Riemann’s Zeros and the Rhythm of the Primes

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November 18, 2009
“On the Number of Primes Less Than a Given Magnitude”
7 page paper offered to the *Monatsberichte der Berliner Akademie* on October 19, 1859. The exact publication date is unknown.
The counting function

We’ll use $p$ to denote a prime number,

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, \ldots \}.$$ 

The prime counting function is

$$\pi(x) := \#\{\text{primes } p \leq x\}.$$
Gauss’s conjecture

On a larger scale, the graph looks remarkably smooth:

In 1792, Gauss conjectured that \( \pi(x) \sim \frac{x}{\log x} \).

(This means that \( \pi(x)/\left(\frac{x}{\log x}\right) \to 1 \) as \( x \to \infty \).)
Gauss could compute $\pi(x)$ up to $x = 3,000,000$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\pi(x)$</th>
<th>$x / \log x$</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>4</td>
<td>0.921</td>
</tr>
<tr>
<td>$10^2$</td>
<td>25</td>
<td>22</td>
<td>1.151</td>
</tr>
<tr>
<td>$10^3$</td>
<td>168</td>
<td>145</td>
<td>1.161</td>
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<td>$10^4$</td>
<td>1229</td>
<td>1086</td>
<td>1.132</td>
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<td>9592</td>
<td>8686</td>
<td>1.104</td>
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<tr>
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<td>78498</td>
<td>72382</td>
<td>1.084</td>
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<td>5428681</td>
<td>1.061</td>
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<tr>
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<td>48254942</td>
<td>1.054</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>455052511</td>
<td>434294482</td>
<td>1.048</td>
</tr>
</tbody>
</table>

Convergence is rather slow...
Gauss later realized that the “log integral” function,

$$\text{Li}(x) := \int_2^x \frac{dt}{\log t} \quad \text{(also } \sim \frac{x}{\log x}),$$

would give a better approximation.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\pi(x)$</th>
<th>Li(x)</th>
<th>$\pi(x) / \text{Li}(x)$</th>
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<td>$10^{10}$</td>
<td>455052511</td>
<td>455055615</td>
<td>0.99999</td>
</tr>
</tbody>
</table>

The claim $\pi(x) \sim \text{Li}(x)$ became the Prime Number Conjecture.
The zeta function

Euler (1748) had studied the series

\[ \zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots , \]

(convergent for \( s > 1 \)). Euler factored this over the primes,

\[ \zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \ldots \right) \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \ldots \right) \left(1 + \frac{1}{5^s} + \ldots \right) \ldots , \]

and used geometric series,

\[ \sum_{k=0}^{\infty} \left(\frac{1}{p^s}\right)^k = \left(1 - \frac{1}{p^s}\right)^{-1} , \]

to conclude

\[ \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} . \]
Before Riemann, the zeta function was studied for real $s > 1$.

From $\lim_{s \to 1} \zeta(s) = \infty$, we can deduce that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty.$$
Riemann’s innovation was to consider $\zeta(s)$ for complex values of $s$.

Euler’s definition works for $\text{Re } s > 1$, and it appears we can bypass the obstacle at $s = 1$. 
Consider a simple example:

\[ \int_{1}^{\infty} t^{-s} \, dt = \frac{1}{s - 1}. \]

This integral converges for Re\( s > 1 \).

But the expression on the right makes sense for any complex \( s \neq 1 \).

We can use this formula to extend the meaning of the integral. For example, we could say

\[ \int_{1}^{\infty} t^2 \, dt = -\frac{1}{3}. \]

Because the extension is an analytic function, complex analysis tells that such an extension is uniquely defined.
Riemann’s zeta function

Riemann extended $\zeta(s)$ to all $s \neq 1$ by deriving the formula

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x},$$

where the integral is taken over a contour in the complex plane:

(The result equals $\sum \frac{1}{n^s}$ for $\text{Re } s > 1$ and is analytic for $s \neq 1$.)

With Riemann’s extension, we can make sense of divergent sums:

$$1 + 2 + 3 + 4 + \cdots = \zeta(-1) = -\frac{1}{12}.$$
Here’s $\text{Re } \zeta(s)$ in the original domain, $\text{Re } s > 1$. 
Here's Re $\zeta(s)$ after Riemann's extension.
Riemann zeros

Using the contour integral formula for $\zeta(s)$, Riemann observes

$$\zeta(s) = 0, \quad \text{for } s = -2, -4, -6, \ldots.$$  

He also shows that all other zeros lie in the strip $0 \leq \Re s \leq 1$. 

