Errata for Spectral Theory

David Borthwick
February 28, 2021

Please email any additional corrections to dborthw@emory.edu.

Chapter 3

p. 51: Lemma 3.27
For the second statement of the lemma, we continue to assume that \( A \) is self-adjoint on \( \mathcal{D}(A) \), and add the hypothesis that \( A \) is essentially self-adjoint on a sub-domain. Here is a corrected wording: “Furthermore, if \( A \) is essentially self-adjoint on a core domain contained in \( \mathcal{D}(A) \), then \( A + B \) is essentially self-adjoint on this core domain.”

Chapter 4

p. 71: Proof of Theorem 4.5
The proof is correct, but the final equation should read
\[ \sigma(M_f) \subset \text{ess-range}(f), \]
i.e., the opposite direction to (4.3).

p. 84: Corollary 4.12
An argument based on the power series expansion (4.26) applies only if \( z \) and \( w \) lie in the same connected component of \( \rho(T) \). One can prove the formula more directly by observing that \( (T - z)^{-1}(T - z) = I \) on \( \mathcal{D}(T) \) and \( (T - z)(T - z)^{-1} = I \) on \( \mathcal{H} \). Thus
\[
(T - z)^{-1} - (T - w)^{-1} = (T - z)^{-1}(T - w)(T - w)^{-1} - (T - z)^{-1}(T - z)(T - w)^{-1} = (T - z)^{-1}(z - w)(T - w)^{-1}.
\]

p. 84: Second resolvent identity
This formula should read,
\[
(S - z)^{-1} - (T - z)^{-1} = (S - z)^{-1}(T - S)(T - z)^{-1},
\]
both here and in Exercise 4.1

p. 85: Corollary 4.13
Technically, the assumption \( \mathcal{H} \neq \{0\} \) should be included here.

p. 91: Fourth equation
This should read: By (4.31),
\[
(I - F(z))Q(z)^{-1}v = 0.
\]

p. 91: Final paragraph of the proof of Thm. 4.19
The specification of \( A \) and \( B \) is a bit unclear.
Chapter 5

p. 107: Final equation
The measure $\nu$ should perhaps have been defined more explicitly. A subset $E \subset Y$ consists of a collection of subsets $E_k \subset \mathbb{S}$. The measure is given by

$$\nu(E) := \sum_k \nu_k(E_k).$$

Since the measures $\nu_k$ are finite, the measure $\nu$ is $\sigma$-finite.

p. 115: Proof of Thm. 5.10
The claim that $f_\varepsilon(A) = \text{right side of (5.21)}$ should be justified. The integral in (5.21) defines an operator

$$B_\varepsilon := \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\lambda) \left[ (A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} \right] d\lambda.$$

This can be interpreted in the weak sense described in §4.2.1, as the unique operator for which

$$\langle u, B_\varepsilon v \rangle = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\lambda) \langle u, (A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1} v \rangle d\lambda,$$

for all $u, v \in \mathcal{H}$. Using the unitary transformation $Q$ provided by the spectral theorem (Theorem 5.6), this can be written

$$\langle u, B_\varepsilon v \rangle = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\lambda) \langle Q^{-1}u, (x - \lambda - i\varepsilon)^{-1} - (x - \lambda + i\varepsilon)^{-1} Q^{-1}v \rangle d\lambda,$$

where the inner product now takes place in $L^2(X, d\nu)$. Since $f$ is integrable and the expression in brackets is bounded, Fubini’s theorem allows us to take the $\lambda$ integral first, yielding

$$\langle u, B_\varepsilon v \rangle = \langle Q^{-1}u, f_\varepsilon(x)Q^{-1}v \rangle = \langle u, f_\varepsilon(A)v \rangle.$$

This proves that $B_\varepsilon = f_\varepsilon(A)$.

p. 116: Proof of Thm 5.11
Missing brackets in the second paragraph. The function $Q^{-1}\phi$ has support on $\{\alpha = \lambda\}$.

