^H \Phi)^{-1} \varphi(\xi), \quad \text{for} \quad l = 1, \ldots, L,$$

we can rewrite (5) as

$$f(\xi) = \sum_{l=1}^L f_l G(\xi, \xi_l).$$
The Optimization Problem If $K > L$, our approximation problem (4) is a nonlinear optimization problem (see e.g. [1, Chap. 1])

$$\min_{a \in \mathbb{R}^K} ||a||_2 \quad \text{with constraint } \Phi a = f.$$ (6)

Again we assume that the matrix $\Phi$ has full rank, i.e. $\text{rank}(\Phi) = L$. Then the unique solution $a$ of (6) is given by

$$a = \Phi^H (\Phi \Phi^H)^{-1} f.$$

Hence,

$$\overline{f}(\xi) = f^H (\Phi \Phi^H)^{-1} \Phi \varphi(\xi).$$

Defining here

$$G(\xi, \xi_l) := [(\Phi \Phi^H)^{-1}]_l \Phi \varphi(\xi), \quad l = 1, \ldots, L,$$

where $[(\Phi \Phi^H)^{-1}]_l$ is the $l$-th row of the matrix, we obtain

$$f(\xi) = \sum_{l=1}^L f_l G(\xi, \xi_l).$$

Note that the matrix $(\Phi \Phi^H)^{-1} \Phi^H$ from the least squares problem and the matrix $\Phi^H (\Phi \Phi^H)^{-1}$ from the optimization problem are also known as Moore-Penrose pseudoinverses.

2.3 Multiscale Decompositions and Wavelet Spaces

**Definition 1.** A sequence $\{\mathcal{V}_j\}_{j \in \mathbb{N}}$ of finite dimensional subspaces of $L^2(S^2)$ will be called a multiresolution analysis of $L^2(S^2)$ if the following conditions are satisfied:

(M1) $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ for all $j \in \mathbb{N},$

(M2) $\text{Closure} \left( \bigcup_{j \in \mathbb{N}} \mathcal{V}_j, \| \cdot \| \right) = L^2(S^2).$

Usually, a definition of multiresolution includes a condition on the intersection of the spaces $\mathcal{V}_j$. In our setting from (M1) it follows immediately $\bigcap_{j \in \mathbb{N}} \mathcal{V}_j = \mathcal{V}_1$. For further references see [30], [7, p. 241], [22], [20].

Now we define wavelet spaces $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, i.e.

$$\mathcal{W}_j := \mathcal{V}_{j+1} \ominus \mathcal{V}_j,$$

which means nothing else than

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j.$$ (7)
Lemma 1. Let \( \{ \mathcal{V}_j \}_{j=1}^{\infty} \) be a multiresolution analysis of \( L^2(\mathbb{S}^2) \) and let \( \mathcal{W}_j \) for \( j \in \mathbb{N} \) be the corresponding wavelet spaces defined by (7). With \( \mathcal{W}_0 := \mathcal{V}_1 \) we have

\[
L^2(\mathbb{S}^2) = \bigoplus_{j=0}^{\infty} \mathcal{W}_j.
\]

Defining the operators \( R_j \) and \( Q_j \) to be the orthogonal projections \( R_j : L^2(\mathbb{S}^2) \to \mathcal{V}_j \) and \( Q_j : L^2(\mathbb{S}^2) \to \mathcal{W}_j \) for \( j \in \mathbb{N} \) we can illustrate the decomposition in the following way:

\[
\begin{array}{c}
L^2(\mathbb{S}^2) \cdots \mathcal{V}_{j+1} R_j \mathcal{V}_j R_{j-1} \mathcal{V}_{j-1} \cdots \mathcal{V}_2 R_1 \mathcal{V}_1 =: \mathcal{W}_0 \\
\cdots \quad \mathcal{W}_j \quad \mathcal{W}_{j-1} \quad \cdots \quad \mathcal{W}_1
\end{array}
\]

Moreover, for \( f \in L^2(\mathbb{S}^2) \) we conclude

\[
R_{j+1}f = R_1f + \sum_{k=1}^{j} Q_kf.
\]

To combine the approximation problems with the multiresolution approach the following notation will be useful.

Definition 2. A sequence of subsets \( \{ \mathcal{N}_j \}_{j \in \mathbb{N}} \) of the unit sphere \( \mathbb{S}^2 \) will be called refinement grid of \( \mathbb{S}^2 \), if every \( \mathcal{N}_j \) satisfies:

(G1) The set \( \mathcal{N}_j \) consists of finitely many points \( \xi_j^i \in \mathbb{S}^2 \), i.e. \( |\mathcal{N}_j| = L_j < \infty \).

(G2) \( \mathcal{N}_j \subset \mathcal{N}_{j+1} \) for \( j \in \mathbb{N} \).

(G3) The union \( \bigcup_{j \in \mathbb{N}} \mathcal{N}_j \) is dense in \( \mathbb{S}^2 \).

Hence, we have a hierarchical sequence of grids on the sphere.

2.4 Legendre Functions

It turns out that the definition of a basic orthogonal system on the sphere is strongly connected to Legendre functions. We start with some straightforward definitions and properties for further use.

Legendre Polynomials One of the standard definitions of the Legendre polynomials is based on the Gram-Schmidt orthonormalization of the monomials \( 1, t, t^2, \ldots \) on \( t \in [-1,1] \) with respect to the inner product

\[
\langle f, g \rangle_{[-1,1]} := \int_{-1}^{1} f(t)g(t) \, dt.
\]
Here we only modify the normalization.

**Definition 3.** The Legendre polynomials \( P_n : [-1, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}_0 \) are uniquely determined by the following conditions:

1. \( P_n \) is a polynomial of degree \( n \),
2. \( \int_{-1}^{1} P_n(t)P_m(t) \, dt = 0 \) for \( m \neq n \),
3. \( P_n(1) = 1 \).

Note that the conditions 1.) and 2.) determine the polynomials up to a constant factor. Based on this one can easily prove that \( P_n \) has \( n \) real simple zeros inside the interval \((-1, 1)\). Hence, the normalization 3.) makes sense because of \( P_n(1) \neq 0 \). Equivalent definitions are given by the following characterizations:

The Rodrigues formula:

\[
P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} ((t^2 - 1)^n).
\]

The three-term-recurrence relation: \( P_0(t) = 1, P_1(t) = t \) and

\[
(n + 1)P_{n+1}(t) = (2n + 1)tP_n(t) - nP_{n-1}(t), \quad n = 1, 2, \ldots
\]

The solution of the Legendre differential equation

\[
(1 - t^2)P_n''(t) - 2tP_n'(t) + n(n + 1)P_n(t) = 0
\]

with boundary condition \( P_n(1) = 1 \).

The explicit representation:

\[
P_n(t) = \sum_{j=0}^{[n/2]} (-1)^j \frac{(2n - 2j)!}{2^n (n - 2j)! (n - j)! j!} t^{n-2j}.
\]

In the following result we establish the generating function for Legendre polynomials.

**Theorem 1.** For all \( t \in [-1, 1] \) and for all \( h \in (-1, 1) \) it holds

\[
\sum_{n=0}^{\infty} h^n P_n(t) = \frac{1}{\sqrt{1 - 2ht + h^2}}
\]

For the proof see [7, p. 43f]. By differentiation with respect to \( h \) and some elementary computations one immediately obtains the following equation.

**Corollary 1.** For all \( t \in [-1, 1] \) and for all \( h \in (-1, 1) \) it holds

\[
G_h(t) := \sum_{n=0}^{\infty} (2n + 1)h^n P_n(t) = \frac{1 - h^2}{(1 - 2ht + h^2)^{3/2}}
\]

This function \( G_h \) for \( h \in (0, 1) \) is called Poisson kernel.
Associated Legendre Functions

Definition 4. The functions $P^k_n : [-1,1] \to \mathbb{R}$ with $k \in \mathbb{N}_0$ and $n = k, k+1, \ldots$

$$P^k_n(t) = 
\left(\frac{(n-k)!}{(n+k)!}\right)^{1/2} \left(1 - t^2\right)^{k/2} \frac{d^k}{dt^k} P_n(t)$$

are called associated Legendre functions.

Note that $P^0_n = P_n$ and that $P^k_n$ is a polynomial for even $k$. A detailed investigation of these associated Legendre functions can be found in [29, Chap. 5] or [18]. Again there are a lot of equivalent characterizations of these functions.

The Rodrigues formula:

$$P^k_n(t) = \frac{1}{2^n n!} \left(\frac{(n-k)!}{(n+k)!}\right)^{1/2} \left(1 - t^2\right)^{k/2} \frac{d^n}{dt^n} (1 - t^2)^n.$$

The recursive representation: $P^0_n(t) := 0$, $P^1_n(t) := \frac{\left(2n\right)!^{1/2}}{2^n n!} (1 - t^2)^n/2$ and

$$\left((n-k+1)(n+k+1)\right)^{1/2} P^k_{n-1}(t) = (2n+1)t P^k_n(t) - ((n-k)(n+k))^{1/2} P^k_{n+1}(t).$$

The solution of the general Legendre differential equation

$$(1 - t^2) \frac{d^2 P^k_n(t)}{dt^2} - 2t \frac{d P^k_n(t)}{dt} + \left(n(n+1) - \frac{k^2}{1 - t^2}\right) P^k_n(t) = 0$$

with boundary condition $P^k_n(1) := \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{otherwise}. \end{cases}$

The explicit representation:

$$P^k_n(t) = \left(\frac{(n-k)!}{(n+k)!}\right)^{1/2} \frac{1}{2^n} \sum_{j=0}^{\left[\frac{n-k}{2}\right]} (-1)^j (1 - t^2)^{k/2} \frac{(2n-2j)!}{(n-2j-k)!(n-j)j!} t^{n-2j-k}. \quad (9)$$

The following result establishes an important orthogonality relation for these functions.

Theorem 2. The associated Legendre functions $P^k_m, P^k_n$ with $k \leq \min(m, n)$ satisfy

$$\int_{-1}^1 P^k_m(t) P^k_n(t) \, dt = \frac{2}{2n+1} \delta_{m,n}.$$
3 Spherical Harmonics

3.1 Definition and Properties

Here we introduce the essential building blocks of our analysis, namely the spherical harmonics as elements of an orthonormal system on the sphere. For further reference cf. [7], [29, Chap. 5] and [6, Chap. 6.3].

**Theorem 3.** The functions \( T_{n,k} \) for \( n \in \mathbb{N}_0 \) and \( k = -n, \ldots, n \) defined by

\[
T_{n,k}(r, \vartheta, \varphi) := r^n P_n^k(\cos \vartheta) e^{ik \varphi}
\]

satisfy \( T_{n,k} \in \text{Harm}_n(\mathbb{R}^3) \).

