**Rigid Cohomology for Algebraic Stacks**

by

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## Abstract

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We extend le Stum's construction of the overconvergent site [lS09] to algebraic stacks. We prove that étale morphisms are morphisms of cohomological descent for finitely presnted crystals on the overconvergent site. Finally, using the notion of an open subtopos of [73] we define a notion of overconvergent cohomology supported in a closed substack and show that it agrees with the classical notion of rigid cohomology supported in a closed subscheme.

# Contents

## Acknowledgments

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# Preface

In this thesis we use a recent and important foundational advance in the theory of rigid cohomology – the overconvergent site of [lS09] – to generalize rigid cohomology to algebraic stacks over fields of positive characteristic. The goal is to prove that this gives a Weil cohomology (i.e., those of [Pet03]).

This thesis is a first step – we define, for a closed substack $Z$, rigid cohomology with supports in $Z$. We prove that these groups enjoy the desired finiteness, functorality, and excision exact sequence one expects. Moreover, we prove that they agree with the classical construction for schemes. We also prove the useful technical result that cohomological descent for finitely presented overconvergent crystals with respect to étale morphisms holds on the overconvergent site, giving a different, shorter (and mildly stronger) proof of the main result of [CT03], (which is 153 pages). Many applications are expected, and will appear in future work.

# Chapter 1

# Introduction

The pursuit of a Weil, or 'topological', cohomology theory in algebraic geometry drove the development of Grothendieck's notion of a scheme and the subsequent ideas which permeate modern geometry and number theory. The initial success was the construction of étale cohomology and the subsequent proof of the Weil conjectures – that for a prime power $q = p^r$ and a variety $X$ over the finite field $\mathbb{F}_q$, the numbers $\{X(\mathbb{F}_{q^n})\}_{n\geq 1}$ of $\mathbb{F}_{q^n}$-points of $X$ (i.e., the number of solutions to the equations defining $X$ with values in $\mathbb{F}_{q^n}$) are goverened by surprising formulas – the surprise is that they depend on the dimensions of the singular cohomology $H^i(X'(\mathbb{C}); \mathbb{C})$ spaces of a lift $X'$ of $X$ to characteristic zero (when, of course, such a lift exists).

Other applications abound. For a prime $p$, the condition that two varieties $X$ and $X'$ with good reduction at $p$ have the *same* reduction at $p$ implies that their Betti numbers agree [Ill94, 1.3.8]. One can use existence of 'exotic torsion' in $p$-adic cohomology to give examples of surfaces over $\mathbb{F}_p$ with no lift to characteristic zero [Ill79, II.7.3]. There are practical algorithms for point counting – i.e., computing the numbers $X(\mathbb{F}_{p^n})$ as $p$ and $n$ vary – via cohomological methods which are much faster than brute force, an important problem in cryptography (see [Ked04] for a survey and [Ked01, Section 5] for precise asymptotics for hyperelliptic curves). One can give explicit examples of K3 surfaces defined over $\mathbb{Q}$ with Picard rank 1 [vL05], [AKR07], a famous problem of Mumford. The theme underlying these examples is that a Weil cohomology allows one to access characteristic 0 topological data associated to a lift $X'$ of $X$ via purely algebraic characteristic $p$ data attached to $X$; a great insight was that when there exists no such lift, a Weil cohomology is a legitimate substitite.

Building on ideas of Dwork [Dwo60] and Monsky and Washnitzer [MW68], Berthelot developed in [Ber86] a theory of *rigid cohomology*, a particular flavor of Weil cohomology. Let $K$ be a field with a non-trivial non-archimedean valuation and denote by $\mathcal{V}$ its valuation ring and by $k$ its residue field; then one can associate to a variety $X$ over $k$ a collection of vector spaces $H^i_{\mathrm{rig}}(X)$ over $K$. One merit is that rigid cohomology is extremely concrete; in contast to étale and crystalline cohomology one can do direct, explicit computer computations (making use of the equations for $X$) of rigid cohomology. A drawback is that the definition of rigid cohomology involves many choices, and independence of those choices and functorality are theorems in their own right.

A recent advance is Bernard le Stum's construction of an 'overconvergent site' [lS09] which gives an alternative, equivalent definition of rigid cohomology as the cohomology of the structure sheaf of a ringed site $(X_{\mathrm{AN}^\dagger}, \mathcal{O}_X^\dagger)$. One makes no choices in his definition (perhaps it is better to say that *all* possible choices are encoded in his definition) and one gets functorality for free. Other theorems come nearly for free too. For instance in this thesis we give a short proof of cohomological descent for finitely presented overconvergent crystals with respect to étale morphisms, one of the main results of [CT03, (153 pages)].

## 1.1 Background

Here we give some background and history so that we can explain in more detail the results of this thesis.

### 1.1.1 Algebraic de Rham cohomology and crystals

Let $X$ be a *smooth* algebraic variety over $\mathbb{C}$. Then the algebraic de Rham cohomology groups $H^i_{\mathrm{dR}}(X) = H^i(X; \Omega^\bullet_X)$ are topological in the sense that there are isomorphisms $H^i_{\mathrm{dR}}(X) \cong H^i(X(\mathbb{C}); \mathbb{C})$, where $X(\mathbb{C})$ is the complex manifold associated to $X$ (in particular it has the analytic topology and not the Zariski topology) and $H^i(X(\mathbb{C}); \mathbb{C})$ is the singular cohomology of $X(\mathbb{C})$.

Now suppose $X$ is *singular*. An idea of Herrera and Lieberman [HL71] (also later developed in [Har75]) is to embed $X$ into a smooth scheme $Z$ (e.g., if $X$ is quasi-projective one may take $X = \mathbb{P}^n$) and define the cohomlogy groups $H^i_{\mathrm{dR}}(X) = H^i(X; \widehat{\Omega^\bullet_Z})$; of course, one must check that this definition is independent of the choice of embedding into a smooth scheme $Z$. Here $\widehat{\Omega^\bullet_Z}$ is the formal completion of the de Rham complex of $Z$. The idea is that since $Z$ is smooth, formal limits of differential forms on $Z$ should behave like differential forms on a smooth variety, and indeed one obtains the analogous theorem that $H^i_{\mathrm{dR}}(X) \cong H^i(X(\mathbb{C}); \mathbb{C})$.

Even when there does not exist an embedding of $X$ into a smooth scheme (which can happen for varieties of dimension 3), there is still hope. Let the category $\mathrm{Inf}\, X$ consist of all pairs $(U, V)$ where $U \subset X$ is an open subset and $U \hookrightarrow V$ is a closed immersion with nilpotent defining ideal. Morphisms $(U', V') \to (U, V)$ are just compatible pairs of morphisms $U' \subset U$ and $V' \to V$. One declares a collection of morphisms $\{(U_i, V_i) \to (U, V)\}$ to be a covering if the morphisms $\{V_i \to V\}$ are an open covering of $V$. One gets an abelian category of sheaves and can study their cohomology. An example of a sheaf is $\mathcal{O} := \mathcal{O}_{X_{\mathrm{inf}}}$, which sends a pair $(U, V)$ to $\mathcal{O}_V(V)$.

We define the infinitesimal cohomology of $X$ to be $H^i(X_{\mathrm{inf}}, \mathcal{O})$. The theorem is then that when $X$ admits an embedding into a smooth variety, there is a natural isomorphism between the infinitesimal and de Rham cohomologies of $X$, and in general one recovers the theorem that $H^i(X_{\mathrm{inf}}, \mathcal{O}) \cong H^i(X(\mathbb{C}); \mathbb{C})$, so that infinitesmal cohomology is a valid replacement for de Rham cohomology. The charm of this theorem is that the site $\mathrm{Inf}\, X$ involves no differentials (though of course one can tease them out of this construction). This is nice because it avoids making a choice of embedding – for instance this makes it easy to see functorality, i.e., that a map $X \to Y$ induces a map $H^i_{\mathrm{dR}}(Y) \to H^i_{\mathrm{dR}}(X)$. Moreover, as we will see below, these ideas generalize, for instance to varieties over finite fields.

### 1.1.2 Weil cohomologies

A *Weil cohomology* is a cohomology theory satisfying certain axioms inspired by theorems about the cohomology of topological spaces (see for instance [Har77]Appendix C, Section 3). Let $p$ and $\ell$ be a pair of possibly equal primes. One then associates to a variety $X$ over the finite field $\mathbb{F}_p$ a $\mathbb{Q}_\ell$-vector space $H^i_\ell(X)$; this contrasts for example $H^i_{\mathrm{dR}}(X)$ which is an $\mathbb{F}_p$-vector space. When $X$ is the reduction mod $p$ of a variety $X'$ over $\mathbb{Q}_p$, one can pick an inclusion $\mathbb{Q}_p \subset \mathbb{C}$ and study $H_{\mathrm{dR}}(X'_{\mathbb{C}})$. One way in which our Weil cohomology is topological is that there is an isomorphism $H^i_\ell(X) \otimes_{\mathbb{Q}_p} \mathbb{C} \cong H^i_{\mathrm{dR}}(X'_{\mathbb{C}}) \cong H^i_\ell(X'(\mathbb{C}), \mathbb{C})$. Thus, while $H^i_\ell(X)$ is created from only characteristic $p$ data, it 'knows' about the singular cohomology of any lift of $X$.

A significant new idea, conjectured by Weil and explored by Grothendieck and Serre, was that, when there exists no such lift $X'$ of $X$, $H^i_\ell(X)$ still exhibits topological properties. One important example is the Lefschetz fixed point theorem, which says that the number of fixed points of a map $\phi$ (with some assumptions on $\phi$ of course) is equal to the alternating sum of the traces of $\phi$ acting on the $H^i_\ell(X)$'s. Applying this to the Frobenius map $F$ (noting that $X(\overline{\mathbb{F}_p})^F = X(\mathbb{F}_p)$) gives a cohomological way to count the number of $\mathbb{F}_p$ points of $X$. Other desired properties are finite dimensionality Poincaré duality, and a Kunneth formula; see [Kle68] and [Pet03] for a larger discussion of the aims of a Weil cohomology.

There are various flavors of Weil cohomologies, each with relative merits. Étale cohomology was the first, and the best understood. However it is difficult to compute, and only works when $\ell \neq p$ (where'works' means 'satisfies the axioms of a Weil cohomology'). Crystalline cohomology, conceived by Grothendieck [Gro68] and executed by Berthelot [Ber74], fills this gap, giving $\mathbb{Q}_p$-vector spaces, but only works when $X$ is smooth and complete. There is a logarithmic version of crystalline cohomology which works for proper, log smooth log schemes; these crystalline flavors are also difficult to compute.

### 1.1.3 Rigid cohomology

Berthelot defined in [Ber86] rigid cohomology as a variant of de Rham cohomology on some rigid space associated to a variety $X$ over a finite field. Its big selling points are that it works (i.e., satisfies the axioms of a Weil cohomology) for arbitrary varieties $X$ over finite fields (e.g., singular or non-proper $X$) and that it is very concrete and amenable to calculation.

To cook up this rigid space one has to make many choices: given a variety $X$ over a perfect field $k$ of characteristic $p$ and an embedding $X \hookrightarrow P$ into a formal scheme $P$ topologically of finite type over $\operatorname{Spf} W(k)$, one sets $K$ equal to the fraction field of $W(k)$ and defines the rigid cohomology of $X$ as the hypercohomology of the overconvergent de Rham complex of the tube $]X[_{P_K}$ of $X$ in the Raynaud generic fiber $P_K$ of $P$. Here the overconvergent de Rham complex is just the restriction (in the sense of [73]) of the de Rham complex of $P_K$. The definition is a bit unwieldy but ultimately familiar, nothing more than a de Rham construction on a geometric object.

Difficulties abound. One must prove that this definition is independent of the embedding $X \hookrightarrow P$. When there does not exist an embedding one must take cohomological descent as a definition. Cohomological descent itself is a difficult theorem because rigid cohomology is not the cohomology of a site. Defining a category of coefficients $\operatorname{Isoc}^\dagger X$ is subtle, and Grothendieck's six operations still need work. Functorality is hard. The relative situation is very difficult too because relative quasi-projectivity of a map to a scheme is not satisfied in general; here there is still much work to be done (e.g., coherence and base change; see [Tsu03b]). Nonetheless one can prove that rigid cohomology is a Weil cohomology, and even give a rigid proof of Deligne's Weil II results [Ked06b].

A powerful new technical tool is le Stum's recent work [lS09] which defines a ringed site $(\mathrm{AN}^\dagger X, \mathcal{O}_X^\dagger)$ whose cohomology computes the rigid cohomology of $X$. More specifically the category $\mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}_X^\dagger$ is equivalent to $\operatorname{Isoc}^\dagger X$ and the cohomology of a finitely presented module $E$ is isomorphic to the rigid cohomlogy of the associated isocrystal. While this approach sacrifices the concreteness of Berthelot's, many foundational results become simple; for example, functorality is now easy and independence of the choice of embedding is no longer a concern – all possible choices are encoded in the definition.

## 1.2 Rigid cohomology for algebraic stacks

In light of le Stum's results, Kiran Kedlaya suggested the problem of *extending rigid cohomology to algebraic stacks*. The application he had in mind was to Lafforgue's work on geometric Langlands for $\mathrm{GL}_n$ of function fields in characteristic $p$ [Laf98], extending to overconvergent isocrystals the correspondence between lisse $\ell$-adic sheaves and automorphic representations; the point here is that central to Lafforgue's work is a singular moduli stack of 'vector bundles with extra structure' (which is not Deligne-Mumford, but still separated!) whose cohomology captures both automorphic and $p$-adic data (including Galois and Frobenius actions). As described in [Ked06b, Section 1.5], a generalization of rigid cohomology to stacks would resolve a conjecture of Deligne. Below we describe other applications.

Le Stum's construction immediately generalizes to give a definition of rigid cohomology for stacks; the work of this thesis is to construct variants with supports which agree with the classical definitions for schemes and to prove duality and excision theorems and verify various compatibilities. The aim is to prove that this satisfies the axioms of a Weil cohomology, and this thesis is the beginning of such work. From this, various finiteness theorems will follow formally. Cohomological descent is another necessary result, useful in particular for computations and comparison to other cohomologies; we give a very short proof of cohomological descent for rigid cohomology with respect to smooth hypercovers.

### 1.2.1 Outline and statement of results:

Here we discuss the contents of individual chapters and state the results of this thesis.

Chapter 2, **Definitions**: We recall the definitions and important theorems about the overconvergent site of [lS09]. Then we give their generalizations to stacks.

Chapter 3, **Cohomological descent**: Cohomological descent is a generalization of Čech cohomology and a key computational tool, often allowing one to give deduce statements about cohomology of general varieties from the case of smooth varieties.

We use le Stum's overconvergent site to give a very short proof that cohomological descent holds for crystals on the overconvergent site with respect to smooth hypercovers of schemes, which is one of the main results of [CT03] (which totals 153 pages). An earlier draft of this thesis included a proof of the proper case (which is one of the main results of [Tsu03a]); we thank Bernard le Stum for pointing out an error in the original argument. We have changed this to a conjecture and highlighted any results which use this.

Chapter 4, **Cohomology with support in a closed subset**: In [73, exposé iv, section 9] there is a very general notion of open and closed subtopos. Applying this to $\mathrm{AN}^\dagger(X)$, one obtains a functorial construction $H^i_Z(X, E)$, and theorems like excision come for free. The work is to check that when $X$ is a scheme this agrees with the classical construction, which we have done. While rigid cohomology with supports is well known (see [lS07, Definition 8.2.5]), cohomology with support in a closed subscheme on the overconvergent site is a new result. We are still able to prove cohomological descent for finite flat hypercovers, which is an ingredient in the proof of the étale case.

**Appendix**: We include a short review of topoi and categorical constructions 5.1 and a review of Berkovich spaces 5.2.

# Chapter 2

# Definitions

In [lS09], le Stum associates to a variety $X$ over a field $k$ of characteristic $p$ a ringed site $(\mathrm{AN}^\dagger_{\mathrm{g}}(X), \mathcal{O}^\dagger_{X_{\mathrm{g}}})$ and proves an equivalence $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{O}^\dagger_{X_{\mathrm{g}}}) \cong \mathrm{Isoc}^\dagger(X)$ between the category of finitely presented $\mathcal{O}^\dagger_{X_{\mathrm{g}}}$-modules and the category of overconvergent isocrystals on $X$. Moreover, he proves that the cohomology of a finitely-presented $\mathcal{O}^\dagger_{X_{\mathrm{g}}}$-module agrees with the usual rigid cohomology of its associated overconvergent isocrystal.

His work is in fact more general, and to any presheaf $T$ on the category $\mathbf{Sch}_k$ of schemes over $k$, he associates a ringed site $(\mathrm{AN}^\dagger_{\mathrm{g}}(T), \mathcal{O}^\dagger_{T_{\mathrm{g}}})$. We will define the category of overconvergent isocrystals on $T$ to be $\mathrm{Isoc}^\dagger(T) := \mathrm{Mod}_{\mathrm{fp}}(\mathcal{O}^\dagger_{T_{\mathrm{g}}})$ and study the resulting cohomology theory when $T$ is an algebraic space. With a little more work this will generalize to a stack (or even a fibered category) $\mathcal{X}$ over $k$ and allow us to define a ringed site $(\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{X}), \mathcal{O}^\dagger_{\mathcal{X}_{\mathrm{g}}})$.

In this chapter we recall the basic definitions of [lS09] and explain their mild generalizations to stacks.

## 2.1 Notations and conventions

Throughout $K$ will denote a field of characteristic 0 that is complete with respect to a non-trivial non-archimedean valuation with valuation ring $\mathcal{V}$, whose maximal ideal and residue field we denote by $\mathfrak{m}$ and $k$. We denote the category of schemes over $k$ by $\mathbf{Sch}_k$. We define an **algebraic variety over** $k$ to be a scheme such that there exists a *locally finite* cover by schemes of finite type over $k$ (recall that a collection $\mathcal{S}$ of subsets of a topological space $X$ is said to be locally finite if every point of $X$ has a neighborhood which only intersects finitely many subsets $X \in \mathcal{S}$). Note that we do not require an algebraic variety to be reduced, quasi-compact, or separated.

**Formal Schemes**: As in [lS09, 1.1] we define a formal $\mathcal{V}$-scheme to be a locally topologically finitely presented formal scheme $P$ over $\mathcal{V}$, i.e., a formal scheme $P$ with a locally finite covering by formal affine schemes Spf $A$, with $A$ topologically of finite type (i.e., a quotient of the ring $\mathcal{V}\{T_1, \cdots, T_n\}$ of convergent power series by an ideal $I + \mathfrak{a}\mathcal{V}\{T_1, \cdots, T_n\}$, with $I$ an ideal of $\mathcal{V}\{T_1, \cdots, T_n\}$ of finite type and $\mathfrak{a}$ an ideal of $\mathcal{V}$). This finiteness property is necessary to define the 'generic fiber' of a formal scheme; see 5.2.6 of the appendix.

We refer to [Gro60, 1.10] for basic properties of formal schemes. The first section of [Ber99] is another good reference. Actually, [lS09, Section 1] contains everything we will need.