Chapter 6

p. 133: First equation
The $\gamma$ is erroneous. The first equation should read

$$|\langle f, v \rangle| \leq \|f\|\|v\| \leq \|f\|\|v\|_{H^1},$$
p. 136: Proof of Thm 6.8
The first sentence of the second paragraph is mixed up. This paragraph should read as follows:

If \( u \in \mathcal{D}(-\Delta_D) \), then by (6.13) we have

\[
\|u\|_{H^1}^2 = \langle u, (-\Delta + 1)u \rangle.
\]

By Cauchy-Schwarz and the fact that \( \|u\| \leq \|u\|_{H^1} \), this implies that

\[
\|u\|_{H^1} \leq \|(-\Delta + 1)u\|.
\]

This shows that \((-\Delta_D + 1)^{-1}\) is bounded as a map \( L^2(\Omega) \rightarrow H^1_0(\Omega) \). Therefore \((-\Delta_D + 1)^{-1}\) is compact as a map \( L^2(\Omega) \rightarrow L^2(\Omega) \) by Theorem 6.9.

p. 144: Corollary 6.16
The assumption on \( \psi \) should read \( \psi \in H^1(\Omega) \).

p. 151: Equation (6.48)
The \( \sigma \) in the brackets should be \( \sigma(-\Delta_D) \).

p. 151: Equation (6.49)
These inequalities are backwards. The equation should read:

\[
N_{\mathcal{R}_1}(t) \leq N_{\Omega}(t) \leq N_{\mathcal{R}_2}(t).
\]

p. 168: Last three equations
The \( \mu_t \) on the left side of the last three equations should be \( \nu_t \), as defined at the beginning of the proof.

p. 170: Sentence after (6.92)
The \( a \) should be capitalized in \( A\nu^{-1}s' \).

p. 172: Proof of Thm 6.34
The definition of \( \psi^\pm_1 \) should read

\[
\psi^\pm_1(x) := \max\{\pm\psi_1(x), 0\}.
\]

p. 177: Proof of Thm 6.36
The first sentence should read: “Let \( \phi_1 \) be the eigenfunction...”. (Since \( \lambda_1 \) is simple, \( \phi_1 \) is uniquely defined.)

p. 193: First eq. after Fig. 7.1
The \( \gamma \) should be \( \omega \):

\[
U_{\omega}f(x) := \omega^{\frac{1}{2}}f(\omega^{\frac{1}{2}}x).
\]
p. 195: Proof of Thm 7.7
In last part of the proof, $B$ was mistakenly assumed to be closed. Here is a clean version of the final two paragraphs:

Now assume that $A$ is merely essential self-adjoint. If $u \in D(\overline{A})$, then there exists a sequence $u_n \to u$ with $u_n \in D(A)$, such that $Au_n$ converges to $\overline{A}u$. By the assumption (7.24), the sequence $Bu_n$ also converges, so that $u \in D(\overline{B})$ (since $B$ is closed). By continuity, we can extend (7.24) to

$$\|\overline{B}u\| \leq \alpha \|\overline{A}u\| + \beta \|u\|$$

for all $u \in D(\overline{A})$. By the first part of the proof, this implies that $\overline{A} + \overline{B}$ is self-adjoint on the domain $D(\overline{A})$.

It remains to check that $\overline{A} + \overline{B} = \overline{A + B}$. Since $\overline{A} + \overline{B}$ is a closed extension of $A + B$, we have $\overline{A} + \overline{B} \subset \overline{A + B}$. On the other hand, the assumption (7.24) gives

$$\|(A + B)u\| \leq (\alpha + 1)\|Au\| + \beta \|u\|.$$ 

For $u \in D(\overline{A})$ this implies that $u \in D(\overline{A + B})$ and that $(\overline{A} + \overline{B})u = (\overline{A + B})u$. In other words,

$$\overline{A + B} \subset \overline{A + B}.$$

We conclude that $\overline{A + B}$ is self-adjoint on $D(A)$.

p. 206: First equation
A factor of $r^2$ is missing in the $h''$ term:

$$r^2h'' + 2rh' + (r + \lambda r^2 - l(l + 1))h = 0.$$ 

Also, the first sentence second paragraph should read “...extract the asymptotic behavior as $r \to \infty$,” not $r \to 0$.

Chapter 8

p. 232: Examples 8.6 and 8.7
The number of edges was notated inconsistently in these two examples. Each instance of $k$ should be replaced by $m$, the number of edges.

Acknowledgments: Thanks to Dieter Engelhardt for pointing out a number of typos. I am also grateful to the students in my Fall 2020 functional analysis class for helping to track down errors.