![Diagram showing the critical strip and trivial zeros]
Riemann’s claims about the zeros

1. The number of zeros $\rho$ in the critical strip with $0 \leq \text{Im} \rho \leq T$ is about

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}.$$ 

(Proven by Mangoldt in 1905.)

2. One finds about this many roots on the critical line, . . .

(Proven by Selberg in 1942: a positive fraction of the zeros lie on the critical line.)

3. . . . and it is very likely that they all do.

The Riemann Hypothesis!

“One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts. . . .”
The Riemann Hypothesis

Any non-trivial zero $\rho$ of $\zeta(s)$ has $\text{Re}\,\rho = \frac{1}{2}$.

We’ve verified this for the first $10,000,000,000,000$ zeros!

In terms of containing the zeros in a vertical strip, the best we can do is to replace $0 \leq \text{Re}\,\rho \leq 1$ by $0 < \text{Re}\,\rho < 1$. 
Chebyshev’s counting function

In counting primes convenient to:

1. Include prime powers $p^k$, for $k = 1, 2, 3, \ldots$.
2. Assign to each $p^k$ the weight $\log p$.

This defines the Chebyshev function

$$\psi(x) := \sum_{p^k \leq x} \log p.$$
The two counting functions are related by

$$\pi(x) + \frac{1}{2} \pi(x^{\frac{1}{2}}) + \frac{1}{3} \pi(x^{\frac{1}{3}}) + \cdots = \int_2^x \frac{d\psi(t)}{\log t}.$$

This gives two equivalent forms for the Prime Number Conjecture:

$$\pi(x) \sim \text{Li}(x) \iff \psi(x) \sim x.$$
Riemann’s explicit formula

Riemann gives a formula for $\pi(x)$ in terms of the non-trivial zeros $\rho$ of the zeta function. This is equivalent to

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_n x^{-2n} \frac{1}{2n} - \log 2\pi.$$  

(Proven by Mangoldt in 1895.)

Riemann’s strategy for the Prime Number Conjecture amounts to estimating $\sum_{\rho} \frac{x^\rho}{\rho}$ to prove that $\psi(x) \sim x.$
Riemann’s formula reveals a deep connection between primes and zeros.

To illustrate this, consider the derivative of $\psi(x)$,

$$\psi'(x) = 1 - \sum_{\rho} x^{\rho-1} - \frac{1}{x(x^2 - 1)}.$$

Since $\psi(x)$ is a step function that jumps at each prime power $p^k$, $\psi'(x)$ should be zero except for spikes at

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, \ldots.$$
In the sum

\[ \psi'(x) = 1 - \sum_{\rho} x^{\rho - 1} - \frac{1}{x(x^2 - 1)}, \]

each conjugate pair \( \{\rho, \bar{\rho}\} \) contributes a waveform

\[ x^{\rho - 1} + x^{\bar{\rho} - 1}. \]

For each waveform, \( \text{Re} \, \rho \sim \text{amplitude} \) and \( \text{Im} \, \rho \sim \text{frequency} \).
By adding up these waveforms, we can see the distribution of prime powers emerge.

Approximating $\psi'(x)$ with 1 pair of zeros:
By adding up these waveforms, we can see the distribution of prime powers emerge.

Approximating \( \psi'(x) \) with 10 pairs of zeros:
By adding up these waveforms, we can see the distribution of prime powers emerge.

Approximating $\psi'(x)$ with 100 pairs of zeros:
Rhythm of the primes

The Riemann zeros define the oscillatory modes that generate the distribution of prime numbers.

The Riemann Hypothesis claims a perfect balance among these modes: all modes have equal amplitudes.
If RH is false, then at least one “note” is louder than the others:

“If anything at all in our universe is correct, it has to be the Riemann Hypothesis, if for no other reasons, so for purely esthetical reasons.”

- A. Selberg
Prime Number Theorem

Riemann’s strategy does work: We can prove that

$$\text{Re} \, \rho < 1 \text{ for all } \rho,$$

which implies

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\rho} x^\rho = 0.$$ 

By the explicit formula for $\psi(x)$, this means

$$\psi(x) \sim x,$$

or

$$\pi(x) \sim \text{Li}(x).$$

(1896: Hadamard and de la Vallée-Poussin)
Cautionary tale

So we’ve checked $10^{13}$ zeros. Is that strong evidence?

Consider another Gauss conjecture: that $\pi(x) \leq \text{Li}(x)$ always.

We now know this to be true for all $x \leq 10^{14}$. However, \ldots

Littlewood proved in 1914 that $\pi(x)$ and $\text{Li}(x)$ cross infinitely often!

No value of $x$ for which $\pi(x) > \text{Li}(x)$ is currently known.