***K*-analytic spaces**: We refer to [Ber93] (as well as the brief discussion in [lS09, 4.2]) for definitions regarding $K$-analytic spaces. As in [lS09, 4.2], we define an **analytic variety** over $K$ to be a locally Hausdorff topological space $V$ together with a maximal affinoid atlas $\tau$ which is locally defined by *strictly* affinoid algebras. In Appendix 5.2 we collect and review necessary facts from $K$-analytic geometry, and in particular we note that an analytic variety $V$ has a Grothendieck topology which is finer than its usual topology, which we denote by $V_G$ and refer to as 'the $G$-topology' on $V$.

**Topoi**: We follow the conventions of [73] (exposited in [lS09, 4.1]) regarding sites, topologies, topoi, and localization; see appendix 5.1 for a review. When there is no confusion we will identify an object of a category with its associated presheaf. For a topos $T$ we denote by $\mathbb{D}_+(T)$ the derived category of bounded below complexes of objects of $\operatorname{Ab} T$. Often a morphism $(f^{-1}, f_*)\colon (T, \mathcal{O}_T) \to (T', \mathcal{O}_{T'})$ of ringed topoi will satisfy $f^{-1}\mathcal{O}_{T'} = \mathcal{O}_T$, so that there is no distinction between the functors $f^{-1}$ and $f^*$; in this case, we will write $f^*$ for both.

**Algebraic Spaces and Stacks**: We refer to [Knu71] and [LMB00] for basic definitions regarding algebraic spaces and stacks. Note in particular the standard convention that a representable morphism of stacks is represented by *algebraic spaces*. Actually, most of the theory works for arbitrary fibered categories, but particular examples and theorems will require algebraicity and finiteness assumptions, which will be clearly stated when necessary.

## 2.2 The overconvergent site

Following [lS09], we make the following series of definitions; see [lS09] for a more detailed discussion of the definitions with some examples.

**Definition 2.2.1** ([lS09], 1.2)**.** Define an **overconvergent variety** over $\mathcal{V}$ to be a pair $(X \subset P, V \xrightarrow{\lambda} P_K)$, where $X \subset P$ is a locally closed immersion of an algebraic variety $X$ over $k$ into the special fiber $P_k$ of a formal scheme $P$ (recall our convention that all formal schemes are topologically finitely presented over Spf $\mathcal{V}$), and $V \xrightarrow{\lambda} P_K$ is a morphism of analytic varieties, where $P_K$ denotes the generic fiber of $P$, which is an analytic space (in contrast to the Raynaud generic fiber, which is a rigid analytic space; see 5.2.6). When there is no confusion we will write $(X, V)$ for $(X \subset P, V \xrightarrow{\lambda} P_K)$ and $(X, P)$ for $(X \subset P, P_K \xrightarrow{\mathrm{id}} P_K)$. Define a **formal morphism** $(X', V') \to (X, V)$ of overconvergent varieties to be a commutative diagram

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \longleftarrow & P'_K & \longleftarrow & V' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\ X & \hookrightarrow & P & \longleftarrow & P_K & \longleftarrow & V \end{array}$$

where $f$ is a morphism of algebraic varieties, $v$ is a morphism of formal schemes, and $u$ is a morphism of analytic varieties.

Finally, define $\mathrm{AN}(\mathcal{V})$ to be the category whose objects are overconvergent varieties and morphisms are formal morphisms. We endow $\mathrm{AN}(\mathcal{V})$ with the **analytic topology**, defined to be the topology generated by families $\{(X_i, V_i) \to (X, V)\}$ such that for each $i$, the maps $X_i \to X$ and $P_i \to P$ are the identity maps, $V_i$ is an open subset of $V$, and $V = \bigcup V_i$ is an open covering (recall that an open subset of an analytic space is admissible in the $G$-topology and thus also an analytic space – this can be checked locally in the $G$-topology, and for an affinoid this is clear because there is a basis for the topology of open affinoid subdomains).

**Definition 2.2.2** ([lS09], Section 1.1)**.** The specialization map $P_K \to P_k$ induces by composition a map $V \to P_k$ and we define the **tube** $]X[_V$ of $X$ in $V$ to be the preimage of $X$ under this map. The tube $]X[_{P_K}$ admits the structure of an analytic space and the inclusion $i_X \colon ]X[_{P_K} \hookrightarrow P_K$ is a locally closed immersion of analytic spaces (and generally not open, in contrast to the rigid case). The tube $]X[_V$ is then the fiber product $]X[_{P_K} \times_{P_K} V$ (as analytic spaces) and in particular is also an analytic space.

*Remark* 2.2.3. A formal morphism $(f, u) \colon (X', V') \to (X, V)$ induces a morphism $]f[_u \colon ]X'[_{V'} \to ]X[_V$ of tubes. Since $]f[_u$ is induced by $u$, when there is no confusion we will sometimes denote it by $u$.

The fundamental topological object in rigid cohomology is the tube $]X[_V$, and the important notions are defined only up to neighborhoods of $]X[_V$. We immediately make this precise by modifying $\mathrm{AN}(\mathcal{V})$.

**Definition 2.2.4** ([lS09], Definition 1.3.3)**.** Define a formal morphism

$$(f, u) \colon (X', V') \to (X, V)$$

to be a **strict neighborhood** if $f$ and $]f[_u$ are isomorphisms and $u$ induces an isomorphism from $V'$ to a neighborhood $W$ of $]X[_V$ in $V$.

**Definition 2.2.5.** We define the category $\mathrm{AN}^\dagger(\mathcal{V})$ of overconvergent varieties to be the localization of $\mathrm{AN}(\mathcal{V})$ by strict neighborhoods (which is possible by [lS09, Proposition 1.3.6]): the objects of $\mathrm{AN}^\dagger(\mathcal{V})$ are the same as those of $\mathrm{AN}(\mathcal{V})$ and a morphism $(X', V') \to (X, V)$ in $\mathrm{AN}^\dagger(\mathcal{V})$ is a pair of formal morphisms

$$(X', V') \leftarrow (X', W) \to (X, V),$$

where $(X', W) \to (X', V')$ is a strict neighborhood.

The functor $\mathrm{AN}(\mathcal{V}) \to \mathrm{AN}^\dagger(\mathcal{V})$ induces the image topology on $\mathrm{AN}^\dagger(\mathcal{V})$ (defined in 5.1.19 to be the largest topology on $\mathrm{AN}^\dagger(\mathcal{V})$ such that the map from $\mathrm{AN}^\dagger(\mathcal{V})$ is continuous. By [lS09, Proposition 1.4.1], the image topology on $\mathrm{AN}^\dagger(\mathcal{V})$ is generated by the pretopology of collections $\{(X, V_i) \to (X, V)\}$ with $\bigcup V_i$ an open covering of a neighborhood of $]X[_V$ in $V$ and $]X[_V = \bigcup ]X[_{V_i}$.

*Remark* 2.2.6. From now on any morphism $(X', V') \to (X, V)$ of overconvergent varieties will denote a morphism in $\mathrm{AN}^\dagger(\mathcal{V})$. One can give a down to earth description of morphisms in $\mathrm{AN}^\dagger(\mathcal{V})$ [lS09, 1.3.9]: to give a morphism $(X', V') \to (X, V)$, it suffices to give a neighborhood $W'$ of $]X'[_{V'}$ in $V'$ and a pair $f\colon X' \to X, u\colon W' \to V$ of morphisms which are *geometrically pointwise compatible*, i.e., such that $u$ induces a map on tubes and the outer square of the diagram

$$\begin{array}{ccc} W' & \xrightarrow{u} & V \\ \cup\!\mid & & \cup\!\mid \\ ]X'[_{W'} & \xrightarrow{]f[_u} & ]X[_V \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

commutes (and continues to do so after any base change by any isometric extension $K'$ of $K$).

**Definition 2.2.7.** For any presheaf $T \in \widehat{\mathrm{AN}^\dagger(\mathcal{V})}$, we define $\mathrm{AN}^\dagger(T)$ to be the localized category $\mathrm{AN}^\dagger(\mathcal{V})_{/T}$ whose objects are morphisms $h_{(X,V)} \to T$ (where $h_{(X,V)}$ is the presheaf associated to $(X, V)$) and morphisms are morphisms $(X', V') \to (X, V)$ which induce a commutative diagram

$$\begin{array}{ccc} h_{(X',V')} & \longrightarrow & h_{(X,V)} \\ & \searrow \quad \swarrow & \\ & T & \end{array} .$$

We may endow $\mathrm{AN}^\dagger(T)$ with the induced topology (see 5.1.19), i.e., the smallest topology making continuous the projection functor $\mathrm{AN}^\dagger(T) \to \mathrm{AN}^\dagger(\mathcal{V})$ [lS09, Definition 1.4.7]; concretely, the covering condition is the same as in 2.2.5. When $T = h_{(X,V)}$ we denote $\mathrm{AN}^\dagger(T)$ by $\mathrm{AN}^\dagger(X, V)$. Since the projection $\mathrm{AN}^\dagger T \to \mathrm{AN}^\dagger \mathcal{V}$ is a fibered category, the projection is also cocontinious with respect to the induced topology. Finally, an algebraic space $X$ over $k$ defines a presheaf $(X', V') \mapsto \mathrm{Hom}(X', X)$, and we denote the resulting site by $\mathrm{AN}^\dagger(X)$.

There will be no confusion in writing $(X, V)$ for an object of $\mathrm{AN}^\dagger(T)$.

We use subscripts to denote topoi and continue the above naming conventions – i.e., we denote the category of sheaves of sets on $\mathrm{AN}^\dagger(T)$ (resp. $\mathrm{AN}^\dagger(X, V), \mathrm{AN}^\dagger(X)$) by $T_{\mathrm{AN}^\dagger}$

(resp. $(X,V)_{\mathrm{AN}^\dagger}, X_{\mathrm{AN}^\dagger}$). Any morphism $f\colon T' \to T$ of presheaves on $\mathrm{AN}^\dagger(\mathcal{V})$ induces a morphism $f_{\mathrm{AN}^\dagger}\colon T'_{\mathrm{AN}^\dagger} \to T_{\mathrm{AN}^\dagger}$ of topoi. In the case of the important example of a morphism $(f,u)\colon (X',V') \to (X,V)$ of overconvergent varieties, we denote the induced morphism of topoi by $(u^*_{\mathrm{AN}^\dagger}, u_{\mathrm{AN}^\dagger *})$.

For an analytic space $V$ we denote by $\operatorname{Open} V$ the category of open subsets of $V$ and by $V_{\mathrm{an}}$ the associated topos of sheaves of sets on $\operatorname{Open} V$. Recall that for an analytic variety $(X,V)$, the topology on the tube $]X[_V$ is induced by the inclusion $i_X\colon ]X[_V \hookrightarrow V$.

**Definition 2.2.8** ([lS09, Corollary 2.1.3])**.** Let $(X,V)$ be an overconvergent variety. Then there is a morphism of sites

$$\varphi_{X,V}\colon\ \mathrm{AN}^\dagger(X,V) \to \operatorname{Open} ]X[_V.$$

The notation as usual is in the 'direction' of the induced morphism of topoi and in particular backward; it is associated to the functor $\operatorname{Open} ]X[_V \to \mathrm{AN}^\dagger(X,V)$ given by $U = W \cap ]X[_V \mapsto (X,W)$ (and is independent of the choice of $W$ up to strict neighborhoods). This induces a morphism of topoi

$$(\varphi^{-1}_{X,V}, \varphi_{X,V*})\colon (X,V)_{\mathrm{AN}^\dagger} \to (]X[_V)_{\mathrm{an}}.$$

**Definition 2.2.9** ([lS09, 2.1.7])**.** Let $(X,V) \in \mathrm{AN}^\dagger(T)$ be an overconvergent variety over $T$ and let $F \in T_{\mathrm{AN}^\dagger}$ be a sheaf on $\mathrm{AN}^\dagger(T)$. We define the **realization** $F_{X,V}$ of $F$ on $]X[_V$ to be $\varphi_{(X,V)*}(F|_{(X,V)_{\mathrm{AN}^\dagger}})$, where $F|_{(X,V)_{\mathrm{AN}^\dagger}}$ is the restriction of $F$ to $\mathrm{AN}^\dagger(X,V)$.

We can describe the category $T_{\mathrm{AN}^\dagger}$ in terms of realizations in a manner similar to sheaves on the crystalline or lisse-étale sites.

**Proposition 2.2.10** ([lS09], Proposition 2.1.8)**.** *Let $T$ be a presheaf on $\mathrm{AN}^\dagger(\mathcal{V})$. Then the category $T_{\mathrm{AN}^\dagger}$ is equivalent to the following category :*

1. *An object is a collection of sheaves $F_{X,V}$ on $]X[_V$ indexed by $(X,V) \in \mathrm{AN}^\dagger(T)$ and, for each $(f,u)\colon (X',V') \to (X,V)$, a morphism $\phi_{f,u} : ]f[_u^{-1} F_{X,V} \to F_{X',V'}$, such that as $(f,u)$ varies, the maps $\phi_{f,u}$ satisfy the usual compatibility condition.*

2. *A morphism is a collection of morphisms $F_{X,V} \to G_{X,V}$ compatible with the morphisms $\phi_{f,u}$.*

To obtain a richer theory we endow our topoi with sheaves of rings and study the resulting theory of modules.

**Definition 2.2.11** ([lS09], Definition 2.3.4)**.** Define the **sheaf of overconvergent functions** on $\mathrm{AN}^\dagger(\mathcal{V})$ to be the presheaf of rings

$$\mathcal{O}^\dagger_{\mathcal{V}}\colon (X,V) \mapsto \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V)$$

where $i_X$ is the inclusion of $]X[_V$ into $V$; this is a sheaf by [lS09, Corollary 2.3.3]. For $T \in \widehat{\mathrm{AN}^\dagger(\mathcal{V})}$ a presheaf on $\mathrm{AN}^\dagger(\mathcal{V})$, define $\mathcal{O}^\dagger_T$ to be the restriction of $\mathcal{O}^\dagger_{\mathcal{V}}$ to $\mathrm{AN}^\dagger(T)$.

We follow our naming conventions above, for instance denoting by $\mathcal{O}^\dagger_{(X,V)}$ the restriction of $\mathcal{O}^\dagger_{\mathcal{V}}$ to $\mathrm{AN}(X,V)$.

*Remark* 2.2.12. By [lS09, Proposition 2.3.5, (i)], the morphism of topoi of Definition 2.2.8 can be promoted to a morphism of ringed sites

$$(\varphi^*_{X,V}, \varphi_{X,V*})\colon (\mathrm{AN}^\dagger(X,V), \mathcal{O}^\dagger_{(X,V)}) \to (]X[_V, i_X^{-1}\mathcal{O}_V).$$

In particular, for $(X,V) \in \mathrm{AN}^\dagger T$ and $M \in \mathcal{O}^\dagger_T$, the realization $M_{X,V}$ is an $i_X^{-1}\mathcal{O}_V$-module. For any morphism $(f,u)\colon (X',V') \to (X,V)$ in $\mathrm{AN}^\dagger(T)$, one has a map

$$(]f[^\dagger_u, ]f[_{u*})\colon (]X'[_{V'}, i^{-1}_{X',V'}\mathcal{O}_{V'}) \to (]X[_V, i^{-1}_{X,V}\mathcal{O}_V).$$

of ringed sites, and functoriality gives transition maps

$$\phi^\dagger_{f,u}\colon ]f[^\dagger_u M_{X,V} \to M_{X',V'}$$

which satisfy the usual cocycle compatibilities.

We can promote the description of $T_{\mathrm{AN}^\dagger}$ in Proposition 2.2.10 to descriptions of the categories $\mathrm{Mod}\,\mathcal{O}^\dagger_T$ of $\mathcal{O}^\dagger_T$-modules, $\mathrm{QCoh}\,\mathcal{O}^\dagger_T$ of quasi-coherent $\mathcal{O}^\dagger_T$-modules (i.e., modules which locally have a presentation), and $\mathrm{Mod}_{\mathrm{fp}}\,\mathcal{O}^\dagger_T$ of locally finitely presented $\mathcal{O}^\dagger_T$-modules.

**Proposition 2.2.13** ([lS09], Proposition 2.3.6)**.** *Let $T$ be a presheaf on $\mathrm{AN}^\dagger(\mathcal{V})$. Then the category $\mathrm{Mod}\,\mathcal{O}^\dagger_T$ (resp. $\mathrm{QCoh}\,\mathcal{O}^\dagger_T$, $\mathrm{Mod}_{\mathrm{fp}}\,\mathcal{O}^\dagger_T$) is equivalent to the following category :*

1. *An object is a collection of sheaves $M_{X,V} \in \mathrm{Mod}\,i_X^{-1}\mathcal{O}_V$ (resp. $\mathrm{QCoh}\,i_X^{-1}\mathcal{O}_V$, $\mathrm{Coh}\,i_X^{-1}\mathcal{O}_V$) on $]X[_V$ indexed by $(X,V) \in \mathrm{AN}^\dagger(T)$ and, for each $(f,u)\colon (X',V') \to (X,V)$, a morphism (resp. isomorphism) $\phi^\dagger_{f,u}\colon ]f[^\dagger_u M_{X,V} \to M_{X',V'}$, such that as $(f,u)$ varies, the maps $\phi^\dagger_{f,u}$ satisfy the usual compatibility condition.*

2. *A morphism is a collection of morphisms $M_{X,V} \to M'_{X,V}$ compatible with the morphisms $\phi^\dagger_{f,u}$.*

**Definition 2.2.14** ([lS09], Definition 2.3.7)**.** Define the **category of overconvergent crystals on** $T$, denoted $\mathrm{Cris}^\dagger T$, to be the full subcategory of $\mathrm{Mod}\,\mathcal{O}^\dagger_T$ such that the transition maps $\phi^\dagger_{f,u}$ are isomorphisms.

**Example 2.2.15.** The sheaf $\mathcal{O}^\dagger_T$ is a crystal, and in fact $\mathrm{QCoh}\,\mathcal{O}^\dagger_T \subset \mathrm{Cris}^\dagger T$.

*Remark* 2.2.16. It follows immediately from the definition of the pair $(\varphi^*_{X,V}, \varphi_{X,V*})$ of functors that $\varphi_{X,V*}$ of a $\mathcal{O}^\dagger_{(X,V)}$-module is a crystal, and that the adjunction $\varphi^*_{X,V}\varphi_{X,V*}E \to E$ is an isomorphism if $E$ is a crystal. If follows that the pair $\varphi^*_{X,V}$ and $\varphi_{X,V*}$ induce an equivalence of categories

$$\mathrm{Cris}^\dagger(X,V) \to \mathrm{Mod}\,i_X^{-1}\mathcal{O}_V;$$

see [lS09, Proposition 2.3.8] for more detail.

*Remark* 2.2.17. An advantage of the use of sites and topoi is that the relative theory is simple. For instance, for a morphism $T' \to T$ of presheaves on $\mathrm{AN}^\dagger(\mathcal{V})$ the associated morphism of sites $\mathrm{AN}^\dagger(T') \to \mathrm{AN}^\dagger(T)$ is isomorphic to the projection morphism associated to the localization $\mathrm{AN}^\dagger(T)_{/T'} \to \mathrm{AN}^\dagger(T)$ (and in particular one gets for free an exact left adjoint $u_!$ to the pullback functor $u^*\colon \mathrm{Ab}(T_{\mathrm{AN}^\dagger}) \to \mathrm{Ab}(T'_{\mathrm{AN}^\dagger})$; see 5.1.24).