Note that for simplicity we use the same symbol \( T_{n,k} \) for the cartesian and the spherical coordinates, respectively.

**Proof.** An elementary calculation shows that \( T_{n,k} \) is harmonic. Now we demonstrate that \( T_{n,k} \) is also a homogeneous polynomial. First we assume \( k \geq 0 \).

The other case \( k < 0 \) differs only by some sign-factors. By definition we have

\[
P_n^k(\cos \vartheta) = \sin^k \vartheta \sum_{j=0}^{\frac{n+k}{2}} a_{n,k,j} \cos^{n-k-2j} \vartheta.
\]

Then

\[
T_{n,k}(r, \vartheta, \varphi) = r^n e^{ik \varphi} P_n^k(\cos \vartheta)
\]

\[
= (re^{i\varphi} \sin \vartheta)^k \sum_{j=0}^{\frac{n+k}{2}} a_{n,k,j} r^{2j} (r \cos \vartheta)^{n-k-2j}
\]

\[
= (r \sin \vartheta \cos \vartheta + ir \sin \vartheta \sin \vartheta)^k \sum_{j=0}^{\frac{n+k}{2}} a_{n,k,j} r^{2j} (r \cos \vartheta)^{n-k-2j}.
\]

With the transformation \( x = (x_1, x_2, x_3) = (r \sin \vartheta \cos \varphi, r \sin \vartheta \sin \varphi, r \cos \vartheta) \) one concludes

\[
T_{n,k}(x_1, x_2, x_3) = (x_1 + ix_2)^k \sum_{j=0}^{\frac{n+k}{2}} a_{n,k,j} (x_1^2 + x_2^2 + x_3^2)^j x_3^{n-k-2j}.
\]

This is a polynomial in \( x \), whose homogeneity is easy to see. \( \square \)

Now we restrict these homogeneous harmonic polynomials of degree \( n \) to the unit sphere \( S^2 \).

**Definition 5.** Let \( \mathcal{H}_n := \text{Harm}_n(S^2) := \text{Harm}_n(\mathbb{R}^3) \big|_{S^2} \). Functions \( Y_n \in \mathcal{H}_n \) will be called spherical harmonics of degree \( n \).
For the dimension of $H_n$ one can prove $\dim(H_n) = \dim(Harm_n(\mathbb{R}^3)) = 2n+1$, (cf. [7, Chap. 2.2]). Using the polynomials from Theorem 3 we obtain immediately an orthonormal basis for $H_n$.

**Theorem 4.** For arbitrary $n \in \mathbb{N}_0$ the functions $\{Y_{n,k}\}_{k=-n}^n$, defined by

$$Y_{n,k}(\vartheta, \varphi) := \sqrt{\frac{2n+1}{4\pi}} P_n^{|k|}(\cos \vartheta) e^{ik\varphi},$$

constitute an orthonormal basis for $H_n$, where the orthogonality is with respect to the $L^2$-inner product given in (1).

**Proof.** We compute

$$\langle Y_{n,j}, Y_{n,k} \rangle = \frac{2n+1}{4\pi} \int_{\mathbb{S}^2} P_n^{|j|}(\cos \vartheta) P_n^{|k|}(\cos \vartheta) e^{i(j-k)\varphi} d\omega(\vartheta, \varphi)$$

$$= \frac{2n+1}{4\pi} \int_0^\pi P_n^{|j|}(\cos \vartheta) P_n^{|k|}(\cos \vartheta) \sin \vartheta d\vartheta \int_0^{2\pi} e^{i(j-k)\varphi} d\varphi$$

$$= \frac{2n+1}{2} \delta_{j,k} \int_0^\pi P_n^{|j|}(\cos \vartheta) P_n^{|k|}(\cos \vartheta) \sin \vartheta d\vartheta.$$

Hence, $\{Y_{n,k}\}_{k=-n}^n$ is an orthogonal system. With the substitution $t = \cos \vartheta$ we obtain

$$\langle Y_{n,k}, Y_{n,k} \rangle = \frac{2n+1}{2} \int_{-1}^1 \left| P_n^{|k|}(t) \right|^2 dt = 1.$$

The orthonormal system consists of $2n+1 = 2n+1$ elements. Thus $\{Y_{n,k}\}_{k=-n}^n$ is an orthonormal basis of $H_n$. \qed

From the representation of the basis given in Theorem 4 and from Theorem 2 we obtain the orthogonality of the spherical harmonics of different polynomial degrees

$$\langle Y_n, Y_m \rangle = \int_0^\pi \int_0^{2\pi} Y_n(\vartheta, \varphi) \overline{Y_m(\vartheta, \varphi)} \sin \vartheta d\vartheta d\varphi = 0 \quad \text{for } m \neq n. \quad (10)$$

Hence, the spaces $H_n$ and $H_m$ for $n \neq m$ are orthogonal to each other.

### 3.2 The Addition Theorem

One of the main tools for the whole theory is the so-called Addition Theorem. Different methods of proof can be found in [18, p. 9ff.], [7, Chap. 2.3] or in a more general version in [28, p. 9ff.].

**Theorem 5 (Addition Theorem), [7, p. 37]** Let $\{Y_{n,k}\}_{k=-n}^n$ be an $L^2(\mathbb{S}^2)$-orthonormal basis of $H_n$. Then for all $\xi, \eta \in \mathbb{S}^2$

$$\sum_{k=-n}^n Y_{n,k}(\xi) \overline{Y_{n,k}(\eta)} = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta).$$
Fig. 2. Real part of the spherical harmonics $Y_{n,k}$ from (13). Left: $Y_{15,7}$, right: $Y_{31,20}$

\textbf{Proof.} For every pair $\xi, \eta \in S^2$ there exists a matrix $T \in SO(3)$ (cf. (3)), such that $T\xi = \eta$. Moreover, $\xi \cdot \eta = T\xi \cdot T\eta$. The elements of the group $SO(3)$ describe the set of rotations of the sphere. The set of rotations $T\xi$ with fixed $\xi$ (i.e. $T\xi\xi = \xi$) constitutes a subgroup. Let $\{Y_{n,k}\}_{k=-n}^{n}$ be an orthonormal basis of $H_n$ and $T \in SO(3)$, yielding $Y_{n,k}(T\omega) \in H_n$. Thus for every $Y_{n,j}$ there exists a representation

$$Y_{n,j}(T\omega) = \sum_{k=-n}^{n} a_{k,j} Y_{n,k} \quad \text{for } j = -n, \ldots, n.$$  

The functions $\{Y_{n,k}\}_{k=-n}^{n}$ form an orthonormal basis of $H_n$, which shows

$$\int_{S^2} Y_{n,i}(T\xi) \overline{Y_{n,j}(T\xi)} d\omega(\xi) = \sum_{k=-n}^{n} a_{k,i} \delta_{k,j}.$$  

As $T$ is a rotation, it holds

$$\sum_{k=-n}^{n} a_{k,i} \overline{a_{k,j}} = \delta_{ij}.$$  

Consider now the functions $K_n : S^2 \times S^2 \to \mathbb{C}$ with

$$K_n(\xi, \eta) := \sum_{k=-n}^{n} Y_{n,k}(\xi) \overline{Y_{n,k}(\eta)}.$$

(11)
For all $\xi, \eta \in \mathbb{S}^2$ and for all $T \in SO(3)$ it follows

$$K_n(T\xi, T\eta) = \sum_{k=-n}^{n} Y_{n,k}(T\xi) \overline{Y_{n,k}(T\eta)}$$

$$= \sum_{k=-n}^{n} \left( \sum_{i=-n}^{n} a_{k,i} Y_{n,i}(\xi) \right) \left( \sum_{j=-n}^{n} a_{k,j} Y_{n,j}(\eta) \right)$$

$$= \sum_{i=-n}^{n} \sum_{j=-n}^{n} a_{k,i} a_{k,j} Y_{n,i}(\xi) \overline{Y_{n,j}(\eta)}$$

$$= \sum_{i=-n}^{n} \sum_{j=-n}^{n} Y_{n,i}(\xi) \overline{Y_{n,j}(\eta)} \sum_{k=-n}^{n} a_{k,i} a_{k,j}$$

$$= \sum_{j=-n}^{n} Y_{n,j}(\xi) \overline{Y_{n,j}(\eta)} = K_n(\xi, \eta).$$

Note that this approach is independent of the particular choice of the orthonormal basis. For arbitrary $\xi, \eta \in \mathbb{S}^2$ it holds $K_n(\xi, \eta) = K_n(T\xi, T\eta) = K_n(\xi, T\xi)$. This implies that $K_n(\xi, \eta)$ depends only on the angle between $\xi$ and $\eta$. So we define $K_n(\xi, \eta) := K_n(\xi, \eta)$. From the equation (11) it follows $K_n(\xi, \eta) = K_n(\eta \cdot \xi) = K_n(\xi, \eta)$. Hence, $K_n$ is real-valued. Furthermore, for a fixed $\eta \in \mathbb{S}^2$ the function $K_n(\cdot, \eta)$ is in $\mathcal{H}_n$. Hence, $K_n(t)$ is a polynomial of degree $n$ in $t$. From (10) we know that $K_n(\cdot, \eta)$ is orthogonal to $K_m(\cdot, \eta)$ for arbitrary but fixed $\eta \in \mathbb{S}^2$ and for $n \neq m$. Hence,

$$\langle K_n(\cdot, \eta), K_m(\cdot, \eta) \rangle = \int_{\mathbb{S}^2} K_n(\xi, \eta) K_m(\xi, \eta) \, d\omega(\xi) = 0.$$  

Here we choose $\eta = (0,0,1)^T$, i.e. in spherical coordinates $(\vartheta, \varphi) = (0,0)$. For any other $\eta$ one would apply a transformation $T \in SO(3)$ which maps $\eta$ onto $(0,0,1)^T$ so that $T\eta = (0,0,1)^T$. Summarizing we have for $n \neq m$

$$0 = \langle K_n(\cdot, \eta), K_m(\cdot, \eta) \rangle = \int_0^\pi \int_0^{2\pi} K_n(\cos \vartheta) K_m(\cos \vartheta) \sin \vartheta \, d\vartheta \, d\varphi.$$  

Using $t = \cos \vartheta$ we obtain

$$\int_{-1}^{1} K_n(t) K_m(t) \, dt = 0 \quad \text{for } n \neq m.$$  

From Definition 3 we have

$$K_n(\xi, \eta) = c_n P_n(\xi \cdot \eta).$$

Finally,

$$K_n(\xi, \xi) = K_n(1) = \sum_{k=-n}^{n} |Y_{n,k}(\xi)|^2 = c_n P_n(1) = c_n.$$
and integration over the sphere \( S^2 \) yields

\[
2n + 1 = 4\pi c_n
\]

which concludes the proof. \( \square \)

The function \( K_n : S^2 \times S^2 \to \mathbb{R} \) with \( K_n(\xi \cdot \eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta) \) is called reproducing kernel. For arbitrary \( f \in \mathcal{H}_n \) one has the reproduction property

\[
\langle f, K_n(\cdot \cdot \eta) \rangle = f(\eta) \quad \text{with} \quad \eta \in S^2. \tag{12}
\]

Using an orthonormal basis \( \{Y_{n,k}\}_{k=-n}^n \) of \( \mathcal{H}_n \) and writing \( f \) as its Fourier series \( f = \sum_{k=-n}^n f_k Y_{n,k} \) we obtain (12) by the Addition Theorem 5 as

\[
\langle f, K_n(\cdot \cdot \eta) \rangle = \sum_{k=-n}^n f_k Y_{n,k}(\eta) \int_{S^2} Y_{n,k}(\xi) \overline{Y_{n,l}(\xi)} \, d\omega(\xi) = f(\eta).
\]

### 3.3 Polynomial Approximation

Now we study an approximation process on the sphere based on polynomials. We refer to [7, Chap. 2.2]. Let \( \Pi_N(S^2) \) be the space of polynomials of maximal degree \( N \) over the sphere \( S^2 \). Then one has the decomposition

\[
\Pi_N(S^2) = \text{Hom}_N(S^2) \oplus \text{Hom}_{N-1}(S^2).
\]

More precisely, with \( \Pi_N(S^2) \) and \( \text{Hom}_N(S^2) \) we mean \( \Pi_N(\mathbb{R}^3)|_{S^2} \) and \( \text{Hom}_N(\mathbb{R}^3)|_{S^2} \), respectively. We obtain the following decomposition of \( \Pi_N \).

