Metric Properties of the Tropicalization of a Berkovich Curve

Joe Rabinoff

joint with

Matt Baker & Sam Payne
Notation

\( K=\bar{K} \) complete w.r.t. nontrivial non-Archimedean
\( \text{val}: K \rightarrow \mathbb{R} \cup \{\infty\} \quad 1 \cdot 1 = \exp(-\text{val}(1)) \)
\( \mathbb{R} \) the ring of integers in \( K \quad k = \mathbb{R}/\mathbb{Z} \mathbb{R} \)
\( \hat{X} \): complete connected smooth curve defined over \( K \)

Overview

\( \hat{X} \) gives rise to:

(i) \( \hat{X}^{\text{Berk}} \): analytic space in the sense of Berkovich. Massively infinite metric graph.

(ii) Given a toric embedding \( \hat{X} \hookrightarrow X(\Delta) \) with \( X = \hat{X} \cap \mathbb{T} \neq \emptyset \) get \( \text{Trop}(X) \subseteq \mathbb{R}^n \).
A (combinatorially) finite metric graph.
Thm (Payne):

\[ \hat{\times} \text{Berk} \cong \lim_{\ell \to \infty} \text{Trop}(X) \text{ canonically} \]

Our results roughly say this homeomorphism is an isometry.

- Nontrivial relation between metric structures on $\hat{\times} \text{Berk}$, $\text{Trop}(X)$
- Allows information to pass both ways.
Berkovich Curves

\( X: \) connected smooth affine curve/\( K \)

\( X^{\text{Berk}} := \{ \text{multiplicative seminorms} \ \| \cdot \| \text{ on} \ K[X] \text{ extending abs value on} \ K \} \)

Has weakest topology st. \( 1.1 \to 1/f1 \) cont \( \forall f \in K[X] \)

\( \bullet \ X(K) \hookrightarrow X^{\text{Berk}}, \quad K[X] \xrightarrow{1.1} K^{\text{RVS}}[x] \)

"Type 1"

Facts: \( X^{\text{Berk}} \) locally compact, Hausdorff, & path connected; \( X(K) \) dense

Other points of \( X = \textrm{Spec}(K[t]) \): \( \{ \text{"closed ball"} \}

\( \bullet \ Type 2: a \in K, \ r \in K^\times \)

\(|f|_B(a,r) := \sup \{|f(x)|: x \in B(a,r)\} \)

\( \bullet \ Type 3: \) likewise for \( r \notin K^\times \)

\( \bullet \ Type 4: \) not appearing in this talk
If $X = A'$ then $X$ is a tree. The path from $1 \cdot 1\beta(a,r)$ to $1 \cdot 1\beta(a',r')$:

- Increase $R$, $1 \cdot 1\beta(a,R)$ until $a_k a_l e 1\beta(a,R)$
- Decrease $R$, $1 \cdot 1\beta(a',R)$

**Metric**:

$$d(1 \cdot 1\beta(a,r), 1 \cdot 1\beta(a,R)) = \log(R) - \log(r)$$

**Subdomains**:

1) Open ball

$$D(0,1) := \bigcup_{r < 1} 1\beta(0,r)$$

A tree that deformation retracts onto $1 \cdot 1\beta(0,1)$
(ii) Open annulus

\[ A(0, r, R) = \{ z \in \mathbb{C} : r < |z| < R \} \]

\[ 0 \leq r < R \]

Contains a canonical line called the skeleton:

\[ \Omega := \{ z \in \mathbb{C}^2 : r < s < R \} \]

\[ A(0, r, R) \setminus \Omega \cong \mathbb{H} \]

So \( A(0, r, R) \) deformation retracts onto \( \Omega \).

\[ \text{NB: } \chi_{\text{Berk}}(\Omega) = \log(R) - \log(r) \]

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**Thm (semi-stable reduction):** General \( X \).

There is a finite set \( V \subseteq X_{\text{Berk}} \) of type-2 points such that

\[ X_{\text{Berk}} \setminus V \cong \underbrace{\bigcup_{r < R} A(0, r, R)}_{\text{finite}} \cong \underbrace{\mathbb{H}}_{\text{infinite}} \]

(includes punctured discs)
The metric structure on $X^{\text{Berk}}$ is thus induced by the metric on $(\mathcal{A}')^{\text{Berk}}$.

**Def:** The skeleton of $X$ is

$$\Gamma := \bigvee \{\text{the skeleta of the annuli}\}$$

This is a (combinatorially) finite metric graph.

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**Tropical Curves**

Now assume $X \subseteq T := \mathbb{G}_m^n$, so

$$K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cong K[T] \twoheadrightarrow K[X]$$

Define

$$\text{trop}: X^{\text{Berk}} \rightarrow \mathbb{R}^n$$

$$\text{trop}(1,0) := (-\log |x_1|, \ldots, -\log |x_n|)$$

$$\text{Trop}(X) := \text{trop}(X^{\text{Berk}})$$

**Thm (Kapranov, others):** $\text{Trop}(X)$ is a connected union of finitely many segments & rays (called edges) with rational slopes.
Example: $X: y = x - 1 \leq c^2$
Let $(x, y) \in X(K)$,
$\omega = (x, y) = (\text{val}(x), \text{val}(y)) \in \text{trop}(X)$
- $x = 0, \ y \geq 0$
- $x \geq 0, \ y = 0$
- $x = y \leq 0$

by the ultrametric inequality

In general if $\mathcal{F} = (\mathcal{F}_1, ..., \mathcal{F}_n) \in X(K)$ then
$\text{val}(a_1 \mathcal{F}) = \text{val}(a_1) + I \cdot \text{trop}(\mathcal{F})$

If $f = \sum a_i x^i \in \mathcal{O}_X$ = ideal for $X$
$\Rightarrow$ can't have unique minimal term among $\{\text{val}(a_1) + I \cdot \text{trop}(\mathcal{F})\}$

Gives piecewise linear conditions.
Determines $\text{Trop}(X)$ in a computable way.