One minor subtlety is the choice of an overconvergent variety as a base.

**Definition 2.2.18.** Let $(C, O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be an overconvergent variety and let $T \to C$ be a morphism from a presheaf on $\mathbf{Sch}_k$ to $C$. Then $T$ defines a presheaf on $\mathrm{AN}^\dagger(C, O)$ which sends $(X, V) \to (C, O)$ to $\mathrm{Hom}_C(X, T)$, which we denote by $T/O$. We denote the associated site by $\mathrm{AN}^\dagger(T/O)$, and when $(C, O) = (S_k, S)$ for some formal $\mathcal{V}$-scheme $S$ we write instead $\mathrm{AN}^\dagger(T/S)$.

The minor subtlety is that there is no morphism $T \to h_{(C,O)}$ of presheaves on $\mathrm{AN}^\dagger(\mathcal{V})$. A key construction is the following.

**Definition 2.2.19** ([lS09, Paragraph after Corollary 1.4.15])**.** Let $(X, V) \to (C, O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be a morphism of overconvergent varieties. We denote by $X_V/O$ the image presheaf of the morphism $(X, V) \to X/O$, considered as a morphism of presheaves. Explicitly, a morphism $(X', V') \to X/O$ lifts to a morphism $(X', V') \to X_V/O$ if and only if there exists a morphism $(X', V') \to (X, V)$ over $X/O$, and in particular different lifts $(X', V') \to (X, V)$ give rise to the same morphism $(X', V') \to X_V/O$. When $(C, O) = (\operatorname{Spec} k, \mathrm{M}(K))$, we may write $X_V$ instead $X_V/\mathrm{M}(K)$.

Many theorems will require the following extra assumption of [lS09, Definition 1.5.10]. Recall that a morphism of formal schemes $P' \to P$ is said to be proper at a subscheme $X \subset P'_k$ if, for every component $Y$ of $\overline{X}$, the map $Y \to P_k$ is proper (see [lS09, Definition 1.1.5]).

**Definition 2.2.20.** Let $(C, O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be an overconvergent variety and let $f\colon X \to C$ be a morphism of $k$-schemes. We say that a formal morphism $(f, u)\colon (X, V) \to (C, O)$, written as

$$\begin{array}{ccccc} X & \hookrightarrow & P & \leftarrow & V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ C & \hookrightarrow & Q & \leftarrow & O \end{array},$$

is a **geometric realization** of $f$ if $v$ is proper at $X$, $v$ is smooth in a neighborhood of $X$, and $V$ is a neighborhood of $]X[_{P_K \times_{Q_K} O}$ in $P_K \times_{Q_K} O$. We say that $f$ is **realizable** if there exists a geometric realization of $f$.

**Example 2.2.21.** Let $Q$ be a formal scheme and let $C$ be a closed subscheme of $Q$. Then any projective morphism $X \to C$ is realizable.

We need a final refinement to $\mathrm{AN}^\dagger(\mathcal{V})$.

**Definition 2.2.22.** We say that an overconvergent variety $(X, V)$ is **good** if there is a good neighborhood $V'$ of $]X[_V$ in $V$ (i.e., every point of $]X[_V$ has an affinoid neighborhood in $V$; see 5.2.5). We say that a formal scheme $S$ is good if the overconvergent variety $(S_k, S_K)$ is good. We define the **good overconvergent site** $\mathrm{AN}^\dagger_{\mathrm{g}}(T)$ to be the full subcategory of $\mathrm{AN}^\dagger(T)$ consisting of good overconvergent varieties. Given a presheaf $T \in \mathrm{AN}^\dagger(\mathcal{V})$, we denote by $T_{\mathrm{g}}$ the restriction of $T$ to $\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$.

Note that localization commutes with passage to good variants of our sites (e.g., there is an isomorphism $\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})_{/T_{\mathrm{g}}} \cong \mathrm{AN}^\dagger_{\mathrm{g}}(T)$). When making further definitions we will often omit the generalization to $\mathrm{AN}^\dagger_{\mathrm{g}}$ when it is clear.

The following proposition will allow us to deduce facts about $\mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}^\dagger_{X_g}$ from results about $(X, V)$ and $X_V$.

**Proposition 2.2.23.** *Let $(C, O) \in \mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$ be a good overconvergent variety and let $(X, V) \to (C, O)$ be a geometric realization of a morphism $X \to C$ of schemes. Then the following are true:*

*(i) The map $(X, V)_{\mathrm{g}} \to (X/O)_{\mathrm{g}}$ is a covering in $\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$.*

*(ii) There is an equivalence of topoi $(X_V/O)_{\mathrm{AN}^\dagger_g} \cong (X/O)_{\mathrm{AN}^\dagger_g}$.*

*(iii) The natural pullback map $\mathrm{Cris}^\dagger_{\mathrm{g}}\, X/O \to \mathrm{Cris}^\dagger_{\mathrm{g}}\, X_V/O$ is an equivalence of categories.*

*(iv) Suppose that $(X, V)$ is good. Then the natural map $\mathrm{Cris}^\dagger\, X_V/O \to \mathrm{Cris}^\dagger_{\mathrm{g}}\, X_V/O$ is an equivalence of categories.*

*Proof.* The first two claims are [lS09, 1.5.14, 1.5.15], the third follows from the second, and the last is clear.

□

In particular, the natural map $\mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}^\dagger_{X_{\mathrm{g}}} \to \mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}^\dagger_{(X_V)_{\mathrm{g}}} \cong \mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}^\dagger_{X_V}$ is an equivalence of categories.

## 2.3 Calculus on the overconvergent site and comparison with the classical theory

Here we compare constructions on the overconvergent site and on the ringed spaces $(]X[_V, i_X^{-1}\mathcal{O}_V)$ to the classical constructions of rigid cohomlogy (exposited for example in

[lS07]), introducing along the way the variants of 'infinitesimal calculus' useful in this thesis.

Let $V$ be an analytic variety. Recall (see 5.2) that $V$ has a Grothendieck topology (generated by affinoid subdomains) which is finer than its usual topology; we refer to this as 'the $G$-topology' on $V$ and write $V_G$ when we consider $V$ with its $G$-topology. The natural morphism $\pi\colon V_G \to V$ (induced by the morphism $\mathrm{id}\colon V \to V_G$ on underlying sets) is a morphism of ringed sites. When $V$ is good, the functor $F \mapsto F_G := \pi^* F$ is fully faithful and induces an equivalence of categories

$$\mathrm{Coh}\,\mathcal{O}_V \cong \mathrm{Coh}\,\mathcal{O}_{V_G}.$$

Indeed, for an admissible $W \in \tau_V$, $\pi^*F(W) = \varinjlim_{W \subset W'} F(W')$, where the limit is taken over all open neighborhoods $W'$ of $W$. The unit $\mathrm{id} \to \pi_*\pi^*F$ of adjunction is then visibly an isomorphism, so by lemma 5.1.4, we conclude that $\pi^*$ is an isomorphism.

Recall also that the set $V_0$ of rigid points of $V$ has the structure of a rigid analytic variety such that the inclusion $V_0 \hookrightarrow V$ induces an equivalence $(V_0, \mathcal{O}_{V_0}) \cong (V_G, \mathcal{O}_{V_G})$ of ringed topoi, in particular inducing equivalences

$$\mathrm{Mod}\,\mathcal{O}_{V_0} \cong \mathrm{Mod}\,\mathcal{O}_{V_G}$$

and

$$\mathrm{Coh}\,\mathcal{O}_{V_0} \cong \mathrm{Coh}\,\mathcal{O}_{V_G} \cong \mathrm{Coh}\,\mathcal{O}_V.$$

We denote by $\pi_0$ the composition $\widetilde{V_0} \cong \widetilde{V_G} \to \widetilde{V}$, and for a bounded below complex of abelian sheaves $E_0 \in \mathbb{D}_+(\widetilde{V_0})$ define $E_0^{\mathrm{an}}$ to be $\mathbb{R}\pi_0 E$. When $V$ is good and $E_0$ is coherent there is an isomorphism $E_0^{\mathrm{an}} \cong \pi_0 E_0$ (this follows from [Ber93, 1.3.6 (ii)]). Moreover, suppose that $(X, V)$ is a good overconvergent variety, $]\overline{X}[_V = V$, and $E_0$ is a coherent $j^\dagger_{X_0}\mathcal{O}_{V_0}$-module (see Definition 2.3.1 below). Then by [lS09, Proposition 3.4.3 (3)], $E_0^{\mathrm{an}} \cong \pi_0 E$.

Now let $(X, V)$ be a good overconvergent variety. We studied above (Proposition 2.2.13) the ringed site $(]X[_V, i_X^{-1}\mathcal{O}_V)$. To study the analogue in the classical rigid theory and to compare the two we first make the following definitions.

**Definition 2.3.1.** Let $(X, V)$ be a good overconvergent variety and assume that the inclusion $i_X\colon ]X[_V \hookrightarrow V$ is closed (which we can do since $(X, V) \cong (X, ]\overline{X}[_V)$ in $\mathrm{AN}^\dagger(\mathcal{V})$, where $\overline{X}$ is the closure of $X$ in $P$). Let $]X[_{V_0}$ be the underlying rigid space $(]X[_V)_0$ of $]X[_V$ (see 5.2.5); alternatively, $]X[_{V_0}$ is isomorphic to the rigid analytic tube, i.e., the preimage of $X$ with respect to the composition $V_0 \to (P_K)_0 \to P_k$, where $(P_K)_0$ is the Raynaud generic fiber of $P$ (see the second paragraph of 5.2.6).

Denote by $i_{X_0}\colon ]X[_{V_0} \hookrightarrow V_0$ the corresponding inclusion of rigid analytic spaces and let $F \in \widetilde{V}$ (resp. $F_0 \in \widetilde{V_0}$). We define functors $j_X^\dagger$ [lS09, Proposition 2.2.12] and $j_{X_0}^\dagger$ [lS07, Proposition 5.1.2] by

$$j_X^\dagger F = i_{X*} i_X^{-1} F$$

and

$$j^\dagger_{X_0} F_0 = \varinjlim j'_{0*} j_0'^{-1} F_0$$

where the limit runs over all strict neighborhoods $V_0'$ of $]X[_{V_0}$ in $V_0$ (recall from [lS07, Definition 3.1.1] that a strict neighborhood of $]X[_{V_0}$ in $V_0$ is an admissible open subset $V_0'$ containing $]X[_{V_0}$ such that the covering $\{V_0', V_0 - ]X[_{V_0}\}$ is an admissible covering of $V_0$).

**Proposition 2.3.2.** *With the notation of Definition 2.3.1, the following are true.*

*(i) There is a natural isomorphism*

$$j^\dagger_X F = \varinjlim j'_* j'^{-1} F$$

*where the limit runs over all immersions of neighborhoods $V'$ of $]X[_V$ in $V$.*

*(ii) The functors $i_X^{-1}$ and $i_{X*}$ induce an equivalence of categories*

$$\mathrm{QCoh}\, j^\dagger_X \mathcal{O}_V \to \mathrm{QCoh}\, i_X^{-1} \mathcal{O}_V$$

*which restricts to give an equivalence on coherent sheaves.*

*(iii) The functors*

$$\mathrm{Coh}\,\mathcal{O}_{V'} \to \mathrm{Coh}\, i_X^{-1}\mathcal{O}_V \to \mathrm{Coh}\, j^\dagger_X \mathcal{O}_V,$$
$$E \mapsto i^{-1}_{X,V'} E \mapsto i_{X*} i^{-1}_{X,V'} E,$$

*where $V'$ ranges over neighborhoods of $]X[_V$ in $V$ and $i_{X,V'}$ denotes the inclusion $]X[_V \hookrightarrow V'$, induce equivalences of categories*

$$\varinjlim \mathrm{Coh}\,\mathcal{O}_{V'} \cong \mathrm{Coh}\, i_X^{-1}\mathcal{O}_V \cong \mathrm{Coh}\, j^\dagger_X \mathcal{O}_V.$$

*(iii') The functors*

$$\mathrm{Coh}\,\mathcal{O}_{V'} \to \mathrm{Coh}\, j^\dagger_{X_0}\mathcal{O}_{V_0},\ E \mapsto i_{X_0*} i^{-1}_{X_0,V'} E,$$

*where $V'$ ranges over strict neighborhoods of $]X[_{V_0}$ in $V_0$ and $i_{X_0,V'}$ denotes the inclusion $]X[_{V_0} \hookrightarrow V'$, induce an equivalence of categories*

$$\varinjlim \mathrm{Coh}\,\mathcal{O}_{V'} \cong \mathrm{Coh}\, j^\dagger_{X_0}\mathcal{O}_{V_0}.$$

*(iv) The map $E \mapsto E^{\mathrm{an}}$ induces an equivalence of categories*

$$\mathrm{Coh}\, j^\dagger_{X_0}\mathcal{O}_{V_0} \cong \mathrm{Coh}\, j^\dagger_X \mathcal{O}_V.$$

In particular, $\mathrm{Mod}_{\mathrm{fp}}\, \mathcal{O}^\dagger_{X,V}$ is equivalent to $\mathrm{Coh}\, j^\dagger_{X_0}\mathcal{O}_{V_0}$.

*Proof.* Claim (i) is [lS09, 2.2.12].

For (ii), it suffices to check that the unit $\mathrm{id} \to i_{X*}i_X^*$ and counit $i_X^* i_{X*} \to \mathrm{id}$ of adjunction are isomorphisms (where $i_X^*$ is the composition of $i_X^{-1}$ and tensoring). By Remark 4.1.1 the inclusion $i_X$ induces an immersion of topoi, and in particular the map $i_{X*}$ is fully faithful, so by Lemma 5.1.4 we conclude that the adjunction $i_X^* i_{X*} \to \mathrm{id}$ is an isomorphism. For the other direction, let $E \in \mathrm{QCoh}\, j_X^\dagger \mathcal{O}_V$. We can check locally that the adjunction is an isomorphism, so we may assume that $E$ has a global presentation. Since $]X[_V$ is closed in $V$, $i_{X*}$ is exact [Ber93, 4.3.2] (and $i_X^{-1}$ is always exact) so that the adjunction induces a diagram

$$\begin{array}{ccccccc} \bigoplus_I j_X^\dagger \mathcal{O}_V & \longrightarrow & \bigoplus_J j_X^\dagger \mathcal{O}_V & \longrightarrow & E & \longrightarrow & 0\,. \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_I i_{X*} i_X^{-1} j_X^\dagger \mathcal{O}_V & \longrightarrow & \bigoplus_J i_{X*} i_X^{-1} j_X^\dagger \mathcal{O}_V & \longrightarrow & i_{X*} i_X^{-1} E & \longrightarrow & 0 \end{array}$$

Thus to prove the claim it is thus enough to check that the adjunction $j_X^\dagger \mathcal{O}_V \to i_{X*} i_X^{-1} j_X^\dagger \mathcal{O}_V$ is an isomorphism, which is true since we can write this as $i_{X*} i_X^{-1} \mathcal{O}_V \to i_{X*} i_X^{-1} i_{X*} i_X^{-1} \mathcal{O}_V$, which is $i_{X*}$ applied to the adjunction $i_X^{-1}\mathcal{O}_V \to i_X^{-1} i_{X*} i_X^{-1} \mathcal{O}_V$ and thus an isomorphism (by the beginning of this paragraph).

Claim (iii) is [lS09, Proposition 2.2.12] (and (ii)) and claim (iii') is [lS07, Theorem 5.4.4].

Finally, note that, from the explicit description of the functor $E \mapsto E^{\mathrm{an}}$, following diagram commutes

$$\begin{array}{ccc} \varinjlim \mathrm{Coh}\, \mathcal{O}_{V_0'} & \longrightarrow & \mathrm{Coh}\, j_{X_0}^\dagger \mathcal{O}_{V_0}\,. \\ \downarrow & & \downarrow \\ \varinjlim \mathrm{Coh}\, \mathcal{O}_{V'} & \longrightarrow & \mathrm{Coh}\, j_X^\dagger \mathcal{O}_V \end{array}$$

Claim (iv) then follows from (iii) and (iii') together with [lS09, Corollary 1.3.2] (which says that there is a cofinal system of neighborhoods $\{V'\}$ such that the system $\{V_0'\}$ is a cofinal system of strict neighborhoods) and the isomorphism

$$\mathrm{Coh}\, \mathcal{O}_{V_0'} \cong \mathrm{Coh}\, \mathcal{O}_{V'}.$$

□

*Remark* 2.3.3. A benefit of using Berkovich spaces instead of rigid analytic spaces is that the analogous construction $i_{X_0}^{-1}\mathcal{O}_{V_0}$ in rigid geometry does not serve the same purpose, since the closed inclusion $i_{X_0}\colon\, ]X[_{V_0} \hookrightarrow V_0$ is also open and so $i_{X_0 *} i_{X_0}^{-1} \mathcal{O}_{V_0}$ is not isomorphic to $j_{X_0}^\dagger \mathcal{O}_{V_0}$. If instead one lets $U$ denote the open complement of $]X[_{V_0} \subset V_0$ and then denotes by $i\colon Z \hookrightarrow \widetilde{V}_0$ the closed complement of $U \subset V_0$ in the sense of Section 4.1, then by [lS07, 5.1.12 (i)] the functor $j_{X_0}^\dagger$ is isomorphic to $i_* i^*$. This is a nice instance of the utility of the abstract notion of an immersion of topoi.

**Infinitesimal Calculus**: We recall here several definitions from [lS09, Section 2.4].

Let $V, V' \to O$ be two morphisms of analytic spaces. Then by [Ber93, Proposition 1.4.1], the fiber product $V \times_O V'$ exists – when $V, V'$ and $O$ are affinoid spaces the fiber product is given by the Gelfand spectrum of the completed tensor product of their underlying algebras, and the global construction is given by glueing this construction. As usual the underlying topological space of $V \times_O V'$ is not the fiber product of their underlying topological spaces.

Let $V \to O$ be a morphism of analytic varieties. Then the diagonal morphism $\Delta \colon V \to V \times_O V$ is a $G$-locally closed immersion (see the comments after the proof of [Ber93, Proposition 1.4.1]). We define the relative sheaf of differentials $\Omega_{V/O}$ to be the conormal sheaf of $\Delta$. When $\Delta$ is a closed immersion defined by an ideal $I$, $\Omega_{V/O}$ is the restriction of $I/I^2$ to $V$; in general one can either define the conormal sheaf locally (and check that it glues) or argue that when $V$ is good, $\Delta$ factors as composition of a closed immersion $i \colon V \hookrightarrow U$ into an admissible open $U$, with $i$ defined by an ideal $J$, and define the conormal sheaf as the restriction of $J/J^2$.

Due to the use of completed tensor products, the sheaf of differentials is generally not isomorphic to the sheaf of Kahler differentials. It does however enjoy all of the usual properties; see [Ber93, 3.3].