**Theorem 6.** Every polynomial of degree \( N \) restricted to the sphere \( S^2 \) can be written as a sum of spherical harmonics of degree \( N \), i.e.

\[
\Pi_N(S^2) = \bigoplus_{n=0}^{N} \mathcal{H}_n.
\]

For the proof see [7, Corollary 2.2.5]. Particularly, any function \( f \in L^2(S^2) \) can be approximated by spherical harmonics in the \( L^2(S^2) \)-sense up to arbitrary precision. From the \( L^2(S^2) \)-orthogonality of the spaces \( \mathcal{H}_n \) for \( n \in \mathbb{N}_0 \), we obtain the dimension

\[
\dim \Pi_N(S^2) = \sum_{n=0}^{N} \dim \mathcal{H}_n
\]

\[
= \sum_{n=0}^{N} (2n + 1) = (N + 1)^2.
\]
In the finite dimensional Hilbert space $\mathcal{H}_n$ with the inner product from $L^2(\mathbb{S}^2)$ any function $Y_n \in \mathcal{H}_n$ can be represented with respect to an orthonormal basis $\{Y_{n,k}\}_{k=-n}^n$ as a Fourier sum

$$Y_n = \sum_{k=-n}^n \langle Y_n, Y_{n,k} \rangle Y_{n,k}.$$ 

Finally, any $Y \in \Pi_N$ can be written as

$$Y = \sum_{n=0}^N \sum_{k=-n}^n \langle Y, Y_{n,k} \rangle Y_{n,k}.$$

Note that from Theorem 4 we obtain immediately the following orthonormal basis for $\Pi_N(\mathbb{S}^2)$

$$Y_{n,k}(\theta, \phi) = \frac{2n+1}{4\pi} P_n^{|k|}(\cos \theta) e^{ik\phi}, \quad \text{for } n = 0, \ldots, N \text{ and } k = -n, \ldots, n.$$

**Remark.** For the approximation problem from Section 2.2, choosing the space $\mathcal{V} = \Pi_N(\mathbb{S}^2)$ for an $N \in \mathbb{N}_0$ with a basis $\{\varphi_i\}_{i=1}^{N+1} = \{Y_{n,k}\}_{n=0,k=-n}^N$ we obtain

$$\Phi = \begin{pmatrix} Y_{0,0}(\xi_1) & Y_{1,-1}(\xi_1) & Y_{1,0}(\xi_1) & \cdots & Y_{N,N-1}(\xi_1) & Y_{N,N}(\xi_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{0,L}(\xi_L) & Y_{1,-1}(\xi_L) & Y_{1,0}(\xi_L) & \cdots & Y_{N,N-1}(\xi_L) & Y_{N,N}(\xi_L) \end{pmatrix} \in \mathbb{R}^{L \times (N+1)^2}.$$

The matrix $\Phi \Phi^H$ can be computed with the help of the Addition Theorem 5 as

$$\Phi \Phi^H = \frac{1}{4\pi} \begin{pmatrix} \sum_{n=0}^N (2n+1) P_n(\xi_1 \cdot \xi_1) & \cdots & \sum_{n=0}^N (2n+1) P_n(\xi_1 \cdot \xi_L) \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^N (2n+1) P_n(\xi_L \cdot \xi_1) & \cdots & \sum_{n=0}^N (2n+1) P_n(\xi_L \cdot \xi_L) \end{pmatrix}.$$

### 3.4 The Polynomial Multiscale Decomposition

The above discussion can be summarized as follows.

**Theorem 7.** Let $\{m_j\}_{j=1}^\infty$ be a strictly monotone increasing sequence of positive integers. The spaces $\mathcal{V}_j := \Pi_{m_j-1}(\mathbb{S}^2)$ with $j \in \mathbb{N}$ satisfy the following conditions:

1. $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ for $j \in \mathbb{N}$.
2. Closure $\left( \bigcup_{j=1}^\infty \mathcal{V}_j, \| \cdot \| \right) = L^2(\mathbb{S}^2)$.

Thus $\{\mathcal{V}_j\}_{j=1}^\infty$ is a multiscale decomposition of $L^2(\mathbb{S}^2)$. 


Possible scaling functions \( \varphi \in \mathcal{V}_1 \) are the functions \( \{ \varphi_{m,n} \}_{m,n=0}^{m=1,n} \) with
\[
\varphi_{n,k}(\theta, \varphi) := \sqrt{\frac{2n+1}{4\pi}} \varphi_n^k(\cos \theta) e^{ik \varphi}.
\]
from Theorem 4. Their advantage for fast computations lies in the orthonormality as well as in the separation of the variables (cf. [23] and [17]). These functions however, are not localized on the sphere which is a big drawback (cf. Figure 2).

An interesting alternative is described in [22]. Following their notation with \( m_j = 2^{j-1} \) we write
\[
\mathcal{V}_j := H_{2j-1}^1(\mathbb{S}^2) = \bigoplus_{n=0}^{2^{j-1}-1} \mathcal{H}_n \quad \text{for } j \in \mathbb{N}.
\]
Hence, the sequence \( \{ \mathcal{V}_j \}_{j=0}^\infty \) is a multiscale decomposition of \( L^2(\mathbb{S}^2) \) (see Theorem 7). Furthermore let \((\theta^j_l,\varphi^j_m)\) be the spherical coordinates of \( \xi^j_{l,m} \in S^2 \). We define the sets \( \mathcal{N}_j \) for \( j \in \mathbb{N} \) as follows
\[
\mathcal{N}_j := \left\{ (\theta^j_l,\varphi^j_m) : \left\lfloor \frac{l \pi}{2^{j+1}} \right\rfloor \leq l \leq \left\lfloor \frac{m \pi}{2^{j+1}} \right\rfloor \quad \text{and} \quad m = 0, \ldots, 2^{j+1} - 1 \right\}
\]
\[
\cup \left\{ \xi^j_{0,0}, \xi^j_{2,0} \right\},
\]
where \( \xi^j_{0,0} := (0,0) \) is the north pole and \( \xi^j_{2,0} := (\pi,0) \) is the south pole of the sphere. It is easy to see that the sets \( \mathcal{N}_j \) satisfy the conditions (G1)–(G3) from Definition 2. Moreover, they can be seen as a refinement grid of \( S^2 \) (cf. Fig. 3). Furthermore we define \( \xi^j_{l,m} := \xi^j_{0,0} \) and \( \xi^j_{2l,m} := \xi^j_{2,0} \) for \( m = 1, \ldots, 2^{j+1} - 1 \) and \( \mathcal{M}_j := \{(n,k) : n = 0, \ldots, 2^{j-1} - 1 \text{ and } k = -n, \ldots, n\} \). In this section we use exclusively the orthonormal basis given in (13). For this basis consisting of spherical harmonics from \( H_{2j-1}^1(\mathbb{S}^2) \) we study a quadrature formula of Clenshaw-Curtis-type. For the proof see [22] and [8, Chap. 3.7].

**Theorem 8.** Let \( f \in \mathcal{V}_j \), then the Fourier coefficients \( \alpha_{n,k}(f) \) of \( f \) with \( (n,k) \in \mathcal{M}_j \) can be written as
\[
\alpha_{n,k}(f) = \frac{4\pi}{2^n} \sum_{l=0}^{2^n} \sum_{m=0}^{2^{n+1}-1} \varepsilon^{(j)}_l \chi^{(j)}_l f(\xi^j_{l,m}) \varphi_{n,k}(\xi^j_{l,m})
\]
with
\[
\varepsilon^{(j)}_l = \varepsilon^{(j)}_0 = 1/2 \quad \text{and} \quad \varepsilon^{(j)}_l := 1 \quad \text{for } l = 1, \ldots, 2^{j-1} - 1
\]
and the Clenshaw-Curtis weights
\[
\chi^{(j)}_l := \frac{1}{2^n} \sum_{s=0}^{2^n-1} \varepsilon^{(j-1)}_s \frac{2}{1 - 4s^2} \cos \left( \frac{ls \pi}{2^{j-1}} \right), \quad \text{for } l = 0, \ldots, 2^j.
\]
Fig. 3. The two grids $\mathcal{N}_3$ and $\mathcal{N}_4$ with equidistant angles seen from the north pole. The grid $\mathcal{N}_3$ is plotted with $\times$ and the grid $\mathcal{N}_4$ with $+$

Now we introduce the scaling function $\varphi_j$ for the scale $\mathcal{V}_j$ as

$$\varphi_j := 2^{2-j} \sqrt{\pi} \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} Y_{n,k}$$

and the “weighted” scaling functions from $\mathcal{V}_j$

$$\varphi_j(\cdot \cdot \xi_{l,m}^j) := 2^{2-j} \sqrt{\pi} \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} Y_{n,k}(\xi_{l,m}^j) Y_{n,k}.$$  \hspace{1cm} (15)

The Addition Theorem 5 yields

$$\varphi_j(\cdot \cdot \xi_{l,m}^j) = \frac{1}{2^j \sqrt{\pi}} \sum_{n=0}^{2^{j-1}-1} (2n + 1) P_n(\cdot \cdot \xi_{l,m}^j).$$  \hspace{1cm} (16)
Note that the functions \( \varphi_j(\cdot \cdot E_j) \) for \( E_j \in \mathcal{N}_j \) are real-valued, whereas the scaling functions \( \tilde{\varphi}_j \) are in general complex-valued. We summarize some properties of the functions \( \varphi_j(\cdot \cdot E_j) \) in the next result. Let us mention that we do not obtain a basis. However, we have a good localization behaviour on the sphere and the system of functions spans the space \( \mathcal{V}_j \). The oversampling leads to the concept of frames. We do not go into the details but refer to the properties 4.) and 5.) of the following theorem.