Summary: $\text{Trop}(X) = \{ \omega \in \mathbb{R}^n : \text{the ultrametric inequality does not rule out the existence of } \mathcal{F} \in X(K) \text{ such that } \text{trop}(\mathcal{F}) = \omega \}$
Def: Let $\text{val}(k^n) \leq \mathbb{R}^n$. Define

$$R[T]^w := \{ \Sigma a_i x^i : \text{val}(a_i) + i \cdot \omega \geq 0 \}$$

An $R$-model for $T$.

$$X^w := \text{Spec}(R[T]^w/\text{ann}(R[T]^w))$$ (closure of $X$)

$\overline{X}^w := X^w \otimes_{\text{Rk}}$, the initial degeneration

$\overline{X}^w$ is easy to compute.

Thm (Speyer, Sturmfels, others): $\text{Trop}(X)$ is a finite union of open edges $e$ and vertices such that $\overline{X}^0$ is "constant" along $w_e$.

Def: The tropical multiplicity $m_{\text{Trop}}(e)$ of $e$ is the number of irreducible components of $\overline{X}^w$ (counted with multiplicity) for any $w_e$.

Def: The lattice length $l_{\text{Trop}}(e)$ is the length along a primitive lattice vector.

$\Rightarrow \text{Trop}(X)$ a finite metric graph

$l_{\text{Trop}}(e) = 2$
Theorems

\( X \subseteq \mathcal{G}_{\text{sm}} \) connected smooth curve
\& its compactification, \( \Gamma \subseteq X^\text{Berk} \) skeleton

**Thm (i):** Can assume \( \tau : \Gamma \to \text{Trop}(X) \)

  (i) \( \tau : X^\text{Berk} \to \text{Trop}(X) \) factors thru \( \Gamma \)

  (ii) \( \tau \) is affine linear with integer slope on each edge in \( \Gamma \)

  \[ (iii) \ e \in \Gamma \text{ edge, } s_i = \text{slope of } \log |x_i| \text{ on } e \]

  \[ \text{ex}(e) := \gcd(s_1, \ldots, s_n) \quad \text{"expansion factor"} \]

  Then \( \text{log}_X \tau(e) = \text{ex}(e) \cdot \text{ex}(e) \)

  \[ (iv) \ e \in \text{Trop}(X) \text{ edge } \Rightarrow \tau^{-1}(e) = \sum_{i=1}^{\text{i.e.}} e_i, \text{ and } \]

  \[ m_{\text{Trop}}(e') = \sum_{i=1}^{\text{i.e.}} \text{ex}(e_i) \]

**Cor:** \( \hat{X}^\omega \) integral for \( \omega e \) then \( \exists ! e \) above \( e' \), and \( e \to e' \) is an isometry.
Example: $K = \mathbb{C}_p$, $X \subseteq \mathbb{G}_m^2$, $X: y^2 + (2-p)y = x + (p+1)$

rational curve:

$x(t) = t(t-p)$  \quad  y(t) = t - 1$

\[\Gamma\]

\[\text{Trop}(X)\]

Thm\(\text{2}\): For any finite subgraph $\Gamma \subseteq X_{\text{Berk}}$

\[\exists X \hookrightarrow X(\Delta)\text{ s.t. } X = X \wedge \Xi \neq \emptyset\text{ and }\]

\[\text{trop}: \Gamma \hookrightarrow \text{Trop}(X)\]

is an isometry.

Thm\(\text{3}\): Let $\Lambda \subseteq \text{Trop}(X)$ closed & connected.

If $m_{\text{Trop}}(w) = 1$ for all $w \in \Lambda$ then $\exists$

\[\Lambda \longmapsto X_{\text{Berk}}\text{ isometric section}\]

Describes $X_{\text{Berk}}$ as $\lim$ finite metric subgraphs.
Application: \( E/K \) elliptic curve, \( \text{val}(j(E)) < 0 \) (bad reduction). Tate Uniformization: \( \exists! q \in K^*, \text{val}(q) = -\text{val}(j(E)) > 0 \) such that \( E \cong C_m/q\mathcal{O}_m \)

\[ \Omega := \text{skeleton = circle, circumference = } \text{val}(q) = -\text{val}(j(E)) \]

If \( \Omega \hookrightarrow \text{Trop}(E) \) an isometry \( \Rightarrow \exists \text{loop in Trop}(E), \text{lattice length} = -\text{val}(j(E)) \) ! Explains, strengthens results of Katz-Markwig-Markwig

Example (K-M-M): \( E: x^3y + xy^3 + \frac{1}{p} xy + x + ty = 0 \) \( \text{val}(j(E)) = -8 \). This embedding: trop isom.