Checking the Poisson formula against the Selberg trace formula

David Borthwick
October 2007

In this note we will confirm the appearance of the extra $\frac{n_{\mathrm{c}}}{4}$ in the Poisson formula (Theorem 11.3 of [1]), by comparing this to the finite-area case of the Selberg trace formula. Let $\mathcal{R}_X$ be the resonance set for X a finite-area hyperbolic surface. For the sake of comparison, assume that Δ_X has no eigenvalue at 1/4 (which is guaranteed in the infinite-area case). Let $\mathcal{L}_X$ denote the primitive length spectrum of X.

The Selberg trace formula as given in [2, eq. (5.27)] says the following. For $g \in C_0^\infty(\mathbb{R})$ let $h = \widehat{g}$ and define

$$h_+(z) := \int_0^\infty g(t) > e^{zt}\, dt.$$

Then

$$\begin{aligned}\sum_{\zeta\in\mathcal{R}_X} h_+(\zeta - 1/2) &= \sum_{\ell\in\mathcal{L}_X}\sum_{k=1}^\infty \frac{\ell}{2\sinh(k\ell/2)} g(k\ell) \\ &\quad - \frac{\chi(X)}{2}\int_{-\infty}^\infty h(\xi)\,\xi\tanh\pi\xi\,d\xi - \frac{n_{\mathrm{c}}}{2\pi}\int_{-\infty}^\infty h(\xi)\,\Psi(1+i\xi)\,d\xi.\end{aligned} \tag{1}$$

The Poisson-wave trace formula from [1, eq. (12.1)] gives

$$\begin{aligned}\sum_{\zeta\in\mathcal{R}_X} e^{(\zeta-\frac{1}{2})t} &= \sum_{\ell\in\Lambda}\sum_{k=1}^\infty \frac{\ell}{2\sinh(k\ell/2)}\delta(t-k\ell) \\ &\quad + \frac{\chi(X)}{2}\frac{\cosh(t/2)}{\sinh^2(t/2)} + \frac{n_{\mathrm{c}}}{2}(\coth(t/2)-1).\end{aligned} \tag{2}$$

To compare the formulas, we take $\varphi \in C_0^\infty(\mathbb{R}_+)$ and set $g(t) = \varphi(|t|)$. Pairing φ with (2) gives

$$\begin{aligned}\sum_{\zeta\in\mathcal{R}_X} h_+(\zeta - 1/2) &= \sum_{\ell\in\mathcal{L}_X}\sum_{k=1}^\infty \frac{\ell}{2\sinh(k\ell/2)} g(k\ell) \\ &\quad + \frac{\chi(X)}{2}\int_0^\infty \varphi(t)\,\frac{\cosh(t/2)}{\sinh^2(t/2)}\,dt + \frac{n_{\mathrm{c}}}{2}\int_0^\infty \varphi(t)\,(\coth(t/2)-1)\,dt.\end{aligned}$$

The first line agrees with (1), so we just need to check the integrals.

For the $\chi(X)$ term, we start by computing the contour integral,

$$\int_{-\infty}^\infty e^{-i\xi t}\tanh\pi\xi\,d\xi = \frac{-i}{\sinh t/2},$$

for $t \neq 0$. This shows that the distributional Fourier transform of $\xi\tanh\pi\xi$ is

$$i\frac{d}{dt}\frac{-i}{\sinh t/2} = -\frac{1}{2}\frac{\cosh t/2}{\sinh^2 t/2},$$

away from $t = 0$. Since g is not supported near zero, we have

$$\begin{aligned} -\frac{\chi(X)}{2} \int_{-\infty}^{\infty} h(\xi)\, \xi \tanh \pi\xi \, d\xi &= \frac{\chi(X)}{4} \int_{-\infty}^{\infty} g(t) \frac{\cosh t/2}{\sinh^2 t/2}\, dt \\ &= \frac{\chi(X)}{2} \int_{0}^{\infty} \varphi(t) \frac{\cosh t/2}{\sinh^2 t/2}\, dt. \end{aligned}$$

Hence the $\chi(X)$ terms in (1) and (2) match.

Now for the n_{c} term. Using the formula,

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2},$$

we can compute the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-i\xi t}\, \Psi'(1+i\xi)\, d\xi = \begin{cases} 0 & t > 0 \\ -2\pi t \frac{e^t}{1-e^t} & t < 0 \end{cases}$$

Thus, away from $t = 0$, the distributional Fourier transform of $\Psi(1+i\xi)$ is

$$\begin{cases} 0 & t > 0 \\ -2\pi \frac{e^t}{1-e^t} & t < 0 \end{cases}$$

We find that

$$\begin{aligned} -\frac{n_{\mathrm{c}}}{2\pi} \int_{-\infty}^{\infty} h(\xi)\, \Psi(1+i\xi)\, d\xi &= n_{\mathrm{c}} \int_{-\infty}^{0} g(t) \frac{e^t}{1-e^t}\, dt \\ &= n_{\mathrm{c}} \int_{0}^{\infty} \varphi(t) \frac{e^{-t}}{1-e^{-t}}\, dt \\ &= \frac{n_{\mathrm{c}}}{2} \int_{0}^{\infty} \varphi(t)\, (\coth t/2 - 1)\, dt \end{aligned}$$

Thus the n_{c} terms agree also.

References

[1] D. Borthwick, *Scattering Theory for Infinite-Area Hyperbolic Surfaces*, Birkhäuser, Boston, 2007.

[2] W. Müller, Spectral geometry and scattering theory for certain complete surfaces of finite volume, *Invent. Math.* **109** (1992), 265–305.