**Theorem 9.** Let \( E_j \in \mathcal{N}_j \).

1. The functions \( \varphi_j(\cdot \cdot E_j) \) have the reproducing property
   \[
   \langle f, \varphi_j(\cdot \cdot E_j) \rangle = 2^{-j+2} \sqrt{\pi} f(E_j) \quad \text{for all } f \in \mathcal{V}_j.
   \]
2. It holds \( \|\varphi_j(\cdot \cdot E_j)\| = 1 \) and \( \varphi_j(E_j; E_j) = 2^{-j} \).
3. The function \( \varphi_j(\cdot \cdot E_j) \) is localized around \( E_j \), i.e.
   \[
   \frac{\|\varphi_j(\cdot \cdot E_j)\|}{\varphi_j(E_j; E_j)} = \min \left\{ \|f\| : f \in \mathcal{V}_j, f(E_j) = 1 \right\}.
   \]
4. We have \( \text{span} \left\{ \varphi_j(\cdot \cdot E_j) : E_j \in \mathcal{N}_j \right\} = \mathcal{V}_j \).
5. The set \( \left\{ (2^{j-2} \epsilon_j^{(j)})^2 \varphi_j(\cdot \cdot E_j) : E_j \in \mathcal{N}_j \right\} \) is a tight frame in \( \mathcal{V}_j \), i.e. for every function \( f \in \mathcal{V}_j \) we have
   \[
   2^{j-2} \sum_{l=0}^{2^j} \sum_{m=0}^{2^{j+1}-1} \epsilon_j^{l} \chi_j^{(j)} \langle f, \varphi_j(\cdot \cdot E_j) \rangle^2 = \|f\|^2.
   \]
6. We have
   \[
   \alpha_{n,k}(\varphi_j(\cdot \cdot E_j)) = \frac{Y_{n,k}(E_j)}{\sqrt{\pi} Y_{n,k}(E_j)} \alpha_{n,k}(\tilde{\varphi}_j) = \begin{cases} 2^{j-2} \sqrt{\pi} Y_{n,k}(E_j) & \text{for } (n,k) \in \mathcal{M}_j, \\ 0 & \text{otherwise} \end{cases}
   \]
   and the two-scale-relation
   \[
   \alpha_{n,k}(\tilde{\varphi}_j) = \begin{cases} 2 \alpha_{n,k}(\tilde{\varphi}_{j+1}) & \text{for } (n,k) \in \mathcal{M}_j, \\ 0 & \text{otherwise} \end{cases}
   \]
7. We have
   \[
   \int_{S^2} \varphi_j(\cdot \cdot E_j) d\omega(\xi) = 2^{j-2} \sqrt{\pi}.
   \]

**Proof.** We will only sketch the main ideas shortly.
1. Using $\varphi_j(o \cdot \xi_{l,m}^j)$ from (15) and rewriting $f$ as a Fourier series with respect to orthonormal basis $\{Y_{n,k}\}_{(n,k) \in M_j}$ we obtain

$$
\langle f, \varphi_j(o \cdot \xi_{l,m}^j) \rangle = \sum_{n=0}^{2^j-1} \sum_{k=-n}^{n} 2^{-j+2}\sqrt{\pi} \alpha_{n,k}(f) Y_{n,k}(\xi_{l,m}^j) = 2^{-j+2}\sqrt{\pi} f(\xi_{l,m}^j).
$$

2. Starting from (15) we have

$$
||\varphi_j(o \cdot \xi_{l,m}^j)||^2 = \langle \varphi_j(o \cdot \xi_{l,m}^j), \varphi_j(o \cdot \xi_{l,m}^j) \rangle = \sum_{n=0}^{2^j-1} \sum_{k=-n}^{n} 2^{-2j+4}\pi Y_{n,k}(\xi_{l,m}^j) Y_{n,k}(\xi_{l,m}^j).
$$

By the Addition Theorem 5 we conclude

$$
||\varphi_j(o \cdot \xi_{l,m}^j)||^2 = 2^{-2j+4}\pi \sum_{n=0}^{2^j-1} \frac{2n+1}{4\pi} P_n(1)
$$

$$
= 2^{-2j+2} \sum_{n=0}^{2^j-1} (2n+1)
$$

$$
= 2^{-2j+2} 2^{2j-2} = 1.
$$

Hence, $||\varphi_j(o \cdot \xi_{l,m}^j)|| = 1$. The second statement follows from (16) and

$$
\varphi_j(\xi_{l,m}^j : \xi_{l,m}^j) = \varphi_j(1) = \frac{1}{2^{j-1}} \sum_{n=0}^{2^j-1} (2n+1) P_n(1)
$$

$$
= \frac{1}{2^{j-1}} \frac{2^{2j-2}}{\sqrt{\pi}}.
$$

3. For all $f \in V_j$ with $f(\xi_{l,m}^j) = 1$ it holds

$$
1 = \sum_{n=0}^{2^j-1} \sum_{k=-n}^{n} \alpha_{n,k}(f) Y_{n,k}(\xi_{l,m}^j).
$$

Applying the Cauchy-Schwarz inequality gives

$$
1 \leq \sum_{n=0}^{2^j-1} \sum_{k=-n}^{n} |\alpha_{n,k}(f)|^2 \sum_{n=0}^{2^j-1} \sum_{k=-n}^{n} Y_{n,k}(\xi_{l,m}^j) Y_{n,k}(\xi_{l,m}^j).
$$

Equality in (17) will be attained for $\hat{f}$ iff the vectors $\{\alpha_{n,k}(\hat{f})\}_{(n,k) \in M_j}$ and $\{Y_{n,k}(\xi_{l,m}^j)\}_{(n,k) \in M_j}$ are linearly independent, which means
\[ \alpha_{n,k}(\hat{f}) = \alpha Y_{n,k}(\xi_{l,m}^j) \] for \( n = 0, \ldots, 2^{j-1} - 1 \) and \( k = -n, \ldots, n \) with a constant \( \alpha \in \mathbb{Q} \). From item 2) we deduce \( \alpha = 2^{j-2} \pi \) and (15) reads as

\[ \hat{f} = 2^{j-2} \pi \sum_{n=0}^{2^{j-1} - 1} \sum_{k=-n}^{n} \alpha Y_{n,k}(\xi_{l,m}^j) Y_{n,k} \]

\[ = 2^{j-2} \sqrt{\pi} \varphi_j(\alpha \cdot \xi_{l,m}^j). \]

From (18) for all \( f \in \mathcal{V}_j \) with \( f(\xi_{l,m}^j) = 1 \) and Parseval's equation it follows

\[ \|f\|^2 \geq 2^{j-2} \pi = \|\hat{f}\|^2. \]

Thus,

\[ \min \left\{ \|f\| : f \in \mathcal{V}_j, f(\xi_{l,m}^j) = 1 \right\} = 2^{j-2} \sqrt{\pi} = \frac{\|\varphi_j(\alpha \cdot \xi_{l,m}^j)\|}{\varphi_j(\xi_{l,m}^j, \xi_{l,m}^j)}. \]

4. Writing \( f \in \mathcal{V}_j \) as its Fourier sum

\[ f = \sum_{n=0}^{2^{j-1} - 1} \sum_{k=-n}^{n} \alpha_{n,k}(f) Y_{n,k} \]

we can compute the coefficients \( \alpha_{n,k}(f) \) using Theorem 8. Together with (15) we obtain

\[ f = \sqrt{\pi} \sum_{l=0}^{2^{j} - 1} \sum_{m=0}^{2^{j+1} - 1} \varepsilon_l^{(j)} \chi_l^{(j)} f(\xi_{l,m}^j) \varphi_j(\alpha \cdot \xi_{l,m}^j). \]  

(19)

Hence, \( \mathcal{V}_j \) is spanned by the functions \( \varphi_j(\alpha \cdot \xi_{l,m}^j) \) with \( \xi_{l,m}^j \in \mathcal{N}_j \).

5. Again we write

\[ \|f\|^2 = \langle f, f \rangle = \sum_{n=0}^{2^{j-1} - 1} \sum_{k=-n}^{n} \overline{\alpha_{n,k}(f)} \langle f, Y_{n,k} \rangle. \]

From (19) it follows

\[ \|f\|^2 = \sum_{n=0}^{2^{j-1} - 1} \sum_{k=-n}^{n} \sum_{l=0}^{2^{j} - 1} \sum_{m=0}^{2^{j+1} - 1} \overline{\alpha_{n,k}(f)} \sqrt{\pi} \varepsilon_l^{(j)} \chi_l^{(j)} f(\xi_{l,m}^j) \langle \varphi_j(\alpha \cdot \xi_{l,m}^j), Y_{n,k} \rangle. \]
Using item 1.) for \( f(\xi_{l,m}^j) \) and \( \varphi_j(\alpha \cdot \xi_{l,m}^j), Y_{n,k} \) we compute

\[
\|f\|^2 = \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} \sum_{l=0}^{2^j} \sum_{m=0}^{2^j+1-1} \alpha_{n,k}(f) \sqrt{\pi} \varepsilon^{(j)}_l \chi_l^{(j)}(f, \varphi_j(\alpha \cdot \xi_{l,m}^j)) Y_{n,k}(\xi_{l,m}^j)
\]

\[
= \sum_{l=0}^{2^j} \sum_{m=0}^{2^j+1-1} \sqrt{\pi} \varepsilon^{(j)}_l \chi_l^{(j)}(f, \varphi_j(\alpha \cdot \xi_{l,m}^j)) \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} \alpha_{n,k}(f) Y_{n,k}(\xi_{l,m}^j)
\]

\[
= \sum_{l=0}^{2^j} \sum_{m=0}^{2^j+1-1} 2^{j-2} \varepsilon^{(j)}_l \chi_l^{(j)}(f, \varphi_j(\alpha \cdot \xi_{l,m}^j)) f(\xi_{l,m}^j)
\]

where for the last equality we used the reproduction property from item 1.)