**Definition 2.3.4.** Let $(X, V) \to (C, O)$ be a morphism of overconvergent varieties. Suppose that $V$ is good and that $i_X^{-1} \colon ]X[_V \hookrightarrow V$ is closed. We define the category $\mathrm{MIC}(X, V/O)$ of **overconvergent modules with integrable connection** to be the category of pairs $(M, \nabla)$, where $M \in \mathrm{Mod}\, i_X^{-1}\mathcal{O}_V$ and $\nabla \colon M \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O}$ is an $i_C^{-1}\mathcal{O}_O$-linear map satisfying the Leibniz rule and such that the induced map $\nabla \circ \nabla \colon M \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^2_{V/O}$ is zero. Morphisms $(M, \nabla) \to (M', \nabla')$ are morphisms $M \to M'$ as $i_X^{-1}\mathcal{O}_V$-modules which respect the connections (see [lS09, Definition 2.4.5]). Similarly, we define a category $\mathrm{MIC}(X_0, V_0/O_0)$ of such pairs $(M_0, \nabla_0)$ with $M_0 \in \mathrm{Coh}\, j^\dagger_{X_0}\mathcal{O}_{V_0}$ (see [lS07, Definition 6.1.8]).

Let $(M, \nabla) \in \mathrm{MIC}(X, V/O)$. Then $\nabla$ extends to a complex

$$M \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O} \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^2_{V/O} \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^3_{V/O} \to \ldots$$

of abelian sheaves, which we call the **de Rham complex** of $(E, \nabla)$ and write as $M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}$. We define the de Rham complex of $(M_0, \nabla_0) \in \mathrm{MIC}(X_0, V_0/O_0)$ similarly.

The bridge between crystals and modules with integrable connection is the notion of a stratification, which we now define.

**Definition 2.3.5.** Let $(X, V) \to (C, O)$ be a morphism of overconvergent varieties. Set $V^2 = V \times_O V$ and denote by

$$p_1, p_2 \colon (X, V^2) \to (X, V)$$

the two projections. We define an **overconvergent stratification** on an $i_X^{-1}\mathcal{O}_V$-module $M$ to be an isomorphism

$$\epsilon \colon p_2^\dagger M \cong p_1^\dagger M$$

of $i_X^{-1}\mathcal{O}_{V^2}$-modules satisfying the evident cocycle condition on triple products (see for example [BO78, Definition 2.10]). We denote the category of such pairs $(M, \epsilon)$ by $\operatorname{Strat}^\dagger(X, V/O)$, where morphisms are morphisms of $i_X^{-1}\mathcal{O}_V$-modules which respect the stratification. We define the rigid variant $\operatorname{Strat}^\dagger(X_0, V_0/O_0)$ analogously.

We omit a discussion of the notion of more general (than overconvergent) stratifications.

*Remark* 2.3.6. One can relate crystals and overconvergent stratifications as follows. Let $(X, V) \to (C, O)$ be a morphism of overconvergent varieties and suppose that $(X, V)$ is a good overconvergent variety. Let $E \in \operatorname{Cris}^\dagger X_V/O$ and consider the diagram

$$(X, V^2) \cong (X, V) \times_{X_V/O} (X, V) \underset{p_2}{\overset{p_1}{\rightrightarrows}} (X, V) \xrightarrow{p} X_V/O .$$

Then the composition $\epsilon$ of the two isomorphisms

$$\epsilon\colon p_2^\dagger E_{X,V} \cong E_{X,V^2} \cong p_1^\dagger E_{X,V}$$

(which exist by applying the condition that $E$ is a crystal to the maps $p_i$) defines a stratification on $E_{X,V}$ and thus a functor

$$\operatorname{Cris}^\dagger X_V/O \to \operatorname{Strat}^\dagger(X, V/O), \tag{2.3.6.1}$$

given by $E \mapsto (E_{X,V}, \epsilon)$. On the other hand, a stratification on $E \in \operatorname{Mod} i_X^{-1}\mathcal{O}_V$ defines descent data on the crystal $\varphi_{X,V}^* E$ with respect to the map $p\colon (X, V) \to X_V/O$; by definition the map $p$ is a surjection of presheaves and thus a covering (in the canonical topology). By descent theory, the map 2.3.6.1 is an equivalence of categories (see [lS09, 2.5.3]).

*Remark* 2.3.7. Here we relate the notion of a stratification and a module with connection. There is a map

$$\operatorname{Strat}^\dagger(X, V/O) \to \operatorname{MIC}(X, V/O)$$

defined via the usual yoga of 'infinitesimal calculus', which we now recall. Let $V \hookrightarrow V^2 := V \times_O V$ be the diagonal morphism and denote by $V^{(n)}$ the $n^{\text{th}}$ infinitesimal neighborhood of the diagonal (when $V^{(0)} = V \hookrightarrow V^2$ is defined by an ideal $I$, and $V^{(n)} \hookrightarrow V^2$ is defined by the ideal $I^{n+1}$; in general one defines $V^{(n)}$ locally and glues). By definition the sequence

$$0 \to \Omega^1_{V/O} \to \mathcal{O}_V^{(1)} \to \mathcal{O}_V \to 0$$

is exact. We denote by $p_1^{(n)}$ and $p_2^{(n)}$ the two compositions

$$V^{(n)} \longrightarrow V^2 \underset{p_2}{\overset{p_1}{\rightrightarrows}} V .$$

Let $(M, \epsilon) \in \mathrm{Strat}^\dagger(X, V/O)$ be a module with an overconvergent stratification. Then $\epsilon$ restricts to give a compatible system $\{\epsilon^{(n)} \colon p_2^{(n)\dagger} M \cong p_1^{(n)\dagger} M\}$ of isomorphisms on $]X[_{V^{(n)}}$.

Denote by $\theta_i$ the natural map

$$\theta_i \colon M \to p_i^{(1)\dagger} M = M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(1)}}$$

given by tensoring (noting that the underlying topological spaces of $V^{(i)}$ are the same). We define a connection $\nabla$ on $M$ by the formula

$$\nabla = (\epsilon \circ \theta_2) - (\theta_1) \colon M \to p_1^{(1)\dagger} M = M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(1)}}$$

The map $\nabla$ lands in $M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_{V/O}$ by the description above of $\Omega^1_{V/O}$ together with the observation that since the two compositions

$$V \longrightarrow V^{(n)} \underset{p_2}{\overset{p_1}{\rightrightarrows}} V$$

are equal, the composition

$$M \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(1)}} \to M \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_V$$

is zero. Integrability of $\nabla$ follows from the cocycle condition.

Next we mildly refine the notion of a connection.

**Definition 2.3.8.** Let $(M, \nabla) \in \mathrm{MIC}(X, V/O)$. As in [lS09, Definition 2.4.6], we say that $\nabla$ is **overconvergent** if $M$ is coherent and $(M, \nabla)$ is in the image of the map

$$\mathrm{Strat}^\dagger(X, V/O) \to \mathrm{MIC}(X, V/O)$$

and we denote the category of overconvergent modules with integrable connection by $\mathrm{MIC}^\dagger(X, V/O)$. We define $\mathrm{MIC}^\dagger(X_0, V_0/O_0)$ similarly, where $E_0$ is a $j^\dagger_{X_0}\mathcal{O}_{V_0}$-module and the connection is a map $\nabla \colon E_0 \to E_0 \otimes_{\mathcal{O}_{V_0}} \Omega^1_{V_0/O_0}$.

When $(C, O) = (S_k, S_K)$ for a formal scheme $S$ and $V = P_K$ for a formal embedding $X \hookrightarrow P$ of $X$ into some formal scheme over $S$, we denote $\mathrm{MIC}^\dagger(X_0, V_0/O_0)$ by $\mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$; by [lS07, Corollary 8.1.9] this category only depends on the closure $\overline{X}$ of $X$ in $P$ and is independent of the choice of $P$, which we therefore omitted from the notation.

**Corollary 2.3.9.** *The natural map* $\mathrm{MIC}^\dagger(X_0, V_0/O_0) \to \mathrm{MIC}^\dagger(X, V/O)$ *is an equivalence of categories.*

*Proof.* It suffices to prove this for coherent modules with overconvergent stratification, which follows from Proposition 2.3.2 (iii) and (iv).

□

*Remark* 2.3.10. The composition

$$\mathrm{Cris}^\dagger X_V/O \cong \mathrm{Strat}^\dagger(X, V/O) \to \mathrm{MIC}(X, V/O)$$

induces an equivalence of categories

$$\mathrm{Mod}^\dagger_{\mathrm{fp}}(X_V/O) \cong \mathrm{MIC}^\dagger(X, V/O);$$

see [lS09, remark after 2.4.5].

The following theorem of le Stum ties this discussion together with Proposition 2.2.23 to give an intrinsic characterization of isocrystals via the good overconvergent site and in particular gives a new proof of the independence of $\mathrm{Isoc}^\dagger(X \subset \overline{X})$ from the choice of compactification $\overline{X}$.

**Theorem 2.3.11** ([lS09], Corollary 2.5.11)**.** *Let $S$ be a formal $\mathcal{V}$-scheme and let $X/S_k$ be a realizable algebraic variety. Then there is an equivalence of categories* $\mathrm{Mod}_{\mathrm{fp}}(\mathcal{O}^\dagger_{X_{\mathrm{g}}/S}) \cong \mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$*.*

Le Stum proves a similar result for cohomology, which we recall below.

**Definition 2.3.12** ([lS09, Definition 3.5.1])**.** Let $(C, O)$ be an overconvergent variety, let $f\colon X' \to X$ be a morphism of schemes over $C$. Then $f$ induces a morphism of topoi $f_{\mathrm{AN}^\dagger_{\mathrm{g}}}\colon X'/O_{\mathrm{AN}^\dagger_{\mathrm{g}}} \to X/O_{\mathrm{AN}^\dagger_{\mathrm{g}}}$. For $F \in (X'/O)_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ be a sheaf of abelian groups (or more generally any bounded below complex of abelian sheaves) we define the **relative rigid cohomology** of $F$ to be $\mathbb{R}f_{\mathrm{AN}^\dagger_{\mathrm{g}}*}F$.

When $(C, O) = (\operatorname{Spec} k, \mathcal{M}(K))$ and $X = \operatorname{Spec} k$, for an integer $i \geq 0$ we define the **absolute rigid cohomology** of $F$ to be the $K$-vector space $H^i(\mathrm{AN}^\dagger_{\mathrm{g}} X', F) := (\mathbb{R}^i f_{\mathrm{AN}^\dagger_{\mathrm{g}}*}F)_{(\operatorname{Spec} k, \mathcal{M}(K))}$; since the realization functor is exact this is isomorphic to the $i^{\mathrm{th}}$ derived functor of the global sections functor. When $F = \mathcal{O}^\dagger_{X'_{\mathrm{g}}}$, we write $H^i(\mathrm{AN}^\dagger_{\mathrm{g}} X') := H^i(\mathrm{AN}^\dagger_{\mathrm{g}} X', \mathcal{O}^\dagger_{X'_{\mathrm{g}}})$.

*Remark* 2.3.13. The functor $F \mapsto F_{\mathrm{g}}$ is exact on abelian sheaves (since goodness of $(X, V)$ is local on $V$), so when computing cohomology we can derive either of the functors $f_{\mathrm{AN}^\dagger *}$ or $f_{\mathrm{AN}^\dagger_{\mathrm{g}}*}$.

Now we explain how to compare the cohomology on the overconvergent site to classical rigid cohomology. Let $j_{X,V}\colon (X, V) \to X/O$ be an overconvergent variety over $X/O$ and let $E \in \mathrm{Cris}^\dagger_{\mathrm{g}} X/O$ be a crystal. Then the adjunction

$$E \to j_{X,V*}j^*_{X,V}E \cong j_{X,V*}\varphi^*_{X,V}E_{X,V}$$

(where the second map is an isomorphism by Remark 2.2.16) induces by [lS09, Proposition 3.3.10 (ii)] a map

$$E \to \mathbb{R}j_{X,V*}\left(\varphi^*_{X,V}\left(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}\right)\right)$$

of complexes of $\mathcal{O}^\dagger_{(X/O)_{\mathrm{g}}}$-modules. In the following nice situation this map is a quasi-isomorphism and we can thus compute the cohomology of $E$ via the cohomology of the de Rham complex $E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}$.

**Theorem 2.3.14.** *Let $(C, O)$ be an overconvergent variety and suppose that $(X, V)$ is a geometric realization of the morphism $X \to C$, and denote by $p_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ the morphism of topoi $p_{\mathrm{AN}^\dagger_{\mathrm{g}}} \colon (X/O)_{\mathrm{AN}^\dagger_{\mathrm{g}}} \to (C, O)_{\mathrm{AN}^\dagger_{\mathrm{g}}}$. Then the following are true.*

*(i) The augmentation*

$$E \to \mathbb{R}j_{X,V*}\left(\varphi^*_{X,V}\left(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}\right)\right)$$

*is an isomorphism.*

*(ii) The natural map*

$$\left(\mathbb{R}p_{\mathrm{AN}^\dagger_{\mathrm{g}}*}E\right)_{C,O} \to \mathbb{R}p_{]X[_V*}\left(E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^\bullet_{V/O}\right)$$

*(induced by part (i)) is a quasi-isomorphism.*

Of course, one can compute any other realization $\left(\mathbb{R}p_{\mathrm{AN}^\dagger_{\mathrm{g}}*}E\right)_{C',O'}$ of relative rigid cohomology from (ii) by base change [lS09, Corollary 3.5.7].

*Proof.* Claim (i) is [lS09, Proposition 3.5.4] and claim (ii) is [lS09, Theorem 3.5.3] (which follows from (i) by [lS09, Proposition 3.3.9]).

□

One can compare this with the classical notions of rigid cohomology, which we now recall.

**Definition 2.3.15** ([lS07, Definition 8.2.5])**.** Let $S$ be a formal $\mathcal{V}$-scheme, let $f \colon X \to S_k$ be a morphism of algebraic varieties, let $X \hookrightarrow P$ be a formal embedding over $S$ and denote by $g$ the map $]\overline{X}[_{(P_K)_0} \to (S_K)_0$. Let $E_0 \in \mathrm{Isoc}^\dagger(X \subset \overline{X}/S) := \mathrm{MIC}^\dagger(X_0, (P_K)_0/(S_K)_0)$. We define the classical rigid cohomology $\mathbb{R}f_{\mathrm{rig}}E_0$ of $E_0$ to be the higher direct image $\mathbb{R}g_*(E_0 \otimes \Omega^\bullet_{]\overline{X}[_{(P_K)_0}/(S_K)_0})$ of the de Rham complex associated to $(E_0, \nabla)$ (considered as a complex of abelian sheaves). When $S = \mathrm{Spf}\,\mathcal{V}$, we call this the absolute rigid cohomology and denote its $i^{\mathrm{th}}$ homology by $H^i_{\mathrm{rig}}(X, E_0)$.

Actually, rigid cohomology is independent of the choice of $P$ and $\overline{X}$ [lS07, Proposition 8.2.1], which we thus do not mention in the following theorem. When no choice of $P$ exists one can define $\mathrm{Isoc}^\dagger(X)$ and rigid cohomology by cohomological descent [CT03].

**Theorem 2.3.16** ([lS09], Proposition 3.5.8)**.** *Let $S$ be a formal $\mathcal{V}$-scheme such that $(S_k, S_K)$ is a good overconvergent variety and let $f\colon X \to S_k$ be a morphism of algebraic varieties. Let $(X, P)$ be a geometric realization of $X \to S_k$ and denote by $\overline{X}$ the closure of $X$ in $P$. Then for any $E \in \mathrm{Mod}^\dagger_{\mathrm{fp,g}}(X/S)$ and $E_0 \in \mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$ such that $E_{X,P} \cong i_X^{-1} E_0^{\mathrm{an}}$, there is a natural map (in the derived category)*

$$i_{S_k}^{-1}(\mathbb{R}f_{\mathrm{rig}}E_0)^{\mathrm{an}} \to (\mathbb{R}f_{\mathrm{AN}^\dagger_{\mathrm{g}}\,*}E)_{(S_k,S_K)}$$

*which is a quasi-isomorphism.*

The natural map is constructed as follows. Denote by $V$ the tube $]\overline{X}[_{P_K}$, by $O$ the analytic space $S_K$, and by $u$ the map $V \to O$. There is a natural map

$$(\mathbb{R}f_{\mathrm{rig}}E_0)^{\mathrm{an}} = \left(\mathbb{R}u_{0*}\left(E_0 \otimes_{\mathcal{O}_{O_0}} \Omega^\bullet_{V_0/O_0}\right)\right)^{\mathrm{an}} \to \mathbb{R}u_*\left(E_0 \otimes_{\mathcal{O}_{O_0}} \Omega^\bullet_{V_0/O_0}\right)^{\mathrm{an}}.$$

Since $V$ is smooth in a neighborhood of the tube $]X[_{P_K}$, $\Omega^\bullet_{V_0/O_0}$ is locally free in such a neighborhood. Thus the tensor product $E_0 \otimes_{\mathcal{O}_{O_0}} \Omega^\bullet_{V_0/O_0}$ has coherent terms and analytifies to $E' \otimes_{\mathcal{O}_O} \Omega^\bullet_{V/O}$, where $i_X^{-1}E' \cong E_{(S_k,S_K)}$. Furthermore, since $i_X^{-1}$ and $i_{S_k}^{-1}$ are exact, there are isomorphisms

$$i_{S_k}^{-1}\mathbb{R}u_*\left(E' \otimes_{\mathcal{O}_O} \Omega^\bullet_{V/O}\right) \cong \mathbb{R}_{]f[_*}i_X^{-1}\left(E' \otimes_{\mathcal{O}_O} \Omega^\bullet_{V/O}\right) \cong \mathbb{R}_{]f[_*}\left(E_{(S_k,S_K)} \otimes_{i_X^{-1}\mathcal{O}_O} i_X^{-1}\Omega^\bullet_{V/O}\right)$$

By Theorem 2.3.14 (ii), the last term is isomorphic to $\left(\mathbb{R}f_{\mathrm{AN}^\dagger_{\mathrm{g}}\,*}E\right)_{(S_k,S_K)}$. Applying $i_{S_k}^{-1}$ and composing these isomorphisms gives the natural map.

We end by stating a corollary of the comparison theorem.

**Theorem 2.3.17.** *Let $X$ be an algebraic variety over $k$. Let $(X, P)$ be a geometric realization of $X \to S_k$ and denote by $\overline{X}$ the closure of $X$ in $P$. Then for any $E \in \mathrm{Mod}^\dagger_{\mathrm{fp}}(X_{\mathrm{g}})$ and $E_0 \in \mathrm{Isoc}^\dagger(X \subset \overline{X}/k)$ such that $E_{X,P} \cong i_X^{-1}E_0^{\mathrm{an}}$, there is a natural map*

$$H^i_{\mathrm{rig}}(X, E_0) \to H^i(\mathrm{AN}^\dagger_{\mathrm{g}}\,X, E)$$

*which is an isomorphism.*

## 2.4 The overconvergent site for stacks

Let $C$ be a site and let $u\colon D \to C$ be a fibered category. Suppose moreover that every arrow of $C$ is cartesian; it then follows that that $u$ commutes with fiber products. Recall from 5.1.20 of the appendix that if we endow $D$ with the induced topology, then the functor

$u$ is then both continuous and cocontinuous. In particular, by Appendix 5.1.20 we get a triple of morphisms

$$\widetilde{D} \underset{u_*}{\overset{u_!}{\rightrightarrows}} \widetilde{C} \quad (u^*: \widetilde{C} \to \widetilde{D}) .$$

such that each arrow is left adjoint to the arrow below. Since $u^*$ has both a left and right adjoint it is exact and thus the pair $(u^*, u_*)\colon \widetilde{D} \to \widetilde{C}$ defines a morphism of topoi. On the other hand, $u_!$ is generally not exact (almost any non-trivial example will exhibit this). Finally, we remark that $u_!$ does not take abelian sheaves to a abelian sheaves; nonetheless $u^*\colon \operatorname{Ab}\widetilde{C} \to \operatorname{Ab}\widetilde{D}$ has a *different* left adjoint, $u_!^{\mathrm{ab}}$, which we also denote by $u_!$ when there is no confusion (see appendix 5.1.24). In particular, $u^*$ takes injective abelian sheaves to injective abelian sheaves.

