Specialization of Divisors from Curves to Graphs

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Outline

1. The secret life of graphs
2. Tropical curves and their Jacobians
3. Graphs and arithmetic surfaces
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2. Tropical curves and their Jacobians

3. Graphs and arithmetic surfaces
By a graph $G$, we mean a connected, finite, undirected multigraph without loop edges.
Divisors

The group $\text{Div}(G)$ of divisors on $G$ is the free abelian group on $V(G)$.

We write elements of $\text{Div}(G)$ as formal sums

$$D = \sum_{v \in V(G)} a_v(v)$$

with $a_v \in \mathbb{Z}$.

A divisor $D$ is effective if $a_v \geq 0$ for all $v$.

The degree of $D = \sum a_v(v)$ is $\deg(D) = \sum a_v$.

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$$\text{Div}^0(G) = \{D \in \text{Div}(G) : \deg(D) = 0\}.$$
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Rational functions and principal divisors

- The group of **rational functions** on $G$ is
  \[ \mathcal{M}(G) = \{ \text{functions } f : V(G) \to \mathbb{Z} \} . \]

- The **Laplacian operator** $\Delta : \mathcal{M}(G) \to \text{Div}^0(G)$ is defined by
  \[ \Delta f = \sum_{v \in V(G)} \left( \sum_{e = vw} (f(v) - f(w)) \right) (v). \]

- The group of **principal divisors** on $G$ is the subgroup
  \[ \text{Prin}(G) = \{ \Delta f : f \in \mathcal{M}(G) \} \]
  of $\text{Div}^0(G)$. 
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The Jacobian

- Divisors $D, D' \in \text{Div}(G)$ are **linearly equivalent**, written $D \sim D'$, if $D - D'$ is principal.
- The Jacobian (or Picard group) of $G$ is

  $$\text{Jac}(G) = \text{Div}^0(G)/\text{Prin}(G).$$

- This is a finite abelian group whose cardinality is the number of spanning trees in $G$ (**Kirchhoff’s Matrix-Tree Theorem**).
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The Riemann-Roch theorem

- The **canonical divisor** on $G$ is

\[ K_G = \sum_{v \in V(G)} (\deg(v) - 2)(v). \]

Its degree is $2g - 2$, where $g = \dim_{\mathbb{R}} H_1(G, \mathbb{R})$ is the **genus** of $G$.

- Define $r(D)$ to be $-1$ iff $D$ is not equivalent to an effective divisor, and to be at least $k$ iff $D - E$ is equivalent to an effective divisor for every effective divisor of degree $k$.

**Theorem ("Riemann-Roch for graphs", B.–Norine)**

For every $D \in \text{Div}(G)$, we have

\[ r(D) - r(K_G - D) = \deg(D) + 1 - g. \]
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A word about the proof

The proof of the Riemann-Roch theorem for graphs is based on a combinatorial study of reduced divisors, which are distinguished coset representatives for the elements of $\text{Div}^0(G)/\text{Prin}(G)$. They are also known in the literature as $G$-parking functions.
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Tropical geometry

- Let $K$ be an algebraically closed field which is complete with respect to a (non-trivial) non-archimedean valuation $\text{val}$.
- Examples: $K = \mathbb{C}_p$ or $K = \text{the Puiseux series field } \mathbb{C}\{T\}$.
- If $X$ is a $d$-dimensional irreducible algebraic subvariety of the torus $(K^*)^n$, then

$$\text{Trop}(X) = \overline{\text{val}(X)} \subseteq \mathbb{R}^n$$

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A tropical cubic curve in $\mathbb{R}^2$
Metric graphs

- A **weighted graph** is a graph $G$ together with an assignment of a “length” $\ell(e) > 0$ to each edge $e \in E(G)$.

- A (compact) **metric graph** $\Gamma$ is just the “geometric realization” of a weighted graph: it is obtained from a weighted graph $G$ by identifying each edge $e$ with a line segment of length $\ell(e)$. In particular, $\Gamma$ is a compact metric space.

- A weighted graph $G$ whose geometric realization is $\Gamma$ will be called a **model** for $\Gamma$. 

![Diagram of metric graphs](image)
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![Diagram of weighted graphs](image-url)
Abstract tropical curves

Following Mikhalkin, an abstract tropical curve is just a “metric graph with a finite number of unbounded ends”.

Convention

We will ignore the unbounded ends and use the terms “tropical curve” and “metric graph” interchangeably.
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We will ignore the unbounded ends and use the terms “tropical curve” and “metric graph” interchangeably.
For a tropical curve $\Gamma$, we make the following definitions:

- \textbf{Div}(\Gamma) is the free abelian group on $\Gamma$.
- \textbf{M}(\Gamma) consists of all continuous piecewise affine functions $f : \Gamma \to \mathbb{R}$ with integer slopes.
- The Laplacian operator $\Delta : \mathcal{M}(\Gamma) \to \text{Div}^0(\Gamma)$ is defined by $-\Delta f = \sum_{p \in \Gamma} \sigma_p(f)(p)$, where $\sigma_p(f)$ is the sum of the slopes of $f$ in all tangent directions emanating from $p$.
- \textbf{Prin}(\Gamma) = \{ \Delta f : f \in \mathcal{M}(\Gamma) \}$.
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Notation and terminology

- $K$: a field which is complete with respect to a (nontrivial) non-archimedean valuation
- $R$: the valuation ring of $K$
- $k$: the residue field of $K$ (which we assume to be algebraically closed)
- $\mathfrak{X}$: an arithmetic surface, i.e., a flat proper scheme over $R$ whose generic fiber $X$ is a smooth (proper) geometrically connected curve over $K$. We call $\mathfrak{X}$ a model for $X$. 
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An arithmetic surface $\mathcal{X}/R$ is called **semistable** if its special fiber $\mathcal{X}_k$ is reduced and all singularities of $\mathcal{X}_k$ are ordinary double points. It is called **strongly semistable** if in addition every irreducible component of $\mathcal{X}_k$ is smooth.