6. The proof follows directly from the Fourier representation in spherical harmonics \( Y_{n,k} \).

7. As the integral over the sphere \( S^2 \) is invariant with respect to rotations we can fix \( \xi_{l,m}^j = (0,0,1)^T \). Substituting \( t = \cos \vartheta \) we obtain

\[
\int_{S^2} \varphi_j(\xi \cdot \xi_{l,m}^j) \, d\omega(\xi) = \int_0^{2\pi} d\varphi \int_0^\pi \varphi_j(\cos \vartheta) \sin \vartheta d\vartheta
\]

\[
= 2\pi \int_{-1}^{1} \varphi_j(t) \, dt.
\]

From (16) we conclude

\[
\int_{S^2} \varphi_j(\xi \cdot \xi_{l,m}^j) \, d\omega(\xi) = 2^{1-j} \sqrt{\pi} \sum_{n=0}^{2^{j-1}-1} (2n + 1) \int_{-1}^{1} P_n(t) \, dt
\]

\[
= 2^{1-j} \sqrt{\pi} \left( 2 + \sum_{n=1}^{2^{j-1}-1} \int_{-1}^{1} P_n'(t) - P_{n-1}'(t) \, dt \right).
\]

With \( P_n(1) = 1 \) and \( P_n(-1) = (-1)^n \) the assertion follows. \( \square \)

3.5 The Polynomial Wavelet Space

We define the wavelet spaces \( \mathcal{W}_j \) with \( j \in \mathbb{N} \) as the direct sum

\[
\mathcal{W}_j = \bigoplus_{n=2^{j-1}}^{2^j-1} \mathcal{H}_n.
\]
For the dimension of $\mathcal{W}_j$ we have
\[
\dim \mathcal{W}_j = \dim \mathcal{V}_{j+1} - \dim \mathcal{V}_j = 3 \cdot 2^{2j-2}.
\]
Now we define the wavelets $\tilde{\psi}_j$ for the space $\mathcal{W}_j$ as
\[
\tilde{\psi}_j := 2^{2-j}\sqrt{\pi} \sum_{n=2j-1}^{2j-1} \sum_{k=-n}^{n} Y_{n,k}.
\]  
and the “weighted” wavelets
\[
\psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1}) := 2^{2-j}\sqrt{\pi} \sum_{n=2j-1}^{2j-1} \sum_{k=-n}^{n} Y_{n,k}(\xi_{i,m}^{j+1})Y_{n,k} \in \mathcal{W}_j.
\]
Again we have from the Addition Theorem 5 that
\[
\psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1}) = \frac{1}{2j} \sqrt{\pi} \sum_{n=2j-1}^{2j-1} (2n+1)P_n(\cdot \cdot \cdot \xi_{i,m}^{j+1}).
\]
Note that the functions $\psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1})$ with $\xi_{i,m}^{j+1} \in \mathcal{N}_{j+1}$ are real-valued and the wavelets $\tilde{\psi}_j$ are complex-valued. Again we summarize the main properties of the functions $\psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1})$ with $j \in \mathbb{N}$ in the following theorem.

**Theorem 10.** Let $\xi_{i,m}^{j+1} \in \mathcal{N}_{j+1}$.

1. The functions $\psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1})$ have the reproducing property
\[
\langle f, \psi_j(\cdot \cdot \cdot \xi_{i,m}^{j+1}) \rangle = 2^{-j+2}\sqrt{\pi}f(\xi_{i,m}^{j+1}) \text{ for all } f \in \mathcal{W}_j.
\]
2. We have the orthogonality
\[ \langle \varphi_j(o \cdot \xi_{j,m}^{+1}), \psi_j(o \cdot \xi_{j,m}^{+1}) \rangle = 0 \quad \text{for all } \xi_{j,m}^{+1} \in \mathcal{N}_j. \]

3. It holds \( \| \psi_j(o \cdot \xi_{j,m}^{+1}) \| = 3^{1/2} \) and \( \psi_j(o \cdot \xi_{j,m}^{+1}) = \frac{3^{2j-2}}{\sqrt{\pi}}. \)

4. The function \( \psi_j(o \cdot \xi_{j,m}^{+1}) \) is localized around \( \xi_{j,m}^{+1} \), i.e.
\[
\frac{\| \psi_j(o \cdot \xi_{j,m}^{+1}) \|}{\psi_j(o \cdot \xi_{j,m}^{+1})} = \min \left\{ \| f \| : f \in \mathcal{W}_j, f(\xi_{j,m}^{+1}) = 1 \right\}.
\]

5. We have \( \text{span} \left\{ \psi_j(o \cdot \xi_{j,m}^{+1}) : \xi_{j,m}^{+1} \in \mathcal{N}_{j+1} \right\} = \mathcal{W}_j. \)

6. The set \( \left\{ (2j-2)\xi_{j,m}^{+1}, \chi^{(j+1)}_l(o \cdot \xi_{j,m}^{+1}) : \xi_{j,m}^{+1} \in \mathcal{N}_{j+1} \right\} \) is a tight frame in \( \mathcal{W}_j \), i.e., for every function \( f \in \mathcal{W}_j \) we have
\[
2^{j-2} \sum_{l=0}^{2^{j+1} - 2^{j+2} - 1} \sum_{m=0}^{2^{j+1}} \xi_{j,m}^{+1} \chi^{(j+1)}_l(o \cdot \xi_{j,m}^{+1}) \| f(\psi_j(o \cdot \xi_{j,m}^{+1})) \| = \| f \|^2.
\]

7. We have
\[
\alpha_{n,k}(\psi_j(o \cdot \xi_{j,m}^{+1})) = \overline{Y_{n,k}(\xi_{j,m}^{+1})} \alpha_{n,k}(\tilde{\psi}_j) = \begin{cases} 2^{j-2} \sqrt{\pi} Y_{n,k}(\xi_{j,m}^{+1}) & \text{for } (n,k) \in \mathcal{M}_{j+1} \setminus \mathcal{M}_j, \\ 0 & \text{otherwise} \end{cases}
\]

and the two-scale-relation
\[
\alpha_{n,k}(\tilde{\psi}_j) = \begin{cases} 2\alpha_{n,k}(\varphi_{j+1}) & \text{for } (n,k) \in \mathcal{M}_{j+1} \setminus \mathcal{M}_j, \\ 0 & \text{otherwise}. \end{cases}
\]

8. We have
\[
\int_{\mathbb{S}^2} \psi_j(o \cdot \xi_{j,m}^{+1}) \, d\omega(\xi) = 0.
\]

\textbf{Proof.} The items 1.) and 3.) - 8.) can be proved analogously to the proof of Theorem 9. The orthogonality 2.) follows directly from the definition of \( \psi_j(o \cdot \xi_{j,m}^{+1}) \). \( \Box \)
3.6 Algorithms for Reconstruction and Decomposition

To describe the algorithms we use functions $v_{j+1} \in V_{j+1}$, $v_j \in V_j$ and $w_j \in W_j$ represented by

$$v_{j+1} = \sqrt{2} \sum_{l=0}^{2^{j+1} - 1} \sum_{m=0}^{2^{j+1} - 1} \epsilon_l^{(j+1)}(\xi_l^{(j+1)} \cdot \delta) \psi_{j+1}(\xi_{l,m}^{j+1} \cdot \delta),$$

$$v_j = \sqrt{2} \sum_{l=0}^{2^j} \sum_{m=0}^{2^{j+1} - 1} \epsilon_l^{(j)}(\xi_l^{(j)} \cdot \delta) \psi_j(\xi_{l,m}^j \cdot \delta),$$

$$w_j = \sqrt{2} \sum_{l=0}^{2^j} \sum_{m=0}^{2^{j+1} - 1} \epsilon_l^{(j+1)}(\xi_l^{(j+1)} \cdot \delta) \psi_j(\xi_{l,m}^{j+1} \cdot \delta).$$

Decomposing the function $v_{j+1} \in V_{j+1}$ into functions $v_j \in V_j$ and $w_j \in W_j$ means

$$v_{j+1} = v_j + w_j.$$

Such an orthogonal decomposition is unique and the Fourier coefficients $\alpha_{n,k}(v_{j+1})$ with respect to the spherical harmonics $Y_{n,k}$ defined in (13) with $(n,k) \in M_{j+1}$ satisfy

$$\alpha_{n,k}(v_{j+1}) = \alpha_{n,k}(v_j) + \alpha_{n,k}(w_j)$$

and clearly $\alpha_{n,k}(v_{j+1}) = 0$ for $(n,k) \notin M_{j+1}$. Hence,

$$\alpha_{n,k}(v_j) = \alpha_{n,k}(v_{j+1})$$

for $(n,k) \in M_j$, and

$$\alpha_{n,k}(w_j) = \alpha_{n,k}(v_{j+1})$$

for $(n,k) \in M_{j+1} \setminus M_j$. 

---

**Fig. 5.** The weighted wavelet $2/(2^j - 1)\psi_j(\delta \cdot \xi_{l,m}^j)$ with $\xi_{l,m}^j = (0,0,1)^T$. Left: $2/15\psi_4(\cos \phi)$, right: $2/31\psi_5(\cos \phi)$.
These results are derived in [22]. From Theorem 8 it follows now

$$v_j(\xi) = \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} \alpha_{n,k}(v_{j+1})Y_{n,k}(\xi)$$

$$= \sum_{n=0}^{2^{j-1}-1} \sum_{k=-n}^{n} \frac{4\pi}{2^{j+1}} \sum_{l=0}^{2^{j+1}} \sum_{m=0}^{2^{j+2}-1} \epsilon_l^{(j+1)} \chi_l^{(j+1)} v_{j+1}(\xi_l^{j+1}) Y_{n,k}(\xi_l^{j+1}) Y_{n,k}(\xi)$$

for all $\xi \in S^2$, particularly for $\xi = \xi_{l,m}^j \in N_j$ and

$$w_j(\xi)$$

$$= \sum_{n=2^{j-1}}^{2^j-1} \sum_{k=-n}^{n} \alpha_{n,k}(v_{j+1})Y_{n,k}(\xi)$$

$$= \sum_{n=2^{j-1}}^{2^j-1} \sum_{k=-n}^{n} \frac{4\pi}{2^{j+1}} \sum_{l=0}^{2^{j+1}} \sum_{m=0}^{2^{j+2}-1} \epsilon_l^{(j+1)} \chi_l^{(j+1)} v_{j+1}(\xi_l^{j+1}) Y_{n,k}(\xi_l^{j+1}) Y_{n,k}(\xi)$$

for all $\xi \in S^2$ and also particularly for all $\xi_{l,m}^{j+1} \in N_{j+1}$. Now let $R$ be the operator

$$R(\nu^{(j+1)}) = \nu^{(j)}$$

mapping

$$\nu^{(j+1)} := \left\{ v_{j+1}(\xi_l^{j+1}) \right\}_{\xi_l^{j+1} \in N_{j+1}}$$

onto

$$\nu^{(j)} := \left\{ v_j(\xi_l^j) \right\}_{\xi_l^j \in N_j}$$

as defined in (21). Analogously, we introduce the operator $Q$ defined from (22) by

$$Q(\nu^{(j+1)}) = W^{(j)} := \left\{ w_j(\xi_l^{j+1}) \right\}_{\xi_l^{j+1} \in N_{j+1}} .$$