**Definition 2.4.1.** Let $\mathcal{X} \to \mathbf{Sch}_k$ be a fibered cateory. We define the **overconvergent site** $\mathrm{AN}^\dagger(\mathcal{X})$ of $\mathcal{X}$ to be the category $\mathrm{AN}^\dagger(\mathcal{V}) \times_{\mathbf{Sch}_k} \mathcal{X}$ with the topology induced by the projection $\mathrm{AN}^\dagger(\mathcal{X}) \to \mathrm{AN}^\dagger(\mathcal{V})$. We define the **good overconvergent site** $\mathrm{AN}^\dagger_g(\mathcal{X})$ similarly.

*Remark* 2.4.2. Concretely, an object of $\mathrm{AN}^\dagger(\mathcal{X})$ is an overconvergent variety $(X, V)$ together with an object of the fiber category $\mathcal{X}(X)$; by the 2-Yoneda lemma this data is equivalent to an overconvergent variety $(X, V)$ and a map of categories $\mathbf{Sch}_X \to \mathcal{X}$ fibered over $\mathbf{Sch}_k$. In particular, for a presheaf $T$ on $\mathbf{Sch}_k$ with associated fibered category $\mathbf{Sch}_T \to \mathbf{Sch}_k$, $\mathrm{AN}^\dagger(T)$ defined as before is equivalent to $\mathrm{AN}^\dagger(\mathbf{Sch}_T)$.

**Definition 2.4.3.** Let $\mathcal{X}$ be a fibered category over $k$. We define the **sheaf of overconvergent functions** $\mathcal{O}^\dagger_{\mathcal{X}_{\mathrm{g}}}$ to be the pullback $u^{-1}\mathcal{O}^\dagger_{\mathcal{V}_{\mathrm{g}}}$ with respect to the projection $u\colon \mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{X}) \to \mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$.

Of course, the main case we consider will be when $\mathcal{X}$ is an algebraic stack over $k$. As before we will consider the categories $\operatorname{Mod}\mathcal{O}^\dagger_{\mathcal{X}_{\mathrm{g}}}$, $\operatorname{Mod}_{\mathrm{fp}}\mathcal{O}^\dagger_{\mathcal{X}_{\mathrm{g}}}$ and $\mathrm{Cris}^\dagger\,\mathcal{X}_{\mathrm{g}}$. A map $f\colon \mathcal{X} \to \mathcal{Y}$ of fibered categories induces a morphism $f_{\mathrm{AN}^\dagger_{\mathrm{g}}}\colon \mathcal{X}_{\mathrm{AN}^\dagger_{\mathrm{g}}} \to \mathcal{Y}_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ of topoi, and to any abelian sheaf $F \in \mathcal{X}_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ and for any good overconvergent variety $(X, V) \in \mathrm{AN}^\dagger_g\,\mathcal{X}$ one can study the cohomology $\mathbb{R}f_{\mathrm{AN}^\dagger_g\,*}F$ and the realization $F_{X,V}$.

*Remark* 2.4.4. It is worth noting here that the usual issues of functorality surrounding the lisse-étale site (e.g. functorality of crystalline cohomology for stacks [Ols07]) are not issues here.

# Chapter 3

# Cohomological descent

Cohomological descent is a robust computational and theoretical tool, central to rigid cohomology. On one hand, it facilitates explicit calculations (analogous to the computation of coherent cohomology in scheme theory via Čech cohomology); on another, it allows one to deduce results about singular schemes (e.g., finiteness of rigid cohomology) from results about smooth schemes. Moreover, for a scheme $X$ which fails to embed into a formal scheme smooth near $X$, one actually *defines* rigid cohomology via cohomological descent. For algebraic stacks it is doubly important, allowing one to reduce results and constructions to the case of schemes; in fact, cohomological descent is needed even for basic calculations (e.g., rigid cohomology of $BG$ for a finite group $G$).

The main result of the series of papers [CT03],[Tsu03a], and [Tsu04] is that cohomological descent for overconvergent isocrystals holds with respect to both étale and proper hypercovers. The burden of choices in the definition of rigid cohomology makes their proofs of cohomological descent very difficult, totaling to over 200 pages. Even after the main cohomological descent theorems [CT03, Theorems 7.3.1 and 7.4.1] are proved one still has to work a bit to get a spectral sequence [CT03, Theorem 11.7.1]. Actually, even to state what one means by cohomological descent (without a site) is subtle.

The situation is more favorable for the overconvergent site, since one may apply the abstract machinery of [72, Exposé Vbis and VI]. Indeed, the main result of this chapter (see Theorem 3.2.9) is a short proof of the following.

**Theorem 3.0.5.** *Cohomological descent for locally finitely presented modules on the overconvergent site holds with respect to smooth hypercovers.*

By le Stum's comparison theorems between rigid and overconvergent cohomology (see Theorems 2.3.11) and 2.3.16), we obtain a spectral sequence (see Remark 3.1.9) computing rigid cohomology, which gives a shorter proof of Theorem 11.7.1 of [CT03].

The following variant of Theorem 3.0.5 is also expected.

**Conjecture 3.0.6.** Cohomological descent for locally finitely presented modules on the overconvergent site holds with respect to proper hypercovers.

Finally, we note that the proof of Theorem 3.0.5 is not merely a formal consequence of the techniques of [72, Exposé Vbis and VI], in contrast to, for example, cohomological descent for abelian sheaves on the étale site with respect to smooth hypercovers (which is simply Čech theory; see Theorem 3.1.16 (i)).

## 3.1 Background on cohomological descent

Here we recall the definitions and facts about cohomological descent that we will need. The standard reference is [72, Exposé Vbis and VI]; some alternatives are Deligne's paper [Del74] and Brian Conrad's notes [Con]; the latter has a lengthy introduction with a lot of motivation and gives more detailed proofs of some theorems of [72] and [Del74].

We refer to Appendix 5.1 for a review of categorical constructions (e.g., comma categories, fibered categories, the canonical topology, etc.).

3.1.1. We denote by $\Delta$ the simplicial category whose objects the are the sets $[n] := \{0, 1, \ldots, n\}$, $n \geq 0$, and whose morphisms are monotonic maps of sets $\phi\colon [n] \to [m]$ (i.e., for $i \leq j$, $\phi(i) \leq \phi(j)$). We define the augmented simplicial category to be $\Delta^+ := \Delta \cup \{\varnothing\}$. A **simplicial** (resp. **augmented simplicial**) object $X_\bullet$ of a category $C$ is a functor $X_\bullet\colon \Delta^{\mathrm{op}} \to C$ (resp. $X_\bullet\colon (\Delta^+)^{\mathrm{op}} \to C$); one denotes by $X_n$ the image of $n$ under $X_\bullet$. We will typically write an augmented simplicial object as $X_\bullet \to X_{-1}$, where $X_\bullet$ is the associated simplicial object. A morphism between two simplicial or augmented simplicial objects is simply a natural transformation of functors. We denote these two categories by $\operatorname{Simp} C$ and $\operatorname{Simp}^+ C$.

Similarly, we define the truncated simplicial categories $\Delta_{\leq n} \subset \Delta$ and $\Delta^+_{\leq n} \subset \Delta^+$ to be the full subcategories consisting of objects $[m]$ with $m \leq n$ (with the convention that $[-1] = \varnothing$). We define the category $\operatorname{Simp}_n C$ of $n$-**truncated simplicial objects** of $C$ to be the category of functors $X_\bullet\colon \Delta^{\mathrm{op}}_{\leq n} \to C$ (and define $\operatorname{Simp}^+_n C$ analogously).

3.1.2. Any morphism $p_0\colon X \to Y$ in a category $C$ gives rise to an augmented simplicial object $p\colon X_\bullet \to Y$ with $X_n$ the fiber product of $n+1$ many copies of the morphism $p_0$; in this case we denote by $p_n$ the morphism $X_n \to Y$ and by $p_i^j$ the $j^{\text{th}}$ projection map $X_i \to X_{i-1}$ which forgets the $j^{\text{th}}$ component.

3.1.3. This last construction is right adjoint to the forgetful functor $X_\bullet \mapsto (X_0 \to X_{-1})$ from $\operatorname{Simp}^+ C \to \operatorname{Simp}^+_{\leq 0} C$. We can generalize this point of view to construct an augmented simplicial object out of an $n$-truncated simplicial object as follows. We first define the $n$-skeleton functor

$$\operatorname{sk}_n\colon \operatorname{Simp} C \to \operatorname{Simp}_{\leq n} C$$

by sending $X_\bullet\colon \Delta^{\mathrm{op}} \to C$ to the composition $\operatorname{sk}_n(X_\bullet)\colon \Delta^{\mathrm{op}}_{\leq n} \subset \Delta^{\mathrm{op}} \to C$. We define an augmented variant

$$\mathrm{sk}_n\colon \operatorname{Simp}^+ C \to \operatorname{Simp}^+_{\leq n} C$$

similarly, which we also also denote by $\mathrm{sk}_n$. When $C$ admits finite limits the functor $\mathrm{sk}_n$ has a right adjoint $\mathrm{cosk}_n$ [Con, Theorem 3.9], which we call the $n$-**coskeleton**. When we denote a truncated augmented simplicial object as $X_\bullet \to Y$, we may also write $\mathrm{cosk}_n(X_\bullet/Y) \to Y$ to denote $\mathrm{cosk}_n(X_\bullet \to Y)$ (so that $\mathrm{cosk}_n(X_\bullet/Y)$ is a simplicial object).

3.1.4. When $C$ is a site we promote these notions a bit. The codomain fibration, i.e., the fibered category $\pi\colon \operatorname{Mor} C \to C$ which sends a morphism $X \to Y \in \operatorname{Ob}(\operatorname{Mor} C)$ to its target $Y$ (see Example 5.1.11), is a prestack if and only if $C$ is subcanonical and a stack if every $F \in \widetilde{C}$ is representable (equivalently, if the Yoneda embedding $C \to \widehat{C}$ induces an isomorphism $C \to \widetilde{C}$). The fibers are the comma categories $C_{/X}$, and the site structure induced by the projection $C_{/X} \to C$ makes $\pi$ into a **fibered site** (i.e., a fibered category with sites as fibers such that for any arrow $f\colon X \to Y$ in the base, any cartesian arrow over $f$ induces a functor $C_{/X} \to C_{/Y}$ which is a continuous morhpism of sites; see [72, Exposé VI]). For a simplicial object $X_\bullet$ of $C$, the 2-categorical fiber product $\Delta^{\mathrm{op}} \times_C \operatorname{Mor} C \to \Delta^{\mathrm{op}}$ also is a fibered site; to abusively notate this fiber product as $X_\bullet$ will cause no confusion. We will call a site fibered over $\Delta^{\mathrm{op}}$ a **simplicial site**. We define a morphism of fibered sites below 3.1.5.

3.1.5. Associated to any fibered site is a **fibered topos**; we explicate this for the fibered site $X_\bullet \to \Delta^{\mathrm{op}}$ associated to a simplicial object $X_\bullet$ of a site $C$. We define first the **total site** $\operatorname{Tot} X_\bullet$ to be the category $X_\bullet$ together with the smallest topology such that for every $n$, the inclusion of the fiber $X_n$ into $X_\bullet$ is continuous. The **total topos** of $X_\bullet$ is then defined to be the category $\widetilde{X_\bullet}$ of sheaves on $\operatorname{Tot} X_\bullet$. We can define a morphism of fibered sites to be a morphism of fibered categories which induces a continuous morphism of total sites.

For $F_\bullet \in \widetilde{X_\bullet}$ denote by $F_n$ the restriction of $F_\bullet$ to $X_n$; as usual for any cartesian arrow $f$ over a map $d' \to d$ in $\Delta^{\mathrm{op}}$ one has an induced map $f^*F_d \to F_{d'}$ and as one varies $d' \to d$, these maps enjoy a cocycle compatibility. The total topos $\widetilde{X_\bullet}$ is equivalent to the category of such data. One can package this data as sections of a fibered topos $T_\bullet \to \Delta^{\mathrm{op}}$ (with fibers $T_n = \widetilde{C_n}$), i.e., a fibered category whose fibers are topoi such that cartesian arrows induce morphisms of topoi (or rather, the pullback functor of a morphism of topoi) on fibers. The total topos $\widetilde{X_\bullet}$ is then equivalent to the category of sections of $T_\bullet \to \Delta^{\mathrm{op}}$. When the topology on each fiber $X_n$ is subcanonical (i.e., representable objects are sheaves), the topology on $\operatorname{Tot} X_\bullet$ also is subcanonical and the inclusion $X_\bullet \subset T_\bullet$ of fibered sites (where one endows each fiber $T_n$ of the the fibered topos $T_\bullet$ with its canonical topology) induces an equivalence of categories of total topoi.

3.1.6. Let $p_0\colon X \to Y$ be a morphism of presheaves on a site $C$. As before, this gives rise to an augmented simplicial presheaf $p\colon X_\bullet \to Y$. Denoting by $\widehat{C}$ the category of presheaves on $C$, we may again promote $X_\bullet$ to a fibered site and study its fibered topos as in 3.1.4 above. Indeed, Yoneda's lemma permits one to consider the fibered site $\mathrm{Mor}'\,\widehat{C} \to \widehat{C}$ (where $\mathrm{Mor}'\,\widehat{C}$ is the subcategory of $\mathrm{Mor}\,\widehat{C}$ whose objects are arrows with source in $C$ and target in $\widehat{C}$), and again the 2-categorical fiber product $\Delta^{\mathrm{op}} \times_{\widehat{C}} \mathrm{Mor}'\,\widehat{C}$ is a fibered site. We also remark that passing to the presheaf category allows one to augment any simplicial object in $C$ by sending $\varnothing$ to the final object of $\widehat{C}$ (which is represented by the punctual sheaf).

3.1.7. A morphism $f\colon X_\bullet \to Y_\bullet$ of simplicial sites induces a morphism $(f^*, f_*)\colon \widetilde{X_\bullet} \to \widetilde{Y_\bullet}$ of their total topoi; concretely, the morphisms of topoi $(f_n^*, f_{n*})\colon \widetilde{X_n} \to \widetilde{Y_n}$ induce for instance a map $\{F_n\} \mapsto \{f_{n*}F_n\}$ which respects the cocycle compatibilities.

To an augmented simplicial site $p\colon X_\bullet \to S$ one associates a morphism $(p^*, p_*)\colon \widetilde{X_\bullet} \to \widetilde{S}$ of topoi as follows. The pullback functor $p^*$ sends a sheaf of sets $\mathcal{F}$ on $S$ to the collection $\{p_n^*\mathcal{F}\}$ together with the canonical isomorphisms $p_{n+1}^{j*}p_n^*\mathcal{F}_n \cong p_{n+1}^*\mathcal{F}$ induced by the canonical isomorphism of functors $p_{n+1}^{j*} \circ p_n^* \cong p_{n+1}^*$ associated to the equality $p_{n+1} = p_n \circ p_{n+1}^j$. Its right adjoint $p_*$ sends the collection $\{\mathcal{F}_n\}$ to the equalizer of the cosimplicial sheaf

$$\cdots p_{(n-1)*}\mathcal{F}_{n-1} \Longrightarrow p_{n*}\mathcal{F}_n \Rrightarrow p_{(n+1)*}\mathcal{F}_{n+1} \cdots \tag{3.1.7.1}$$

where the $n+2$ maps between $p_{n*}\mathcal{F}_n$ and $p_{n+1*}\mathcal{F}_{n+1}$ are the pushforwards $p_{n*}$ of the adjoints $\mathcal{F}_n \to p_{n+1*}^j\mathcal{F}_{n+1}$ to $p_{n+1}^{j*}\mathcal{F}_n \to \mathcal{F}_{n+1}$ (using the equality $p_{(n+1)*} = p_{n*} \circ p_{n+1*}^j$). It follows from an elementary manipulation of the simplicial relations that the equalizer of 3.1.7.1 only depends on the first two terms; i.e., it is equal to the equalizer of

$$p_{0*}\mathcal{F}_0 \rightrightarrows p_{1*}\mathcal{F}_1 .$$

One can of course derive these functors, and we remark that while, for an augmented simplicial site $p\colon X_\bullet \to S$ and an abelian sheaf $\mathcal{F} \in \mathrm{Ab}\,X_\bullet$, the sheaf $p_*\mathcal{F}$ only depends on the first two terms of the cosimplicial sheaf of 3.1.7.1, the cohomology $\mathbb{R}p_*\mathcal{F}$ depends on the entire cosimplicial sheaf. Finally, we note the standard indexing convention that for a complex $\mathcal{F}_{\bullet,\bullet}$ of sheaves on $X_\bullet$, for any $i$ we have that $\mathcal{F}_{\bullet,i} \in \mathrm{Ab}\,X_\bullet$.

**Example 3.1.8** ([Con, Examples 2.9 and 6.7])**.** Let $S \in C$ be an object of a site and let $q\colon S_\bullet \to S$ be the constant augmented simplicial site associated to the identity morphism $\mathrm{id}\colon S \to S$. The total topos $\widetilde{S_\bullet}$ is then equivalent to the category $\mathrm{Cosimp}\,\widetilde{S} = \mathrm{Hom}(\Delta, \widetilde{S})$ of co-simplicial sheaves on $S$ and $\mathrm{Ab}(S_\bullet)$ is equivalent to $\mathrm{Cosimp}\,\mathrm{Ab}(S)$.

(i) It is useful to consider the functor

$$\mathrm{ch}\colon\ \mathrm{Cosimp}\,\mathrm{Ab}(S) \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab}(S))$$

to the category of chain complexes concentrated in non-negative degree which sends a cosimplicial sheaf to the chain complex whose morphisms are given by alternating sums of the simplicial maps. The direct image functor $q_*$ is then given by

$$\mathcal{F}_\bullet \mapsto H^0(\mathrm{ch}\,\mathcal{F}_\bullet) = \ker(\mathcal{F}_0 \to \mathcal{F}_1).$$

Let $I_\bullet \in \mathrm{Ab}\, S_\bullet$. Then $I_\bullet$ is injective if and only if $\mathrm{ch}\, I_\bullet$ is a split exact complex of injectives (this is a mild correction of [Con, Corollary 2.13]). Furthermore, for $I_\bullet \in \mathrm{Ab}\, S_\bullet$ injective, the natural map

$$\mathbb{R}q_* I_\bullet := q_* I_\bullet \to \mathrm{ch}\, I_\bullet$$

is a quasi-isomorphism and thus $\mathbb{R}^i q_* I_\bullet = H^i(\mathrm{ch}\, I_\bullet)$. One concludes by [Har77, Theorem 1.3A] that the collection of functors $\mathcal{F}_\bullet \mapsto H^i(\mathrm{ch}\, \mathcal{F}_\bullet)$ (the $i^{\text{th}}$ homology of the complex $\mathrm{ch}\, \mathcal{F}_\bullet$) forms a universal $\delta$ functor and thus that $\mathbb{R}^i q_* \mathcal{F}_\bullet \cong H^i(\mathrm{ch}(\mathcal{F}_\bullet))$.