By the **Semistable Reduction Theorem**, there exists a finite extension $L/K$ such that $X_L := X \times_K L$ has a strongly semistable model.

The **dual graph** of a strongly semistable arithmetic surface $\mathcal{X}$ is the weighted graph $G = G_\mathcal{X}$ whose vertices correspond to the irreducible components of $\mathcal{X}_k$ and whose edges correspond to the singular points of $\mathcal{X}_k$, with $\ell(e)$ equal to the thickness of the corresponding singularity.
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Semistability and dual graphs

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If $R$ is a DVR with maximal ideal $(\pi)$, the **thickness** of $z \in X^\text{sing}_k \subset X$ is the unique natural number $k$ such that $z$ has an analytic local equation of the form $xy = \pi^k$.

- $z$ is a **regular** point of $X$ iff its thickness is 1.
- In general, the **formal fiber** $\text{red}^{-1}(z) \subset X(\bar{K})$ is isomorphic (as a rigid analytic space) to an open annulus

$$A = \{ x \in \bar{K} : |x| < R \} \setminus \{ x \in \bar{K} : |x| \leq r \}$$

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Thickness of singularities
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The reduction graph

- The **reduction graph** $\Gamma = \Gamma_X$ of $X$ is the metric graph realization of $G_X$.

- **Remark:** $\Gamma_X$ sits naturally inside the Berkovich analytic space $X^{an}$ associated to $X$, and there is a canonical deformation retraction $X^{an} \to \Gamma_X$. Berkovich calls $\Gamma_X$ the **skeleton** of the model $X$. 
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Models and their reductions

\[ p \text{ odd prime} \]
\[ X / \mathbb{Q}_p^{nr} : (x^2 - 2y^2 + z^2)(x^2 - z^2) + p y^3 z = 0 \]

Stable model:

Minimal regular model:

Dual graph:

Dual graph:
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Let $Z_1, \ldots, Z_n$ be the irreducible components of $\mathcal{X}_k$, corresponding to the vertices $v_1, \ldots, v_n$ of $G_\mathcal{X}$.

Given a Cartier divisor $\mathcal{D} \in \text{Div}(\mathcal{X})$, let $\mathcal{L}$ be the associated line bundle, and define a homomorphism $\rho_\mathcal{X} : \text{Div}(\mathcal{X}) \to \text{Div}(G)$ by

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\rho_\mathcal{X}(\mathcal{D}) = \sum_i \deg(\mathcal{L}|_{Z_i})(v_i) \in \text{Div}(G).
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This extends to a homomorphism $\rho : \text{Div}(X) \to \text{Div}(G)$ by taking Zariski closures.
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$$\rho_{\mathcal{X}}(D) = \sum_{i} \text{deg}(\mathcal{L}|_{Z_i})(v_i) \in \text{Div}(G).$$

This extends to a homomorphism $\rho : \text{Div}(X) \rightarrow \text{Div}(G)$ by taking Zariski closures.
Specialization of divisors

- Suppose $R$ is a DVR, and let $\mathcal{X}/R$ be a regular strongly semistable arithmetic surface.
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- \( \deg(\rho(D)) = \deg(D) \).
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For every $D \in \text{Div}(X)$, we have $r_G(\rho(D)) \geq r_X(D)$.

For general $R$, there is also a specialization map $\tau : \text{Div}(X_{\overline{K}}) \rightarrow \text{Div}(\Gamma)$ which can be thought of in terms of Berkovich spaces as ‘retraction to the skeleton’. The analogous specialization inequality holds:

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Tropical Brill-Noether theory I

Theorem ("Tropical Brill-Noether theorem Part I", B.)

If \( g - (r + 1)(g - d + r) \geq 0 \), then every tropical curve \( \Gamma \) of genus \( g \) has a special divisor (a divisor \( D \) with \( \deg(D) \leq d \) and \( r(D) \geq r \)).

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- By a semicontinuity argument on the moduli space of metric graphs of genus $g$, we may assume without loss of generality that $\Gamma$ has rational edge lengths.
- Rescaling, we may assume that $\Gamma$ has a “regular model” $G$ (i.e., a model with all edge lengths equal to 1).
- By deformation theory, there exists a regular, totally degenerate, strongly semistable arithmetic surface $X$ with dual graph $G$.
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**Theorem (“Tropical Brill-Noether theorem Part II”, Cools-Draisma-Payne-Robeva)**

If \( g - (r + 1)(g - d + r) < 0 \), then there exist (infinitely many) tropical curves \( \Gamma \) of genus \( g \) having no special divisor.

Combining this combinatorial result with our specialization inequality gives a new proof of the Brill-Noether-Griffiths-Harris theorem in classical algebraic geometry: If \( g - (r + 1)(g - d + r) < 0 \), then on a general smooth algebraic curve \( X/\mathbb{C} \) of genus \( g \) there is no divisor \( D \) with \( \deg(D) \leq d \) and \( r(D) \geq r \).
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