Summarizing, the operators $R$ and $Q$ describe the decomposition of a function $v_{j+1} \in \mathcal{V}_{j+1}$. 
\[
\text{Decomposition Algorithm}
\]

Input:
\[\psi^{(j+1)}\]

Compute for \(i = j, \ldots, 1\):
\[
\begin{align*}
\psi^{(i)} & := R(\psi^{(i+1)}) \\
\omega^{(i)} & := Q(\psi^{(i+1)})
\end{align*}
\]

Output:
\[
\begin{align*}
\psi^{(1)} \\
\omega^{(i)} \quad (i = j, \ldots, 1)
\end{align*}
\]

\[
\begin{array}{c}
\vspace{1em}
\begin{array}{c}
\psi^{(j+1)} \\
\downarrow R \\
\psi^{(j)} \\
\downarrow Q \\
\omega^{(j)}
\end{array}
\quad \\
\begin{array}{c}
\psi^{(j-1)} \\
\downarrow R \\
\psi^{(2)} \\
\downarrow Q \\
\omega^{(j-1)}
\end{array}
\quad \\
\vdots
\end{array}
\]

Now we want to describe the reconstruction of a function \(v_{j+1} \in \mathcal{V}_{j+1}\) from \(v_j \in \mathcal{V}_j\) and \(w_j \in \mathcal{W}_j\), i.e.

\[v_j + w_j = v_{j+1}.
\]

The Fourier coefficients \(\alpha_{n,k}(v_{j+1})\) can be computed from the \(\alpha_{n,k}(v_j)\) and \(\alpha_{n,k}(w_j)\). For this reconstruction assume that \(v_j(\xi^j_{l,m})\) for \(\xi^j_{l,m} \in \mathcal{N}_j\) and \(w_j(\xi^{j+1}_{l,m})\) for \(\xi^{j+1}_{l,m} \in \mathcal{N}_{j+1}\) are given. We obtain

\[
v^{j+1}_{j+1}(\xi) = \sum_{n=0}^{2^{j+1}-1} \sum_{k=-n}^{n} \frac{4\pi}{2^j} \sum_{l=0}^{2^j} \sum_{m=0}^{2^{j+1}-1} \varepsilon^{(j)}_l \chi^{(j)}_l v_j(\xi^j_{l,m}) Y_{n,k}(\xi_{l,m}) Y_{n,k}(\xi)
\]

\[
+ \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{k=-n}^{n} \frac{4\pi}{2^{j+1}} \sum_{l=0}^{2^{j+1}} \sum_{m=0}^{2^{j+2}-1} \varepsilon^{(j+1)}_l \chi^{(j+1)}_l w_j(\xi^{j+1}_{l,m}) Y_{n,k}(\xi^{j+1}_{l,m}) Y_{n,k}(\xi).
\]
Especially, the equality holds true for $\xi = \xi_{p,q}^{j+1} \in \mathcal{N}_{j+1}$. Defining the operator $R^*$ now by

$$
\left( R^*(\mathbf{v}^{(j)}) \right)_{p,q} := \sum_{n=-2^{j-1}}^{2^{j-1}} \sum_{k=-n}^{n} \frac{4\pi}{2^{j}} \sum_{l=0}^{2^{j+1}-1} \sum_{m=0}^{2^{j+1}-1} \varepsilon_{l}^{(j)} \chi_l^{(j)} \mathbf{v}_j(\xi^{(j)}_{l,m}) Y_{n,k} \overline{Y}_{n,k}(\xi^{j+1}_{p,q})
$$

and the operator $Q^*$ by

$$
\left( Q^*(\mathbf{w}^{(j)}) \right)_{p,q} := \sum_{n=-2^{j-1}}^{2^{j-1}} \sum_{k=-n}^{n} \frac{4\pi}{2^{j+1}} \sum_{l=0}^{2^{j+1}-1} \sum_{m=0}^{2^{j+1}-1} \varepsilon_{l}^{(j+1)} \chi_l^{(j+1)} \mathbf{w}_j(\xi^{(j+1)}_{l,m}) Y_{n,k} \overline{Y}_{n,k}(\xi^{j+1}_{p,q})
$$

we have

$$
\mathbf{v}^{(j+1)} := R^*(\mathbf{v}^{(j)}) + Q^*(\mathbf{w}^{(j)}).
$$

Here the operators $R^*$ and $Q^*$ are the adjoints of $R$ and $Q$, respectively. Now we illustrate the rule for reconstruction.

---

**Reconstruction Algorithm**

**Input:**

$$
\mathbf{v}^{(1)}, \mathbf{w}^{(i)} \quad (i = 1, \ldots, j)
$$

**Compute for** $i = 1, \ldots, j$:

$$
\mathbf{v}^{(i+1)} := R^*(\mathbf{v}^{(i)}) + Q^*(\mathbf{w}^{(i)})
$$

**Output:**

$$
\mathbf{v}^{(j+1)}
$$

---

![Diagram](image.png)
4 Spherical Basis Functions

4.1 Positive Definite Functions

Definition 6. A continuous function $G : [-1, 1] \rightarrow \mathbb{R}$ is called positive definite on $S^m$ with $m \in \mathbb{N}$, if for every $L \in \mathbb{N}$ and any sequence of points $\{\xi_l\}_{l=1}^L$ on $S^m$ the corresponding Gramian matrix $A$ with the entries $A_{ij} = G(\xi_i, \xi_j)$ for $i, j = 1, \ldots, L$, is positive semidefinite. Moreover, if for pairwise distinct points $\{\xi_l\}_{l=1}^L$ with $\xi_l \in S^m$ for $l = 1, \ldots, L$ the matrix $A$ is positive definite, then $G$ is called strongly positive definite.

Every positive definite function $G$ depends only on the angle between the points $\xi, \eta$. The Legendre polynomials $P_n$ constitute an orthogonal basis on $[-1, 1]$. Hence, every positive definite function $G$ on $S^2$ can be written as an $L^2[-1, 1]$-convergent Fourier-Legendre series

$$G(t) = \sum_{n=0}^{\infty} a_n P_n(t) \quad \text{with } a_n \in \mathbb{R}.$$ 

Particularly, for a positive definite function $G$ on $S^2$ and a fixed $\eta \in S^2$ we can write $G(\cdot \cdot \eta)$ in terms of spherical harmonics

$$G(\cdot \cdot \eta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \langle G(\cdot \cdot \eta), Y_{n,k} \rangle Y_{n,k}.$$ 

The following lemma summarizes some basic properties of positive definite functions.

Lemma 2. Let $\{G_k\}_{k \in \mathbb{N}}$ be a sequence of positive definite functions on $S^m$. Then we have:

1. For $b_1, b_2 \geq 0$ the function $H := b_1 G_1 + b_2 G_2$ is positive definite on $S^m$.
2. The product $G_1 G_2$ is positive definite on $S^m$.
3. If the sequence $\{G_k\}_{k \in \mathbb{N}}$ converges pointwise to a continuous function $G$, then $G$ is positive definite on $S^m$.

Proof. The proof is simply based on the fact that a positive semidefinite matrix satisfies by definition

$$0 \leq e^T A e = \sum_{i=1}^{L} \sum_{j=1}^{L} c_i c_j A_{ij} \quad \text{for all } e = (c_1, \ldots, c_L)^T \in \mathbb{R}^L.$$ 

Moreover, the spectral theorem for Hermitian matrices [9, p. 104] ensures the existence of an orthogonal matrix $P \in \mathbb{R}^{L \times L}$ and a diagonal matrix $A = \text{diag}(\lambda_1, \ldots, \lambda_L) \in \mathbb{R}^{L \times L}$ with $\lambda_l \geq 0$ for $l = 1, \ldots, L$, such that $A = P^T A P$. □
We are now able to state the main theorem of this subsection which characterizes positive definite functions by the non-negativity of its Fourier-Legendre coefficients. This celebrated result goes back to Schoenberg [25, Theorem 1].

**Theorem 11 (Schoenberg).** Let \( G(t) = \sum_{n=0}^{\infty} a_n P_n(t) \) with \( \sum_{n=0}^{\infty} |a_n| < \infty \). Then the following conditions are equivalent:

1. The function \( G \) is positive definite on \( S^2 \).
2. In the representation of \( G \)

\[
G(t) = \sum_{n=0}^{\infty} a_n P_n(t)
\]

it holds that \( a_n \geq 0 \) for all \( n \in \mathbb{N}_0 \).

**Proof.** As a first step we show that \( P_n \) is positive definite on \( S^2 \). The Addition Theorem 5 implies

\[
P_n(\xi \cdot \eta) = \frac{4\pi}{2n+1} \sum_{k=-n}^{n} Y_{n,k}(\xi) Y_{n,k}(\eta).
\]

For the points \( \xi_l \in S^2 \) and for arbitrary \( q_l \in \mathbb{R} \) with \( l = 1, \ldots, L \), it holds that

\[
\sum_{l=1}^{L} \sum_{m=1}^{L} q_l q_m P_n(\xi_l \cdot \xi_m) = \frac{4\pi}{2n+1} \sum_{l=1}^{L} \sum_{m=1}^{L} q_l q_m \sum_{k=-n}^{n} Y_{n,k}(\xi_l) Y_{n,k}(\xi_m) = \frac{4\pi}{2n+1} \sum_{k=-n}^{n} \left( \sum_{l=1}^{L} q_l Y_{n,k}(\xi_l) \right) \left( \sum_{m=1}^{L} q_m Y_{n,k}(\xi_m) \right)^2 \geq 0.
\]

Thus the positive definiteness of \( P_n \) on \( S^2 \) is shown. The positive definiteness on \( S^2 \) of the function \( G(t) := \sum_{n=0}^{\infty} a_n P_n(t) \), with \( a_n \geq 0 \) for all \( n \in \mathbb{N}_0 \) and \( \sum_{n=0}^{\infty} a_n < \infty \) follows now from Lemma 2. It remains to prove that any positive definite function \( G \) on \( S^2 \) has a representation of the form (23). Writing \( G \) as

\[
G(t) = \sum_{n=0}^{\infty} a_n \sqrt{\frac{2}{2n+1}} P_n(t) \quad \text{with} \quad \sum_{n=0}^{\infty} |a_n| < \infty
\]

we have to show that \( a_n \geq 0 \) for all \( n \in \mathbb{N}_0 \). The coefficients \( a_n \) of the Fourier-Legendre series are given by

\[
a_n = \sqrt{\frac{2}{2n+1}} \int_{-1}^{1} G(t) P_n(t) \, dt \quad \text{for all} \quad n \in \mathbb{N}_0.
\]
The positive definiteness of $P_n$ on $S^2$ and Lemma 2 implies that it is sufficient to show that
\[ \int_{-1}^{1} H(t) \, dt \geq 0 \]
for every positive definite function $H$ on $S^2$. To this end we infer from
\[ \sum_{i=1}^{L} \sum_{j=1}^{L} c_i c_j H(\xi_i \cdot \xi_j) \geq 0 \quad \text{for any } c = (c_1, \ldots, c_L)^T \in \mathbb{R}^L, \xi_i \in S^2, \]
i.e., the nonnegativity of arbitrary Riemann sums that
\[ 0 \leq \int_{S^2} \int_{S^2} H(\xi \cdot \eta) \, d\omega(\xi) \, d\omega(\eta) = 4\pi \int_{S^2} H(\xi \cdot \eta') \, d\omega(\xi) \]
for any $\eta' \in S^2$. Choosing particularly $\eta' = (0,0,1)^T$, we obtain immediately
\[ \int_{-1}^{1} H(t) \, dt \geq 0 \]
which concludes the proof. \( \square \)

In [31] Xu and Cheney proved the following interesting amplification.