(ii) Actually, a mildly stronger statement is true: for an injective resolution $\mathcal{F}_\bullet \to I_{\bullet,\bullet}$ (where $I_{\bullet,i} \in \mathrm{Cosimp}\, \mathrm{Ab}(S)$), one can show that the map $\mathrm{ch}\, \mathcal{F}_\bullet \to \mathrm{ch}\, I_{\bullet,\bullet}$ induces a quasi-isomorphism $\mathrm{ch}\, \mathcal{F} \to \mathrm{Tot}\, \mathrm{ch}\, I_{\bullet,\bullet}$, where Tot is the total complex constructed by collapsing the double complex $\mathrm{ch}\, I_{\bullet,\bullet}$ along the diagonals. On the other hand the natural map $\mathbb{R}q_* \mathcal{F}_\bullet := q_* I_{\bullet,\bullet} \to \mathrm{Tot}\, \mathrm{ch}\, I_{\bullet,\bullet}$ is an isomorphism. Putting this together we see that the map $\mathrm{ch}\, \mathcal{F}_\bullet \to \mathrm{Tot}\, \mathrm{ch}\, I_{\bullet,\bullet}$ is a quasi-isomorphism.

(iii) We note a final useful computation. Let $I_{\bullet,\bullet} \in \mathbb{D}_+(S_\bullet)$ a complex of injective sheaves. Define $I_{-1,n} = \ker(\mathrm{ch}\, I_{\bullet,n})$; by [Stacks, 015Z] (noting that since $q$ is a morphism of topoi, $Q^*$ is an exact left adjoint to $q_*$) this is an injective sheaf. Then the hypercohomology of $I_{\bullet,\bullet}$ is simply (by definition) $\mathbb{R}q_*\,(I_{\bullet,\bullet}) := q_*\,(I_{\bullet,\bullet}) = I_{-1,\bullet}$.

*Remark* 3.1.9. Let $p\colon X_\bullet \to S$ be an augmented simplicial site, and let $\mathcal{F}_\bullet \in \widetilde{X_\bullet}$ be a sheaf of abelian groups. Using Example 3.1.8, we can clarify the computation of the cohomology $\mathbb{R}p_* \mathcal{F}_\bullet$ via the observation that the associated map of topoi factors as

$$\widetilde{X_\bullet} \xrightarrow{r} \widetilde{S}_\bullet \xrightarrow{q} \widetilde{S},$$

where $r_* \mathcal{F}_\bullet$ is the cosimplicial sheaf given by Equation 3.1.7.1. Therefore, to compute $\mathbb{R}p_* \mathcal{F}_\bullet$ we first study $\mathbb{R}r_* \mathcal{F}_\bullet$.

Set $\mathcal{F}_{-1} = p_* \mathcal{F}_\bullet = \ker \mathrm{ch}\, r_* \mathcal{F}_\bullet$. Viewing $\mathcal{F}_{-1}$ as a complex concentrated in degree 0, we can consider the morphism of complexes $\mathcal{F}_{-1} \to \mathrm{ch}\, r_* \mathcal{F}_\bullet$. When $\mathcal{F}_\bullet$ is injective, $r_* \mathcal{F}_\bullet$ also is injective by [Stacks, 015Z]; applying the description of injective objects of $\mathrm{Ab}(S_\bullet)$ of Example 3.1.8 (i) to $\mathrm{ch}\, r_* \mathcal{F}_\bullet$ we conclude that the map of complexes $\mathcal{F}_{-1} \to \mathrm{ch}\, r_* \mathcal{F}_\bullet$ is a quasi-isomorphism when $\mathcal{F}_\bullet$ is injective.

Let $\mathcal{F}_\bullet \to I_{\bullet,\bullet}$ be an injective resolution of $\mathcal{F}_\bullet$. Then one gets a commutative diagram of chain complexes

$$\begin{array}{ccc} I_{-1,\bullet} & \longrightarrow & \mathrm{ch}\, r_* I_{\bullet,\bullet}\ ; \\ \uparrow & & \uparrow \\ \mathcal{F}_{-1} & \longrightarrow & \mathrm{ch}\, r_* \mathcal{F}_\bullet \end{array} \qquad (3.1.9.1)$$

we can alternatively view Diagram 3.1.9.1 as a double complex, indexed so that the sheaf $\mathcal{F}_{-1}$ lives in bi-degree $(-1,-1)$. By the remark at the end of the preceding paragraph, all rows of Diagram 3.1.9.1 except the bottom are quasi-isomorphisms; the columns are generally not quasi-isomorphisms (since $\mathrm{ch}\, r_* \mathcal{F}_\bullet$ is not exact in positive degree). Now we compute that

$$\mathbb{R}p_* \mathcal{F}_\bullet := p_* I_{\bullet,\bullet} = I_{-1,\bullet}. \tag{3.1.9.2}$$

The output $I_{-1,\bullet}$ is quasi-isomorphic to the total complex $\mathrm{Tot}\, \mathrm{ch}\, r_* I_{\bullet,\bullet}$ (given by collapsing the diagonals); since the $i^{\text{th}}$ column of the double complex $\mathrm{ch}\, r_* I_{\bullet,\bullet}$ computes $\mathbb{R}p_{i*}\mathcal{F}_i$, there is an $E_1$-spectral sequence

$$\mathbb{R}^j p_{i*}\mathcal{F}_i = H^j(p_{i*} I_{\bullet,\bullet}) \Rightarrow H^{i+j}(\mathrm{Tot}\, \mathrm{ch}\, r_* I_{\bullet,\bullet}) \cong H^{i+j}(I_{-1,\bullet}) \cong \mathbb{R}^{i+j} p_* \mathcal{F}_\bullet, \tag{3.1.9.3}$$

where the last isomorphism is the $(i+j)^{\text{th}}$ homology of Equation 3.1.9.2.

Our later computations will rely on the following degenerate case of the preceding remark.

**Corollary 3.1.10.** *Let $p\colon X_\bullet \to S$ be an augmented simplicial site. Then the following are true.*

*(i) Let $\mathcal{F}_\bullet \in \mathrm{Ab}\, X_\bullet$ be a sheaf of abelian groups. Suppose that for $i \geq 0$ and $j > 0$, one has $\mathbb{R}^j p_{i*}\mathcal{F}_i = 0$. There is a quasi-isomorphism $\mathbb{R}p_*\mathcal{F}_\bullet \cong \mathrm{ch}\, r_*\mathcal{F}_\bullet$.*

*(ii) Let $\mathcal{F} \in \widetilde{S}$ be an abelian sheaf such that for $i \geq 0$ and $j > 0$, $\mathbb{R}^j p_{i*} p_i^* \mathcal{F} = 0$, such that $\mathrm{ch}\, r_* p^* \mathcal{F}$ is exact in positive degrees, and such that the adjunction $\mathcal{F} \to \ker(\mathcal{F}_0 \to \mathcal{F}_1)$ is an isomorphism, then $\mathcal{F} \to \mathbb{R}p_* p^* \mathcal{F}$ is a quasi-isomorphism.*

*Proof.* The second claim is a special case of the first claim. By [Con, Lemma 6.4], for any $i, j$, the sheaf $I_{i,j}$ is injective, and thus the $i^{\text{th}}$ column of $I_{\bullet,\bullet}$ is an injective resolution of $\mathcal{F}_i$. For $i \geq 0$, the $i^{\text{th}}$ column of $r_* I_{\bullet,\bullet}$ is the complex $\mathbb{R}p_{i*}\mathcal{F}_i$. Thus by hypothesis the complex $r_{i*}\mathcal{F}_i \to r_{i*} I_{i,\bullet}$ is exact and it follows that the map

$$\mathrm{ch}\, r_* \mathcal{F}_\bullet \to \mathrm{Tot}\, \mathrm{ch}\, r_* I_{\bullet,\bullet} =: \mathbb{R}p_* \mathcal{F}_\bullet$$

of Diagram 3.1.9.1 is a quasi-isomorphism.

□

*Remark* 3.1.11. Let $f\colon X_\bullet \to Y_\bullet$ be a map of simplicial sites, $\mathcal{F}_\bullet \in \widetilde{X_\bullet}$ be a sheaf of abelian groups, and suppose that for every $i \geq 0$, the natural map $f_{i*}\mathcal{F}_i \to \mathbb{R}f_{i*}\mathcal{F}_i$ is a quasi-isomorphism. Then the strategy used in the proof of Corollary 3.1.10 generalizes to prove that the natural map $f_*\mathcal{F}_\bullet \to \mathbb{R}f_*\mathcal{F}_\bullet$ is a quasi-isomorphism.

Finally, we arrive at the main definition.

**Definition 3.1.12.** Let $C$ be a site. We say that an augmented simplicial object $p\colon X_\bullet \to S$ of $C$ is **of cohomological descent** if the adjunction $\mathrm{id} \to \mathbb{R}p_* p^*$ on $\mathbb{D}_+(S)$ is an isomorphism; equivalently, $p$ is of cohomological descent if and only if the map $p^*\colon \mathbb{D}_+(S) \to \mathbb{D}_+(X_\bullet)$ is fully faithful [Con, Lemma 6.8] (this explains the analogue with classical descent theory).

A morphism $X \to S$ of $C$ is of cohomological descent if the associated augmented simplicial site $X_\bullet \to S$ is of cohomological descent (this makes sense even when $C$ does not have fiber products, since we can work in $\widetilde{C}$ instead). We say that an augmented simplicial object $X_\bullet \to S$ of $C$ is **universally** of cohomological descent if for every $S' \to S$, the base change $X_\bullet \times_S S' \to S'$ (viewed in the topos $\widetilde{C}$ in case $C$ fails to admit fiber products) is of cohomological descent.

Similarly, for a sheaf of abelian groups $\mathcal{F} \in \widetilde{S}$ we say that $p$ is of cohomological descent **with respect to** $\mathcal{F}$ if $\mathcal{F} \cong \mathbb{R}p_*p^*\mathcal{F}$, that a morphism $X \to S$ is of cohomological descent with respect to $\mathcal{F}$ if the same is true of the associated augmented simplicial space, and universally of cohomological descent with respect to $\mathcal{F}$ if for every $f \colon S' \to S$, the map $X \times_S S' \to S'$ is of cohomological descent with respect to $f^*\mathcal{F}$.

3.1.13. Once one knows cohomological descent for all $\mathcal{F} \in \mathrm{Ab}\,\widetilde{S}$, one can deduce it for all $\mathcal{F}^\bullet \in \mathbb{D}_+(S)$ via application of the hypercohomology spectral sequence.

3.1.14. The charm of cohomological descent is that there are interesting and useful augmented simplicial sites *other than* 0-coskeletons which are of cohomological descent. Let $C$ be a category with finite limits and let $\mathbf{P}$ be a class of morphisms in $C$ which is stable under base change and composition and contains all isomorphisms. We say that a simplicial object $X_\bullet$ of $C$ is a **P-hypercovering** if for all $n \geq 0$, the natural map

$$X_{n+1} \to (\mathrm{cosk}_n(\mathrm{sk}_n\, X_\bullet))_{n+1}$$

is in $\mathbf{P}$. For an augmented simplicial object $X_\bullet \to Y$ we say that $X_\bullet$ is a $\mathbf{P}$-hypercover of $Y$ if the same condition holds for $n \geq -1$.

**Example 3.1.15.** The 0-coskeleton $\mathrm{cosk}_0(X/Y) \to Y$ of a cover $X \to Y$ is a $\mathbf{P}$-hypercover of $Y$, where $\mathbf{P}$ is the class of covering morphisms.

We record here many examples of morphisms of cohomological descent.

**Theorem 3.1.16.** *Let $C$ be a site. Then the following are true.*

*(i) A covering $p \colon X \to Y$ in $\widetilde{C}$ is universally of cohomological descent.*

*(ii) Any morphism in $C$ which has a section locally (in $C$) is universally of cohomological descent.*

*(iii) The class of morphisms in $C$ universally of cohomological descent form a topology (in the strong sense of* [73, Exposé II]*). In particular, the following are true.*

*(a) For a cartesian diagram of objects*

$$\begin{array}{ccc} X' & \xrightarrow{\pi_0'} & X \\ {\scriptstyle f_0'}\downarrow & & \downarrow{\scriptstyle f_0} \\ S' & \xrightarrow[\pi_0]{} & S \end{array}$$

*in $C$ with $\pi_0$ universally of cohomological descent, $f_0$ is universally of cohomological descent if and only if $f_0'$ is universally of cohomological descent.*

(b) *If $X \to Y$ and $Y \to Z$ are maps in $C$ such that the composition $X \to Z$ is universally of cohomological descent, then so is $Y \to Z$.*

(c) *If $X \to Y$ and $Y \to Z$ are maps in $C$ and are universally of cohomological descent, then so is the composition $X \to Z$.*

(iv) *More generally, let $\boldsymbol{P}$ be the class of morphisms in $C$ which are universally of cohomological descent. Then a $\boldsymbol{P}$-hypercover is universally of cohomological descent.*

*Proof.* Statement (i) is [Ols07, Lemma 1.4.24], (ii) follows from (i) since any morphism with a section is a covering in the canonical topology, (iii) is [Con, Theorem 7.5], and (iv) is [Con, Theorem 7.10].

□

Useful later will be a mild variant applicable to a particular sheaf (as opposed to the entire category of abelian sheaves).

**Theorem 3.1.17.** *Let $C$ be a site. Then the following are true.*

(a) *Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\pi_0'} & X \\ {\scriptstyle f_0'}\downarrow & & \downarrow{\scriptstyle f_0} \\ S' & \xrightarrow[\pi_0]{} & S \end{array}$$

*in $C$ and let $\mathcal{F} \in \widetilde{S}$ be a sheaf of abelian groups. Suppose $\pi_0$ is universally of cohomological descent with respect to $\mathcal{F}$. Then $f_0$ is universally of cohomological descent with respect to $\mathcal{F}$ if and only if $f_0'$ is universally of cohomological descent with respect to $\pi_0^*\mathcal{F}$.*

(b) *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps in $C$ and let $\mathcal{F} \in \widetilde{Z}$ be a sheaf of abelian groups. Suppose that the composition $X \to Z$ is universally of cohomological descent with respect to $\mathcal{F}$. Then is $Y \to Z$ as well.*

(c) *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps in $C$ and let $\mathcal{F} \in \widetilde{Z}$ be a sheaf of abelian groups. If $g$ is universally of cohomological descent with respect to $\mathcal{F}$ and $f$ is universally of cohomological descent with respect to $g^*\mathcal{F}$, then the composition $g \circ f$ is universally of cohomological descent with respect to $\mathcal{F}$.*

(d) *Let $f_i \colon X_i \to Y_i$ be maps in $C$ indexed by some arbitrary set $I$. For each $i \in I$ let $\mathcal{F}_i \in \widetilde{Y_i}$ be a sheaf of abelian groups. Suppose that for each $i$, $f_i$ is of cohomological descent relative to $\mathcal{F}_i$. Then $\coprod f_i \colon \coprod X_i \to \coprod Y_i$ is of cohomological descent relative to $\coprod \mathcal{F}_i$ (where disjoint unions are taken in $\widehat{C}$ as discussed in Appendix 5.1.7).*

*Proof.* The proofs of (a) - (c) are identical to the proof of Theorem 3.1.16 (iii) found in [Con, Theorem 7.5], and (d) follows from the fact that, setting $p_0 = \coprod f_i$, the induced morphism of simplicial topoi

$$p\colon \Big(\widetilde{\coprod X_i}\Big)_\bullet \to \widetilde{\coprod Y_i}$$

is also a morphism of topoi fibered over $I$, so that in particular the natural map

$$\coprod \mathcal{F}_i \to \mathbb{R}p_* p^* \coprod \mathcal{F}_i$$

is an isomorphism if and only if, for all $i \in I$, the map $\mathcal{F}_i \to \mathbb{R} f_{i\bullet *} f_{i\bullet}^* \mathcal{F}_i$ is an isomorphism.

□

## 3.2 Cohomological descent for overconvergent crystals

In this section we prove Theorem 3.0.5. We begin with the case of a Zariski hypercover. Unless otherwise noted, all proofs do not change if we replace objects by their good variants.

**Definition 3.2.1.** Let $T$ be an overconvergent presheaf. A sheaf $\mathcal{F} \in T_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ is said to be **of Zariski type** if, for every overconvergent variety $(X, V) \in \mathrm{AN}^\dagger_{\mathrm{g}} T$ and for every Zariski open immersion $\alpha\colon U \hookrightarrow X$, the induced map $]\alpha[_V^{-1} \mathcal{F}_{X,V} \to \mathcal{F}_{U,V}$ is an isomorphism. We make the analogous definition for $\mathcal{F} \in T_{\mathrm{AN}^\dagger}$.

*Remark* 3.2.2. In the context of Definition 3.2.1, when $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}_T^\dagger$, the natural map $]\alpha[_V^{-1} \mathcal{F}_{X,V} \to ]\alpha[_V^\dagger \mathcal{F}_{X,V}$ is, by [lS09, Corollary 2.3.2], an isomorphism. It follows that an overconvergent crystal is of Zariski type.

One can restate the main result of [lS09, Section 3.6] as the statement that a Zariski covering is universally of cohomological descent (see Definition 3.1.12) for overconvergent abelian sheaves of Zariski type. More precisely:

**Theorem 3.2.3** ([lS09, Section 3.6])**.** *Let $(C, O)$ be an overconvergent variety and let $X \to C$ be a morphism of algebraic varieties. Let $\{U_i\}_{i \in I}$ be a locally finite covering of $X$ by open subschemes and denote by $\alpha_0\colon U = \coprod_{i \in I} U_i \to X$ the induced morphism of schemes. Denote by $\alpha\colon U_\bullet \to X$ the 0-coskeleton of $\alpha_0$. Let $\mathcal{F} \in X/O_{\mathrm{AN}^\dagger}$ be a sheaf of abelian groups of Zariski type. Then the morphism of topoi $U_\bullet/O_{\mathrm{AN}^\dagger} \to X/O_{\mathrm{AN}^\dagger}$ is universally of cohomological descent with respect to $\mathcal{F}$. The same statement holds for $\mathcal{F} \in X/O_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ and $\alpha_{\mathrm{AN}^\dagger_{\mathrm{g}}}$.*

*Proof.* The proof for $\alpha_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ is identical to the proof for $\alpha_{\mathrm{AN}^\dagger}$. By [lS09, Corollary 2.1.11] the maps $(\alpha_i)_{\mathrm{AN}^\dagger *}$ (induced from the $i$-fold fiber products) are exact, and by [lS09, Proposition 3.6.3] the 'ordered Čech complex' is exact. It is a standard fact (whose truth is often noted without proof in the literature) that it follows that the unordered Čech complex is exact (i.e.,

$\mathcal{F} \cong \alpha_{\mathrm{AN}^\dagger *}\alpha^*_{\mathrm{AN}^\dagger}\mathcal{F}$), and so by Corollary 3.1.10 (ii), we conclude that $\mathcal{F} \cong \mathbb{R}\alpha_{\mathrm{AN}^\dagger *}\alpha^*_{\mathrm{AN}^\dagger}\mathcal{F}$. Universality follows since our hypotheses are preserved under base change.