**Theorem 12.** Let $a_n \geq 0$ for all $n \in \mathbb{N}_0$, and $a_n > 0$ for $n = 0, \ldots, L-1$ with $\sum_{n=0}^\infty a_n < \infty$ be given. Furthermore let $G(t) = \sum_{n=-\infty}^\infty a_n P_n(t)$.

Then the $L \times L$ Gramian matrix $A$ with entries $A_{ij} = G(\xi_i \cdot \xi_j)$ is positive definite for any pairwise different points $\{\xi_l\}_{-1}^L$.

**Proof.** Let $\eta \in S^2$ be such that $\xi_i \cdot \eta$ are pairwise different. Then take a $T \in SO(3)$ with $T\eta = (0,0,1)^T$ and $T\xi_l = (\tilde{\theta}_l, \tilde{\phi}_l)$ for $l = 1, \ldots, L$. Hence, the values
\[ \cos \tilde{\theta}_l = T\xi_l \cdot (0,0,1)^T = T\xi_l \cdot T\eta = \xi_l \cdot \eta \quad \text{for } l = 1, \ldots, L \]
are pairwise different and we obtain for $i, j = 1, \ldots, L$
\[ P_n(\xi_i \cdot \xi_j)
\]
\[ = P_n(T\xi_i \cdot T\xi_j) = \sum_{k=-n}^{n} P_n^k(\cos \tilde{\phi}_i) P_n^k(\cos \tilde{\phi}_j) \cos(k(\tilde{\phi}_i - \tilde{\phi}_j))
\]
\[ = P_n(\cos \tilde{\phi}_i) P_n(\cos \tilde{\phi}_j) + 2 \sum_{k=1}^{n} P_n^k(\cos \tilde{\phi}_i) P_n^k(\cos \tilde{\phi}_j) \cos(k(\tilde{\phi}_i - \tilde{\phi}_j)).
\]

The Gramian matrix $A$ has the entries
\[ A_{ij} = G(\xi_i \cdot \xi_j) = G(T\xi_i \cdot T\xi_j)
\]
\[ = \sum_{n=0}^\infty a_n P_n(\cos \tilde{\phi}_i) P_n(\cos \tilde{\phi}_j)
\]
\[ + \sum_{n=0}^\infty 2a_n \sum_{k=1}^{n} P_n^k(\cos \tilde{\phi}_i) P_n^k(\cos \tilde{\phi}_j) \cos(k(\tilde{\phi}_i - \tilde{\phi}_j)).
\]
We decompose \( A = B + D \) in two positive semidefinite matrices \( B, D \) with entries

\[
B_{ij} = \sum_{n=0}^{\infty} a_n P_n(\cos \tilde{\theta}_i) P_n(\cos \tilde{\theta}_j),
\]
\[
D_{ij} = \sum_{n=0}^{\infty} 2a_n \sum_{k=1}^{n} P_n^{(k)}(\cos \tilde{\theta}_i) P_n^{(k)}(\cos \tilde{\theta}_j) \cos(k(\tilde{\theta}_i - \tilde{\theta}_j)).
\]

Thus, \( c^T A c = 0 \) iff \( c^T B c = 0 \) and \( c^T D c = 0 \). So let \( c^T B c = 0 \), i.e.,

\[
0 = \sum_{i=1}^{L} \sum_{j=1}^{L} c_i c_j \sum_{n=0}^{\infty} a_n P_n(\cos \tilde{\theta}_i) P_n(\cos \tilde{\theta}_j) = \sum_{n=0}^{\infty} a_n \left( \sum_{i=1}^{L} c_i P_n(\cos \tilde{\theta}_i) \right)^2.
\]

By assumption we have now

\[
\sum_{i=1}^{L} c_i P_n(\cos \tilde{\theta}_i) = 0, \quad \text{for } n = 0, \ldots, L - 1,
\]

which implies the existence of a polynomial of degree \( L - 1 \)

\[
\sum_{n=0}^{L-1} \beta_n P_n(\cos \vartheta) \quad \text{with } \vartheta \in [0, \pi],
\]

which satisfies the \( L \) interpolation conditions

\[
\sum_{n=0}^{L-1} \beta_n P_n(\cos \tilde{\theta}_i) = c_i \quad \text{for } i = 1, \ldots, L.
\]

It now follows easily that

\[
\sum_{i=1}^{L} c_i^2 = \sum_{i=1}^{L} c_i \sum_{n=0}^{L-1} \beta_n P_n(\cos \tilde{\theta}_i) = \sum_{n=0}^{L-1} \beta_n \sum_{i=1}^{L} c_i P_n(\cos \tilde{\theta}_i) = 0
\]

which yields \( c_i = 0 \). \( \square \)

Taking point systems \( \{ \xi_j \}_{j=1}^{L} \) on \( S^2 \) with arbitrary \( L \) we obtain immediately the following final result.

**Corollary 2.** Let \( G : [-1, 1] \to \mathbb{R} \) be given by

\[
G(t) = \sum_{n=0}^{\infty} a_n P_n(t) \quad \text{with } a_n > 0 \text{ for all } n \in \mathbb{N}_0 \text{ and } \sum_{n=0}^{\infty} a_n < \infty.
\]

Then \( G \) is strongly positive definite on \( S^2 \).
Such $G$ will be called spherical basis functions (cf. [31]). Often they are also called spherical radial basis functions.

In this section we looked on functions on $\mathbb{S}^2$ in particular. Using the inclusion

$$\mathbb{S}^1 \subset \mathbb{S}^2 \subset \cdots \subset \mathbb{S}^\infty$$

it is immediately clear that (strongly) positive definite functions on $\mathbb{S}^{m+1}$ are also (strongly) positive definite on $\mathbb{S}^m$. An exact characterization of strongly positive definite functions on $\mathbb{S}^m$ is still under research (cf. [2] [3], [31], [14], [26] and references therein).

### 4.2 Spherical Basis Functions and Approximation

Now we describe some ideas of the approach of Narcowich and Ward [20]. We start with the introduction of a spherical convolution. Let $G$ and $H$ be given by

$$G(t) = \sum_{n=0}^{\infty} a_n P_n(t) \quad \text{with } a_n \geq 0 \text{ for all } n \in \mathbb{N}_0 \text{ and } \sum_{n=0}^{\infty} a_n < \infty$$

and

$$H(t) = \sum_{n=0}^{\infty} b_n P_n(t) \quad \text{with } b_n \geq 0 \text{ for all } n \in \mathbb{N}_0 \text{ and } \sum_{n=0}^{\infty} b_n < \infty.$$

Then we define the spherical convolution $G * H : [-1,1] \to \mathbb{R}$ as

$$G * H(\xi \cdot \eta) := \int_{\mathbb{S}^2} G(\xi \cdot \zeta) H(\eta \cdot \zeta) \, d\omega(\zeta).$$

Here we have to show that the definition makes sense. Namely, the left-hand side depends only on the inner product $\xi \cdot \eta$ which can be seen from the following simple computations

$$G * H(\xi \cdot \eta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_{\mathbb{S}^2} P_n(\xi \cdot \zeta) P_m(\eta \cdot \zeta) \, d\omega(\zeta)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \frac{(4\pi)^2}{(2n+1)(2m+1)} \sum_{k=-n}^{n} \sum_{l=-m}^{m} Y_{n,k}(\xi) Y_{m,l}(\eta)$$

$$\times \int_{\mathbb{S}^2} Y_{n,k}(\zeta) Y_{m,l}(\zeta) \, d\omega(\zeta)$$

$$= \sum_{n=0}^{\infty} a_n b_n \frac{4\pi}{2n+1} P_n(\xi \cdot \eta).$$

With this representation it is also easy to see that the spherical convolution is commutative, i.e., $G * H = H * G$ and also positive definite. This implies that the convolution $G * H$ is also a spherical basis function.
The most prominent example for a spherical basis function is the Poisson kernel defined in (8). By convolution we obtain a chain of Poisson kernels of different localization

\[ G_{h^n}(t) = \frac{1}{4\pi}(G_h * G_h)(t) = \sum_{n=0}^{\infty} (2n + 1)h^{2n}P_n(t). \]

For more examples see [31].

![Fig. 6. The normalized Poisson kernel \( G_h(o \cdot \xi)/G_h(1) \) with center \( \xi = (0, 0, 1)^T \). Left: \( G_{0.9}(\cos \vartheta)/G_{0.9}(1) \), right: \( G_{0.9}(\cos \vartheta)/G_{0.9}(1) \)](image)

Now let us construct out of a spherical basis function \( G \) a set of functions

\[ G(o \cdot \xi_l) : S^2 \rightarrow \mathbb{R} \quad l = 1, \ldots, L, \]

and the linear space

\[ \mathcal{V} := \text{span}\{G(o \cdot \xi_l) : \xi_l \in \mathcal{N}\}. \]

In the sequel \( \mathcal{N} \) denotes an arbitrary set of \( L \) pairwise distinct points on the sphere. Note that these functions \( G(o \cdot \xi_l) \) can be interpreted as rotations of one function. For any \( \xi_l, \xi_k \in S^2 \) there exists an \( T \in SO(3) \), so that \( G(o \cdot T\xi_l) = G(o \cdot \xi_k) \). A fundamental observation for our purpose is the linear independence of the functions.

**Theorem 13.** Let \( G \) be a spherical basis function. Then the functions \( G(o \cdot \xi_l) \) with \( \xi_l \in \mathcal{N} \) constitute a basis of \( \mathcal{V} \).
Proof. By definition the functions $G(o \cdot \xi_l)$ with $\xi_l \in \mathcal{N}$ for $l = 1, \ldots, L$ span the space $\mathcal{V}$. It remains to prove the linear independence. Therefore assume

$$
\sum_{l=1}^{L} \lambda_l G(o \cdot \xi_l) = 0, \quad \text{with } \lambda_l \in \mathbb{C}.
$$

Then

$$
\sum_{l=1}^{L} \lambda_l G(\xi_k \cdot \xi_l) = 0, \quad \text{for } k = 1, \ldots, L.
$$

Now the assertion follows from the positive definiteness of $G$. □

Our aim is now to approximate data given at the points $\xi_l \in \mathcal{N}$ through an element of

$$
\mathcal{V} = \text{span}\{G(o \cdot \xi_l) : \xi_l \in \mathcal{N}\}.
$$

Following the general approach described in Section 2.2 we have to consider the positive definite matrix

$$
\Phi := \begin{pmatrix}
G(\xi_1 \cdot \xi_1) & \cdots & G(\xi_1 \cdot \xi_L) \\
\vdots & \ddots & \vdots \\
G(\xi_L \cdot \xi_1) & \cdots & G(\xi_L \cdot \xi_L)
\end{pmatrix} \in \mathbb{R}^{L \times L}.
$$

In this particular case the least squares problem reduces to a uniquely solvable interpolation problem $\Phi a = f$.

### 4.3 Multiscale Decomposition

Now we want to study a multiscale decomposition of $L^2(S^2)$ based on spherical basis functions. Let $\{\mathcal{N}_j\}_{j \in \mathbb{N}}$ be a refinement grid of $S^2$ (cf. Definition 2). For a fixed spherical basis function $G$ we define the spaces

$$
\mathcal{V}_j := \text{span}\{G(o \cdot \xi_1), \ldots, G(o \cdot \xi_{L_j})\}, \quad \text{with } \xi_1, \ldots, \xi_{L_j} \in \mathcal{N}_j. \quad (24)
$$

The dimensions satisfy $\dim \mathcal{V}_j = L_j$, $\dim \mathcal{V}_{j+1} = L_{j+1}$ and for the orthogonal complement $\mathcal{W}_j = \mathcal{V}_{j+1} \cap \mathcal{V}_j$ of $\mathcal{V}_j$ it follows $\dim \mathcal{W}_j = L_{j+1} - L_j$. To simplify notation we only use one index for the points $\xi_l$. Going from $\mathcal{N}_j$ to $\mathcal{N}_{j+1}$ we add the points $\xi_l$, $l = L_j + 1, \ldots, L_{j+1}$.

If $G$ is a spherical basis function and the spaces $\mathcal{V}_j$ with $j \in \mathbb{N}$ are given by (24), then $\{\mathcal{V}_j\}_{j \in \mathbb{N}}$ is a multiscale decomposition of $L^2(S^2)$. Particularly, we have

1. $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ for $j \in \mathbb{N}$,
2. Closure $\left( \bigcup_{j \in \mathbb{N}} \mathcal{V}_j, \| \cdot \| \right) = L^2(S^2)$. 
The first inclusion follows immediately from the fact that \( \{ N_j \}_{j \in \mathbb{N}} \) is a refinement grid on \( \mathbb{S}^2 \). To prove the second item we assume there is a non-zero \( F \in L^2(\mathbb{S}^2) \) which is orthogonal to any function \( H \in \bigcup_{j \in \mathbb{N}} \mathcal{V}_j \). Because the refinement grid is dense this implies that \( F \) is orthogonal to \( G(\cdot \cdot \xi) \) for all \( \xi \in \mathbb{S}^2 \). With \( G = \sum_{m=0}^{\infty} a_m P_m(\cdot \cdot \xi) \) we rewrite in Fourier-Legendre series
\[
0 = \langle F, G(\cdot \cdot \xi) \rangle = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha_{n,k}(F) a_m \langle Y_n,k, P_m(\cdot \cdot \xi) \rangle \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha_{n,k}(F) a_n \frac{4\pi}{2n+1} Y_n,k(\xi).
\]
The function on the right-hand side is in \( L^2(\mathbb{S}^2) \). Hence, the Fourier coefficients satisfy \( \alpha_{n,k}(F) = 0 \), which contradicts the assumption.

Some easily proved properties of \( G \) are summarized in the following result.

**Theorem 14.** Let \( \xi \in \mathcal{N}_j \) and \( G(\cdot \cdot \xi) \in \mathcal{V}_j \) as given in (24). Then for every function
\[
f = \sum_{l=1}^{L_j} \alpha_l(f) G(\cdot \cdot \xi_l) \in \mathcal{V}_j
\]
it holds
\[
\langle f, G(\cdot \cdot \xi_m) \rangle = \sum_{l=1}^{L_j} \alpha_l(f) G(\xi_l \cdot \xi_m).
\]
Furthermore,
\[
||G(\cdot \cdot \xi)\|^2 = 4\pi \sum_{n=0}^{\infty} \frac{a_n^2}{2n+1} \quad \text{and} \quad G(\xi_l \cdot \xi_l) = \sum_{n=0}^{\infty} a_n.
\]

Again, we investigate now the multiscale algorithms. Therefore we write \( v_j \in \mathcal{V}_j \) and \( w_j \in \mathcal{W}_j \) as
\[
v_j(\xi) = \sum_{l=1}^{L_j} v_{l}^{(j)} G(\xi \cdot \xi_l) \quad \text{with} \quad v_{l}^{(j)} \in \mathcal{G} \tag{25}
\]
and
\[
w_j(\xi) = \sum_{l=1}^{L_{j+1}} w_{l}^{(j)} G(\xi \cdot \xi_l) \quad \text{with} \quad w_{l}^{(j)} \in \mathcal{G} \tag{26}
\]
The orthogonality of the spaces \( \mathcal{V}_j \) and \( \mathcal{W}_j \) implies
\[
0 = \langle v_j, w_j \rangle = \sum_{k=1}^{L_j} \sum_{l=1}^{L_{j+1}} v_{k}^{(j)} w_{l}^{(j)} \langle G(\cdot \cdot \xi_k), G(\cdot \cdot \xi_l) \rangle \\
= \sum_{k=1}^{L_j} \sum_{l=1}^{L_{j+1}} v_{k}^{(j)} w_{l}^{(j)} G(\xi_k \cdot \xi_l). \tag{27}
\]
Now we write
\[ v_j(\xi) + w_j(\xi) = v_{j+1}(\xi) = \sum_{l=1}^{L_{j+1}} v_l^{(j+1)} G_l(\xi - \xi_l) \quad \text{with } v_l^{(j+1)} \in \mathbb{C}. \] (28)

The functions \( v_{j+1} \in \mathcal{V}_{j+1}, v_j \in \mathcal{V}_j \) and \( w_j \in \mathcal{W}_j \) are uniquely described by their Fourier coefficients given in (28), (25) and (26), respectively. Written as vectors we conclude
\[
v_{j+1} = \begin{pmatrix} v_1^{(j+1)} \\ \vdots \\ v_{L_{j+1}}^{(j+1)} \end{pmatrix} = (v_1^{(j+1)}, \ldots, v_{L_{j+1}}^{(j+1)})^T \in \mathbb{C}^{L_{j+1}},
\]
with \( v_1^{(j+1)} \in \mathbb{C}^{L_j} \) and \( v_{L_{j+1}}^{(j+1)} \in \mathbb{C}^{[L_{j+1} - L_j]} \),
\[
\tilde{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix} = (v_j^{(j)}, \ldots, v_j^{(j)}, 0, \ldots, 0)^T \in \mathbb{C}^{L_{j+1}},
\]
and
\[
w_j = \begin{pmatrix} w_1^{(j)} \\ \vdots \\ w_{L_{j+1}}^{(j)} \end{pmatrix} = (w_1^{(j)}, \ldots, w_{L_{j+1}}^{(j)})^T \in \mathbb{C}^{L_{j+1}}
\]
with \( w_1^{(j)} \in \mathbb{C}^{L_j} \) and \( w_{L_{j+1}}^{(j)} \in \mathbb{C}^{[L_{j+1} - L_j]} \). Summarizing we have
\[
\begin{pmatrix} v_1^{(j+1)} \\ v_{L_{j+1}}^{(j+1)} \end{pmatrix} = \begin{pmatrix} v_j \\ 0 \end{pmatrix} + \begin{pmatrix} w_1^{(j)} \\ w_{L_{j+1}}^{(j)} \end{pmatrix}.
\] (29)

Using matrix notation
\[
M_{j+1} := [G * G(\xi_k : \xi_l)]_{k,l=1}^{L_{j+1}},
\]
equation (27) can be written as
\[
\langle v_j, w_j \rangle = \tilde{v}_j^H M_{j+1} w_j = 0.
\]
The matrix \( M_{j+1} \) is regular. Hence, \( w_j \) can be represented as
\[
w_j = M_{j+1}^{-1} \begin{pmatrix} 0 \\ \tilde{w}_j \end{pmatrix}
\]
with some \( \tilde{w}_j \in \mathbb{C}^{[L_{j+1} - L_j]} \). This relation gives rise to a split of the matrix \( M_{j+1} \) into four submatrices \( M_j \in \mathbb{R}^{L_j \times L_j}, C_j \in \mathbb{R}^{(L_{j+1} - L_j) \times (L_{j+1} - L_j)} \) and \( B_j \in \mathbb{R}^{L_j \times (L_{j+1} - L_j)} \),
\[
M_{j+1} = \begin{pmatrix} M_j & B_j \\ C_j & 0 \end{pmatrix}.
\]
Here, $M_j$ as well as $C_j$ are positive definite. Multiplying the block matrix $M_{j+1}$ with the vector $w_j$ we find that $M_j w_j^{(1)} + B_j w_j^{(2)} = 0$, which gives

$$w_j^{(1)} = -M_j^{-1} B_j w_j^{(2)}.$$ 

Summarizing we have

$$w_j^{(2)} = v_j^{(2)},$$
$$w_j^{(1)} = -M_j^{-1} B_j v_j^{(2)},$$
$$v_j = v_j^{(1)} - M_j^{-1} B_j v_j^{(2)}.$$ 

Analogously to Section 3.6 we introduce the operators $R, R^*$ and $Q, Q^*$. Here $R$ is given by $R(v_{j+1}) = v_j$ where the map is defined in (32). Analogously, $Q$ is given by $Q(v_{j+1}) = w_j$ and the rules (30) and (31). These formulas describe the decomposition completely.

For reconstruction we obtain from (29) the splitting

$$v_j^{(1)} = v_j + v_j^{(1)},$$
$$v_j^{(2)} = w_j^{(2)}.$$ 

For $j \in \mathbb{N}$ the operator $R^*$ is given by

$$(R^*(v_j))_l := \begin{cases} (v_j)_l & \text{for } 1 \leq l \leq L_j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q^*(w_j) := w_j.$$ 

The schemes which illustrate these algorithms are graphically analogous to that ones given in Section 3.6. Further questions of localization and stability are discussed e.g. in [20].

References

10. A. Iske. Scattered data modelling using radial basis functions. This volume, Chapter 8.
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