□

*Remark* 3.2.4. Let $\{X_i\}$ be a collection of schemes. Then the presheaf on $\mathrm{AN}^\dagger \mathcal{V}$ represented by the disjoint union $\coprod X_i$ (as schemes) is *not* equal to the disjoint union (as presheaves) of the presheaves represented by each $X_i$. Nonetheless, Theorem 3.1.17 (d) also holds for the map in $\mathrm{AN}^\dagger \mathcal{V}$ represented by a disjoint union $\coprod Y_i \to \coprod X_i$ of morphisms of schemes (taken as a disjoint union of schemes instead of as presheaves on $\mathrm{AN}^\dagger \mathcal{V}$); indeed, the sheafification of $\coprod X_i$ is the same in each case, and in general for a site $C$ and a presheaf $F \in \widehat{C}$ with sheafification $F^a$, there is a natural equivalence

$$\widetilde{C_{/F}} \cong \widetilde{C}_{/F^a}$$

of topoi.

**Definition 3.2.5.** A map $(f, u)\colon (X', V') \to (X, V)$ of overconvergent varieties is said to be **finite** (see [lS09, Definition 3.2.3]) if, up to strict neighborhoods, $u$ is finite (see [Ber93, paragraph after Lemma 1.3.7]) and $u^{-1}(]X[_V) = ]X'[_{V'}$. Moreover, $u$ is said to be **universally flat** if, locally for Grothendieck topology on $V'$ and $V$, $u$ is of the form $\mathcal{M}(A') \to \mathcal{M}(A)$ with $A \to A'$ flat (see [Ber93, Definition 3.2.5]).

**Proposition 3.2.6.** *Let $(f, u)\colon (X', V') \to (X, V)$ be a finite map of overconvergent varieties and suppose that, after possibly shrinking $V'$ and $V$, $u$ is universally flat. Then $(f, u)$ is universally of cohomological descent with respect to finitely presented overconvergent crystals.*

*Proof.* To ease notation we set $p := (f, u)$. Let $F \in \mathrm{Mod}_{\mathrm{fp}}(X, V)$, and shrink $V$ and $V'$ such that $u$ is finite and such that $F_{X,V}$ is isomorphic to $i_X^{-1}G$ for some $G \in \mathrm{Coh}\,\mathcal{O}_V$ (which is possible by [lS09, Proposition 2.2.10]). By Corollary 3.1.10, it suffices to prove that (i) $\mathbb{R}^q p_* p^* F = 0$ $q > 0$, and (ii) that the Čech complex $F \to p_{\bullet,*}p^*_\bullet F$ is exact. To prove (i), as in [lS09, Proof of Proposition 3.2.4] it suffices to prove that $\mathbb{R}^q ]u[_* ]u[^* F_{X,V} = 0$ for $q > 0$. Then $\mathbb{R}^q ]u[_* ]u[^* F_{X,V} = i_X^{-1}\mathbb{R}^q u_* u^* G$; by [Ber93, Corollary 4.3.2] $\mathbb{R}^q u_* u^* G = 0$ and (i) follows.

For (ii), since one can check exactness of a complex of abelian sheaves on the collection of all realizations and since our hypotheses are stable under base change, it suffices to prove that the Čech complex of $F_{X,V}$ with respect to $]u[$ is exact. Since $i_X^{-1}$ is exact, it suffices to prove that the Čech complex of $G$ with respect to $u$ is exact. By [Ber93, Proposition 4.1.2], $G$ is a sheaf in the flat quasi-finite topology, so by part (i) of this proof and Theorem 3.1.16 (i), $G \to u_{\bullet,*}u^*_\bullet G$ is exact.

□

A *monogenic* map of rings is a map of the form $A \to A[t]/f(t)$, where $f \in A[t]$ is a monic polynomial, and a map of affine schemes is said to be monogenic if the associated map on rings is monogenic.

**Corollary 3.2.7.** *Let $p\colon X \to Y$ be a finite flat surjection of schemes. Then $p$ is universally of cohomological descent with respect to finitely presented crystals.*

*Proof.* By the argument of [BLR90, 2.3, Proposition 3], there exists a there exists a (generally non-cartesian) commutative diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & X \\ \downarrow{\scriptstyle \coprod f_i} & & \downarrow \\ \coprod Y_i & \longrightarrow & Y \end{array}$$

such that $\{Y_i\}$ a cover of $Y$ by affine open subschemes of finite type over $k$, $X_i$ is an affine open subscheme of $X$, and each map $f_i\colon X_i \to Y_i$ is monogenic. Since $Y$ is locally of finite type, we may choose $\{Y_i\}$ to be a locally finite covering.

By Theorem 3.1.17 (b), it suffices to prove that the composition $\coprod X_i \to Y$ is universally of cohomological descent with respect to crystals. By Theorem 3.1.17 (c) and Theorem 3.2.3, it suffices to prove that the map $\coprod X_i \to \coprod Y_i$ is universally of cohomological descent with respect to crystals, and by Theorem 3.1.17 (d) and Remark 3.2.4 it suffices to prove that for each $i$, the map $X_i \to Y_i$ is universally of cohomological descent with respect to crystals.

Thus, we may assume that $p\colon X \to Y$ is monogenic. Choose a formal scheme $P$ which is flat and projective over $\mathcal{V}$ and an embedding $Y \hookrightarrow P$ (which exists since $Y$ is affine and of finite type). Lifting the polynomial defining the map $X \to Y$ gives a map finite and flat map $P' \to P$ of formal schemes and an embedding $X \hookrightarrow P'$ which is compatible with the embedding $Y \hookrightarrow P$. Consider the diagram

$$\begin{array}{ccc} (X, P'_K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ (Y, P_K) & \longrightarrow & Y \end{array} .$$

By Theorem 2.2.23, $(Y, P_K) \to Y$ is a covering, so by Theorems 3.1.17 (b) and (c) and Theorem 3.1.16 (i) it suffices to prove that $(X, P'_K) \to (Y, P_K)$ is universally of cohomological descent with respect to finitely presented crystals, which follows from Proposition 3.2.6. □

**Corollary 3.2.8.** *Let $Y$ be an algebraic variety. Let $\{Y_i'\}$ be the set of irreducible components of $Y$ and let $Y_i := (Y_i')_{red}$ be the reduction of $Y_i'$. Then the morphism $\coprod Y_i \to Y$ is universally of cohomological descent with respect to finitely presented modules on the overconvergent site.*

*Proof.* Let $\{X_i\}$ be a locally finite cover of $Y$ by affine opens and set $X_i' := (\coprod Y_i) \times_Y X$. Applying Theorems 3.2.3 and 3.1.17 (a) and (d) to the diagram of sheaves on $\mathrm{AN}^\dagger Y$ induced by the cartesian diagram of schemes

$$\begin{array}{ccc} \coprod X_i' & \longrightarrow & \coprod Y_i \\ \downarrow & & \downarrow \\ \coprod X_i & \longrightarrow & Y \end{array}$$

it suffices to prove our theorem for each of the maps $X_i' \to X_i$.

We may thus assume that $Y$ is affine. Choose an embedding $Y \hookrightarrow P$ into a formal scheme $P$ which is smooth and projective over $\mathcal{V}$ (which exists since $Y$ is affine and of finite type). Denote by $\overline{Y}$ (resp. $\overline{Y_i}$) the closure of $Y$ (resp. $Y_i$) in $P$; since the preimage under specialization of a closed immersion is an open immersion, $\cup ]Y_i[_P$ is an open covering of $]Y[_P$. Consider the diagram

$$\begin{array}{ccc} \coprod (Y_i, ]\overline{Y}_i[_{P_K}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ (Y, P_K) & \longrightarrow & Y \end{array} .$$

By Theorem 2.2.23, $(Y, P_K) \to Y$ is a covering, so by Theorems 3.1.17 (b) and (c) and Theorem 3.1.16 (i) it suffices to prove that $\coprod (Y_i, ]\overline{Y}_i[_{P_K}) \to (Y, P_K)$ is universally of cohomological descent with respect to finitely presented crystals. The map $\coprod (Y_i, ]\overline{Y}_i[_{P_K}) \to (Y, P_K)$ factors as $\coprod (Y_i, ]\overline{Y}_i[_{P_K}) \to \coprod (Y, ]\overline{Y}_i[_{P_K}) \to (Y, P_K)$; by Theorem 3.1.17 (c) it suffices to prove that each of these two maps is universally of cohomological descent with respect to finitely presented crystals. The second map is a covering in $\mathrm{AN}^\dagger Y$ and thus by Theorem 3.1.16 (i) is universally of cohomological descent with respect to all abelian sheaves; the first map is a disjoint union of finite universally flat morphisms of overconvergent varieties, so by Proposition 3.2.6 and Theorem 3.1.17 (d) it is universally of cohomological descent with respect to finitely presented crystals.

□

Our main theorem is the following.

**Theorem 3.2.9.** *Let $p_0 \colon X \to Y$ be an étale surjection of quasi-compact algebraic varieties over $k$. Let $\mathcal{F} \in \mathrm{Cris}^\dagger_{\mathrm{g}} Y$ be an overconvergent crystal. Then the associated morphism*

$$X_{\mathrm{AN}^\dagger_{\mathrm{g}}} \to Y_{\mathrm{AN}^\dagger_{\mathrm{g}}}$$

*of topoi is universally of cohomological descent with respect to $\mathcal{F}$.*

In contrast to most other results, this proof does not generalize to $p_{\mathrm{AN}^\dagger}$ (i.e. the case without goodness hypothesis). Also, to ease notation we will often omit the subscript $\mathrm{AN}^\dagger_{\mathrm{g}}$ from functors.

*Proof.* Any smooth morphism étale locally has a section, the smooth case follows from the étale case and Theorems 3.1.17 (a) and 3.1.16 (ii). Let $p_0 \colon X \to Y$ be an étale morphism of schemes. By [BLR90, 2.3, Proposition 3] there exists a (generally non-cartesian) commutative diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & X \\ \Big\downarrow{\scriptstyle \coprod f_i} & & \Big\downarrow \\ \coprod Y_i & \longrightarrow & Y \end{array}$$

such that $\{Y_i\}$ a cover of $Y$ by affine open subschemes of finite type over $k$, $X_i$ is an affine open subscheme of $X$, and each map $f_i \colon X_i \to Y_i$ factors as

$$X_i \subset \widetilde{X_i} \to Y_i,$$

where the first map is a Zariski open immersion, the second map is finite and locally free, and the composition $f_i$ is surjective. Since $Y$ is locally of finite type, we may choose $\{Y_i\}$ to be a locally finite covering.

By Theorem 3.1.17 (b), it suffices to prove that the composition $\coprod X_i \to Y$ is universally of cohomological descent with respect to crystals. By Theorem 3.1.17 (c) and Theorem 3.2.3, it suffices to prove that the map $\coprod X_i \to \coprod Y_i$ is universally of cohomological descent with respect to crystals, and by Theorem 3.1.17 (d) it suffices to prove that for each $i$, the map $X_i \to Y_i$ is universally of cohomological descent with respect to crystals.

Thus we have reduced to the case of a surjective étale morphism $p_0 \colon X \to Y$ which admits a factorization $X \subset \widetilde{X} \to Y$ such that the first map is aZ ariski open immersion and the second map is finite and locally free. Thus, by Theorem 3.1.17 (a) and Corollary 3.2.7, it suffices to check that $X \to Y$ is universally of cohomological descent with respect to crystals after pulling back by the map $\widetilde{X} \to Y$, and so we may assume that $\widetilde{X} \to Y$ has a section $s$.

Let $\{Y_i'\}$ be the finite set of irreducible components of $Y$, let $Y_i := (Y_i')_{\mathrm{red}}$ be the reduction of $Y_i'$, and set $X_i = Y_i \times_Y X$. Applying Corollary 3.2.8 and Theorems 3.1.17 (a), (d) to the diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & X \\ \Big\downarrow{\scriptstyle \coprod f_i} & & \Big\downarrow \\ \coprod Y_i & \longrightarrow & Y \end{array}$$

we may assume that $Y$ is integral. We now proceed by induction on the degree of the second map. If it has degree 1 then, since the map $X \to Y$ is surjective, it is actually an isomorphism and thus trivially universally of cohomological descent. Suppose it has degree $d > 1$.

Let $\{\widetilde{X_i}'\}$ be the finite set of irreducible components of $\widetilde{X}$; by the valuative [Gro66, Théorème 11.8.1] and local [Gro61, 0, Corollaire 10.2.7] criterions for flatness, $\widetilde{X_i}' \to Y$ is flat. Let $Y_i$ be the image of $\widetilde{X_i}' \cap X$ in $Y$ (which is open since $\widetilde{X_i}' \to Y$ is finite and flat), let

$\widetilde{X_i}$ be $Y_i \times_Y \widetilde{X_i}$, and let $X_i = \widetilde{X_i} \cap X$. Then we get a diagram

$$\begin{array}{ccc} \coprod X_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod \widetilde{X_i} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \coprod Y_i & \longrightarrow & Y \end{array}$$

such that each map $\widetilde{X_i} \to Y_i$ is finite and locally free and $\{Y_i\}$ is a Zariski cover of $Y$. Since $\widetilde{X} \to Y$ is separated, the section $s$ is a closed immersion and $s(Y)$ is an irreducible component of $\widetilde{X}$; in particular $\widetilde{X}$ has more than one irreducible component and thus for each $i$, the degree of $\widetilde{X_i} \to Y_i$ is strictly less than by $d$. By induction, for each $i$ the map $X_i \to Y_i$ is universally of cohomological descent. The étale case of this theorem now follows from Theorems 3.2.3 and 3.1.17 (b), (c), and (d).

□

**Corollary 3.2.10.** *Assume Conjecture 3.0.6 is true. Let $\boldsymbol{P}$ be the class of morphisms of algebraic varieties that are compositions of finitely many smooth morphisms and proper morphisms. Let $p\colon X_\bullet \to Y$ be a $\boldsymbol{P}$-hypercover of quasi-compact algebraic varieties over $k$, and let $\mathcal{F} \in \mathrm{Cris}^\dagger_{\mathrm{g}} Y$. Then the associated morphism of topoi*

$$(X_\bullet)_{\mathrm{AN}^\dagger_{\mathrm{g}}} \to Y_{\mathrm{AN}^\dagger_{\mathrm{g}}}$$

*is universally of cohomological descent with respect to $\mathcal{F}$.*

*Proof.* This follows directly from Theorem 3.2.9, Conjecture 3.0.6, and Theorem 3.1.16 (iv) (or rather, the 'single sheaf' variant analogous to Theorem 3.1.17, which is proved in the same way).

□

*Remark* 3.2.11. The following is a typical application of cohomological descent. By [Con, Theorem 4.16], it follows from de Jong's alterations theorem [dJ96] that for any separated scheme $Y$ of finite type over $k$ there exists a proper hypercover (e.g. a $\mathbb{P}$-hypercover) $X_\bullet \to Y$ such that each $X_i$ is a regular scheme. This allows one to deduce statements about rigid cohomology for general schemes from the the case of regular schemes; see Proposition 4.2.14 for one example, where we prove that Conjecture 3.0.6 implies finiteness of the absolute rigid cohomology of an algebraic stack.

Next we write out the standard argument which generalizes this result to stacks.

**Corollary 3.2.12.** *Assume Conjecture 3.0.6. Then the following are true.*

(a) *Let $(C, O)$ be an overconvergent variety. Let $p_0 \colon \mathcal{X}' \to \mathcal{X}$ be a representable surjection of algebraic stacks over $C$ such that $p_0$ is either a smooth morphism or a proper morphism. Suppose that $\mathcal{X}$ is locally of finite type over $k$ and denote by $p \colon \mathcal{X}'_\bullet \to \mathcal{X}$ its 0-coskeleton. Let $E \in \mathrm{Cris}^\dagger_{\mathrm{g}}\, \mathcal{X}/O$. Then $p_{\mathrm{AN}^\dagger_{\mathrm{g}}} \colon (\mathcal{X}'_{\mathrm{AN}^\dagger}/O)_{\mathrm{g},\bullet} \to (\mathcal{X}_{\mathrm{AN}^\dagger}/O)_{\mathrm{g}}$ is universally of cohomological descent with respect to $E$.*

(b) *More generally, let $\boldsymbol{P}$ be the class of representable morphisms of stacks over $C$ that are compositions of finitely many smooth morphisms and proper morphisms which are of finite type over $k$, and let $\mathcal{X}$ be a stack locally of finite type over $k$. Then $\mathrm{AN}^\dagger_{\mathrm{g}}$ of a $\boldsymbol{P}$-hypercovering of $\mathcal{X}$ is universally of cohomological descent with respect to objects of $\mathrm{Cris}^\dagger_{\mathrm{g}}\, \mathcal{X}/O$.*

*Proof.* For part (a), first note that universality is clear, since smoothness and properness are stable under base change. Let $T \to \mathrm{AN}^\dagger_{\mathrm{g}}(C, O)$ be a fibered category. Then any map $(Y, V) \to T$ of categories fibered over $\mathrm{AN}^\dagger_{\mathrm{g}}(C, O)$ factors through $(Y/O)_{\mathrm{g}}$, and so to check that a morphism of sheaves on $\mathrm{AN}^\dagger_{\mathrm{g}}\, T$ is an isomorphism it suffices to check it after pulling back to $Y/O$, as $Y$ varies over all maps from presheaves represented by schemes to $T$.

Now let $E \in \mathrm{Cris}^\dagger_{\mathrm{g}}\, \mathcal{X}/O$. By the previous paragraph applied to $T = \mathrm{AN}^\dagger_{\mathrm{g}}\, \mathcal{X}/O$, to check that the map $E \to \mathbb{R}p_{\mathrm{AN}^\dagger_{\mathrm{g}}\,*} p^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} E$ is an isomorphism, it suffices to check that, for every morphism $f \colon Y \to \mathcal{X}$ from a variety $Y$ over $C$, the map $f^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} E \to f^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} \mathbb{R}p_{\mathrm{AN}^\dagger_{\mathrm{g}}\,*} p^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} E$ is an isomorphism. The cohomology and base change argument of [lS09, Proposition 3.2.4] works verbatim for fibered categories over $\mathrm{AN}^\dagger_{\mathrm{g}}(C, O)$, and so the previous adjunction is isomorphic to the adjunction $f^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} E \to \mathbb{R}p'_{\mathrm{AN}^\dagger_{\mathrm{g}}\,*} p^{*'}_{\mathrm{AN}^\dagger_{\mathrm{g}}} f^*_{\mathrm{AN}^\dagger_{\mathrm{g}}} E$ where, setting $p'_0 \colon Y' := Y \times_{\mathcal{X}} \mathcal{X}' \to Y$, $p'$ is $\mathrm{cosk}_0(p'_0)$.

Thus to prove the corollary it suffices to do the case when $\mathcal{X}'$ is an algebraic space $X'$ and $\mathcal{X}$ is an algebraic variety $X$. By Chow's lemma [Knu71, Theorem IV.3.1], there exists a separated birational morphism $X'' \to X'$ such that $X''$ is a scheme and the composition $q_0 \colon X'' \to X$ is proper and surjective. By applying Theorem 3.1.17 (b) to the composition $X''/O \to X'/O \to X/O$, it suffices to prove that $(q_0)_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ is of cohomological descent with respect to crystals on $X_{\mathrm{AN}^\dagger_{\mathrm{g}}}$, which is Theorem 3.2.9 and Conjecture 3.0.6. The first claim follows.

Using the strategy of proof of part (a), part (b) reduces to the case of a **P**-hypercovering where everything in sight is an algebraic space (noting that, by their defining universal property, the functors $\mathrm{cosk}_n$ commute with base change on $S$). By part (a) and Theorem 3.1.17 (c) to conclude that a composition of smooth morphisms and proper morphisms of algebraic spaces is of cohomological descent with respect to good crystals, the claim now follows from Theorem 3.1.16 (iv).

□

*Remark* 3.2.13. Theorem 3.2.12 should still be true when $p_0$ is not representable, but one would need to check that the proof of Theorem 3.1.17 still holds in the 2-category of stacks on a site.

# Chapter 4

# Cohomology supported in a closed subspace

While cohomology with compact supports is the star of the story, cohomology supported in a closed subspace also plays a key role (e.g., in Kedlaya's proof of finite dimensionality of rigid cohomlogy with coefficients [Ked06a]).

In this chapter we use the very general notion of excision on topoi to define cohomology supported in a closed subspace: for a stack $\mathcal{X}$, a closed substack $\mathcal{Z}$, and an overconvergent sheaf $F$ on $\mathrm{AN}^\dagger \mathcal{X}$, we define $H^i_{\mathcal{Z}}(\mathcal{X}, F)$ (and a relative counterpart) and deduce that it satisfies the Bloch-Ogus formalism (i.e., functorality and excision).

## 4.1 A quick guide to excision on topoi

Here we recall very general facts about immersions of topoi, open and closed sub-topoi, excision, and cohomology supported in a closed sub-topos.

Let $f\colon T' \to T$ be a morphism of topoi. We say that $f$ is an **immersion** if $f_*$ is fully faithful [73, Definition 9.1.2]; by Yoneda's lemma this is equivalent to the adjunction $\mathrm{id} \to f^{-1}f_*$ being an isomorphism.

Let $T$ be a topos. Then $T$ has a final object (see Appendix 5.1.15), a choice of which we denote by $\mathrm{e}_T$. Following [73, Definition 8.3], we say that an object $U \in T$ is **open** if it is a subobject of $\mathrm{e}_T$ (i.e., if the map $U \to \mathrm{e}_T$ is a monomorphism). Similarly, for $X \in T$ we define an **open** of $U \subset X$ to be an open object $U \subset T_{/X}$ of the topos $T_{/X}$.

Let $U \in T$ be open. The restriction map $j\colon T_{/U} \to T$ induces a morphism $(j^*, j_*)$ of topoi, which is an immersion – indeed, using the explicit description of the pair $(j^*, j_*)$ and that $U \to \mathrm{e}_T$ is a monomorphism one can easily check that the adjunction is an isomorphism. We define $T' \to T$ to be an **open immersion** of topoi if it is isomorphic to $T_{/U} \to T$ with

$U \in T$ open, and we say that $T' \to T$ is an open subtopos, and we say that a morphism of sites is an open immersion if the induced morphism of topoi is an open immersion.

Now let $T' \to T$ be an open immersion, and let $U \in T$ be an open such that $T' \to T$ is isomorphic to $T_{/U} \to T$. As in [73, 9.3.5], we define the **closed complement** $Z$ of $T'$ in $T$ to be the *complement* of $T_{/U}$ in $T$, i.e., the largest sub-topos $Z$ of $T$ such that $T_{/U} \cap Z$ is equivalent to $\{e_{T_{/U}}\}$. Concretely, $Z$ is the full subcategory of objects $F \in T$ such that the projection map $U \times F \to U$ is an isomorphism (i.e., such that $j^*F$ is isomorphic to $e_{T_{/U}}$). The category $Z$ is independent of the choice of $U$. When $T' = T_{/U}$, we will also call $Z$ the closed complement of $U$.

We denote the inclusion $Z \hookrightarrow T$ by $i_*$ and remark that by [73, Proposition 9.3.4], the map $i^*\colon T \to Z$ given by $F \mapsto U \coprod_{U \times F} F$ is adjoint to $i_*$, with adjunction given by the projection morphism $F \to U \coprod_{U \times F} F$, and that together these form a morphism $(i^*, i_*)\colon Z \to T$ of topoi. Since $Z$ is a full subcategory, the inclusion $i$ is an immersion of topoi, and we say that any immersion of topoi isomorphic to an immersion $Z \to T$ arising as the closed complement of an open immersion is a **closed immersion** of topoi and say that $Z$ is a closed sub-topos of $T$.

*Remark* 4.1.1. Let $C$ be a site. Let $X \to X'$ be a monomorphism in $C$. Then the induced map $\widetilde{C}_{/X} \to \widetilde{C}_{/X'}$ is an open immersion. In particular, when $C = \mathbf{Sch}$, two odd examples of 'open immersions' arise from $X \to X'$ a closed immersion or $\operatorname{Spec} \mathcal{O}_{X',x} \to X'$, with $x \in X'$! Remarkably, as above one can still define a notion of 'closed complement' of a closed immersion and deduce an excision theorem (see Proposition 4.1.6).

Of course, the more interesting open immersions are those whose closed complements admit a 'geometric' description. For example, if $U \subset X$ is an open inclusion of topological spaces, then $U$ is an object of the site $\operatorname{Open} X$ and the restriction morphism $j\colon \operatorname{Open} U \cong (\operatorname{Open} X)_{/U} \to \operatorname{Open} X$ induces the usual morphism of topoi induced by the continuous morphism of sites $\operatorname{Open} X \to \operatorname{Open} U$, $U' \mapsto U \cap U'$. If we denote by $Z$ the (topological) complement of $U$ in $T$, then the closed complement of $U$ in $X_{\text{Open}}$ is isomorphic to the usual inclusion induced by the continuous morphism of sites $\operatorname{Open} X \to \operatorname{Open} Z$ given again by intersection.

Another 'geometric' example is le Stum's explication of Berthelot's $j^\dagger$ functor [lS07, Proposition 5.1.12 (a)]; see Remark 2.3.3.

A closed immersion $Z \to T$ of topoi enjoys many of the same properties as the classical case $\operatorname{Open} Z \to \operatorname{Open} X$; see [73, 9.4] for a nice discussion. Here we recall everything relevant to excision.

Let $(T, \mathcal{O}_T)$ now be a ringed topos. Let $U \in T$ be open with closed complement $Z$ and set $\mathcal{O}_U := j^*\mathcal{O}_T$ and $\mathcal{O}_Z := i^*\mathcal{O}_Z$. We have the following diagrams of topoi, where each arrow is left adjoint to the arrow directly below it. The functors $j^*$, and $j_*$ (resp. $i^*$ and $i_*$)

restrict to a pair of adjoint functors (note that tensoring is not necessary!), giving a diagram

$$\operatorname{Mod}\mathcal{O}_U \underset{j_*}{\overset{j_!^{\mathrm{ab}}}{\underset{\xleftarrow{\;j^*\;}}{\rightarrow}}} \operatorname{Mod}\mathcal{O}_T \underset{i^!}{\overset{i^*}{\underset{\xleftarrow{\;i_*\;}}{\rightarrow}}} \operatorname{Mod}\mathcal{O}_Z$$

where the left arrows were defined in Appendix 5.1.27 and the extra adjoint $i^!$ is defined by

$$i^! P = \ker\left(i^* P \to i^* j_* j^* P\right)$$

(see [73, exposé 4, 9.5 and 14.4] for a more intrinsic description of $i^!$). Note that $i_*$ is thus exact as in the case of a closed immersion of schemes.

The functor $j_!^{\mathrm{ab}}$ differs from the usual $j_!$ (see the end of Appendix 5.1.24), but when the context is clear we will write $j_!$; in particular $j_!$ of a sheaf of abelian groups will *always* refer to $j_!^{\mathrm{ab}}$.

**Proposition 4.1.2.** *Let $P \in \operatorname{Mod}\mathcal{O}_T$. Then the following are true.*

*(i)* $0 \to j_! j^* P \to P \to i_* i^* P \to 0$ *is exact;*

*(ii)* $0 \to i_* i^! P \to P \to j_* j^* P$ *is exact;*

*(iii)* $i_* i^! P$ *is the 'largest subsheaf of $P$ supported on $Z$' (see* [73, exposé 4, 9.3.5]*);*

*(iv)* $i_* i^* P \cong i_* \mathcal{O}_Z \otimes_{\mathcal{O}_T} P$*;*

*(v)* $i_* i^! P \cong \mathscr{H}\mathrm{om}_{\mathcal{O}_T}(i_* \mathcal{O}_Z, P)$*.*

*(vi)* $j_* j^* P \cong \mathscr{H}\mathrm{om}_{\mathcal{O}_T}(j_! \mathcal{O}_U, P)$*.*

The proofs of these (and basically any identity involving these 6 functors) follows from a combination of the very simple description of these functors and maps between them via the covering theorem [73, 14.3] and, for $M, N \in \operatorname{Mod}\mathcal{O}$, the two adjunctions (or if one prefers, *definitions*) $\operatorname{Hom}_T(-, \mathscr{H}\mathrm{om}_{\mathcal{O}}(M, N)) \cong \operatorname{Hom}_{\mathcal{O}}(M, \mathscr{H}\mathrm{om}_T(-, N))$ (as functors on $T$) [73, Proposition 12.1] and $\operatorname{Hom}_{\operatorname{Ab} T}(P, M \otimes_{\mathcal{O}} N) \cong \operatorname{Hom}_{\mathcal{O}}(M, \mathscr{H}\mathrm{om}_{\mathbb{Z}}(N, -))$ (as functors on $\operatorname{Ab} T$) [73, Proposition 12.7].

*Proof.* These are in [73, exposé 4]: (i & ii) are 14.6 (account for the typo in (ii)), (iii) is 14.8, (iv) is 14.10, 1, (v) is 14.10, 2, and (vi) follows from the proof of 14.10 (see also 12.6). □

**Definition 4.1.3.** Given $E \in \operatorname{Mod}\mathcal{O}_T$, we define $\mathscr{H}_Z^0 E := i_* i^! E$ to be the **sheaf of sections of $E$ supported on $Z$** and denote the derived functors of $E \mapsto \mathscr{H}_Z^0 E$ by $\mathscr{H}_Z^i E$.

We can derive the functor $\mathscr{H}_Z^0$ either as a functor on $\operatorname{Ab} T$ or on $\operatorname{Mod}\mathcal{O}_T$, because the functors $i_*$ and $i^!$ commute with the (exact) forgetful functor $\operatorname{Mod}\mathcal{O}_T \to \operatorname{Ab} T$.

*Remark* 4.1.4. This is an appropriate name, because $\Gamma(T, \mathscr{H}^0_Z E)$ is the $\Gamma(T, \mathcal{O}_T)$-module of all sections $s$ of $E$ supported on $Z$ (i.e., such that $s|_U = 0$); see [73, Proposition 14.8].

**Definition 4.1.5.** Let $f\colon (T, \mathcal{O}_T) \to (T', \mathcal{O}_{T'})$ be a morphism of ringed topoi. We define the **cohomology** (resp. **relative cohomology**) **of** $E$ **supported on** $Z$ to be the right derived functors of $E \mapsto \Gamma(T, \mathscr{H}^0_Z E)$ by $H^i_Z(T, E)$ and the derived functors of $E \mapsto f_* \mathscr{H}^0_Z E$ by $\mathbb{R} f_{*,Z} E$.

**Proposition 4.1.6.** *Let $E \in \operatorname{Mod} \mathcal{O}_T$ and let the notation be as above. Then the following are true.*

*(i)* $0 \to \mathscr{H}^0_Z E \to E \to j_* j^* E \to \mathscr{H}^1_Z E \to 0$ *is exact.*

*(ii) There is a long exact sequence*

$$\ldots \to H^i_Z(T, E) \to H^i(T, E) \to H^i(U, j^* E) \to H^{i+1}_Z(T, E) \to \ldots$$

*(iii) Let $U' \subset U \subset T$ be a sequence of open immersions with closed complements $Z \hookrightarrow Z' \hookrightarrow T$ and denote by $Z' \cap U$ the closed complement of $U'$ in $U$. Then there is a long exact sequence*

$$\ldots \to H^i_Z(T, E) \to H^i_{Z'}(T, E) \to H^i_{Z' \cap U}(U, j^* E) \to H^{i+1}_Z(T, E) \to \ldots$$

*(iv) There is a spectral sequence*

$$\mathbb{R}^j f_* \mathscr{H}^j_Z E \Rightarrow \mathbb{R}^{i+j} f_{*,Z} E.$$

*Proof.* Claims (i) - (iii) follow directly from Proposition 4.1.2 above; see [72, exposé 5, Proposition 6.5] for (i) and (ii), and for (iii) apply the proof of (ii) but with $P = i_* i^! \mathcal{O}_T$ (instead of $P = \mathcal{O}_T$) in Proposition 4.1.2 (ii). Claim (iv) is just the spectral sequence associated to a composition of derived functors.

□

Lastly, we discuss functorality. For a ringed topos $(T, \mathcal{O}_T)$ and a morphism of topoi $g\colon T' \to T$, we set $\mathcal{O}_{T'} := g^* \mathcal{O}_T$, and if $T' = T_{/X}$ for some $X \in T$ we write $\mathcal{O}_X$ for $\mathcal{O}_{T_{/X}}$.

**Proposition 4.1.7.** *Let $(T, \mathcal{O}_T)$ be a ringed topos, let $f\colon X' \to X$ be a morphism in $\widehat{T}$, let $j\colon U \subset X$ be an open immersion with closed complement $i\colon Z \hookrightarrow T_{/X}$, let $j'\colon U' = U \times_X X' \subset X'$ and denote its closed complement by $i'\colon Z' \hookrightarrow T_{/X'}$. Let $E \in \operatorname{Mod} \mathcal{O}_X$. Then there is a natural map*

$$H^i_Z(T_{/X}, E) \to H^i_{Z'}(T_{/X'}, f^* E).$$

*Proof.* It follows from the commutativity of the diagram

$$\begin{array}{ccc} T_{/U'} & \longrightarrow & T_{/X'} \\ \downarrow & & \downarrow \\ T_{/U} & \longrightarrow & T_{/X} \end{array}$$

of topoi that

$$f^*j_*j^*E \cong j'_*j'^*f^*E$$

and that the composition

$$f^*E \to f^*j_*j^*E \cong j'_*j'^*f^*E$$

is the adjunction. Then there is an isomorphism

$$f^*i_*i^!E = f^*\ker(E \to j_*j^*E) = \ker(f^*E \to f^*j_*j^*E) = i'_*i'^!f^*E,$$

where the second equality follows from exactness of $f^*$ on $\operatorname{Mod}\mathcal{O}_X$ (recall that $\mathcal{O}_{X'} = f^*\mathcal{O}_X$) and the other two follow from Proposition 4.1.2 (ii); the natural map

$$H^0(T_{/X}, i_*i^!E) \to H^0(T_{/X'}, f^*i_*i^!E) \cong H^0(T_{/X'}, i'_*i'^!f^*E)$$

thus induces a map

$$H^i_Z(T_{/X}, E) \to H^i_{Z'}(T_{/X'}, f^*E).$$

□

## 4.2 Excision on the overconvergent site

Here we apply the very general notions of Section 4.1 to the overconvergent site (with our definitions and notation as in Chapter 2). We state everything for $\mathrm{AN}^\dagger$ and for the sake of brevity omit restating definitions for the good variants on $\mathrm{AN}^\dagger_{\mathrm{g}}$, but note that everything carries over without incident.

Let $X \to \mathrm{AN}^\dagger(\mathcal{V})$ be a fibered category over the overconvergent site and let $F \in \operatorname{Mod}\mathcal{O}^\dagger_X$ be an overconvergent module on $X$. Let $U \in X_{\mathrm{AN}^\dagger}$ be open in the sense of Section 4.1 (i.e. $U$ is a subsheaf of the final object of $X_{\mathrm{AN}^\dagger}$). Denote by $j\colon U_{\mathrm{AN}^\dagger} := (X_{\mathrm{AN}^\dagger})_{/U} \hookrightarrow X_{\mathrm{AN}^\dagger}$ the open immersion of topoi induced by restriction, denote by $Z_{\mathrm{AN}^\dagger}$ the closed complement of $U$ in $X_{\mathrm{AN}^\dagger}$, and denote by $\mathcal{O}^\dagger_U$ and $\mathcal{O}^\dagger_Z$ the restrictions of $\mathcal{O}^\dagger_X$ to $U_{\mathrm{AN}^\dagger}$ and $Z_{\mathrm{AN}^\dagger}$. As in Section 4.1 this induces a collage of adjoint functors

$$\operatorname{Mod}\mathcal{O}^\dagger_U \;\underset{j_*}{\overset{j_!}{\rightrightarrows}}\!\!\!\overset{j^*}{\longleftarrow}\; \operatorname{Mod}\mathcal{O}^\dagger_X \;\underset{i^!}{\overset{i^*}{\rightrightarrows}}\!\!\!\overset{i_*}{\longleftarrow}\; \operatorname{Mod}\mathcal{O}^\dagger_Z.$$

**Definition 4.2.1.** Let $E \in \operatorname{Mod} \mathcal{O}_X^\dagger$ be an overconvergent module. We define $\underline{\Gamma}_Z^\dagger E := i_* i^! E$ to be the **subsheaf of** $E$ **of sections supported on** $Z$, and we define $H_Z^0(\mathrm{AN}^\dagger X, E) := H^0(\mathrm{AN}^\dagger X, \underline{\Gamma}_Z^\dagger E)$ to be the $H^0(\mathrm{AN}^\dagger X, \mathcal{O}_X^\dagger)$ submodule of **sections of** $E$ **supported on** $Z$. For a morphism $f\colon X \to Y$ of categories fibered over $\mathrm{AN}^\dagger \mathcal{V}$, we define the **relative cohomology of** $E$ **supported on** $Z$ to be $\mathbb{R} f_{\mathrm{AN}^\dagger *} \underline{\Gamma}_Z^\dagger E$, which we denote by $\mathbb{R} f_{\mathrm{AN}^\dagger *, Z} E$. Since the realization functors are exact, when $Y = \operatorname{Spec} k$ the realization of the $i^{\text{th}}$ cohomology sheaf of $\mathbb{R} f_{\mathrm{AN}^\dagger *, Z} E$ is isomorphic to the $i^{\text{th}}$ derived functor of $H_Z^0(\mathrm{AN}^\dagger X, E)$; consequently we denote both of these $K$-vector spaces by $H_Z^i(\mathrm{AN}^\dagger X, E)$ and $H_Z^i(\mathrm{AN}^\dagger X, \mathcal{O}_X^\dagger)$ by $H^i_{\mathrm{rig},Z}(\mathrm{AN}^\dagger X)$.

This differs slightly from the definition of 4.1, and in addition we have switched from the notation $\mathscr{H}_Z^0$ of [73] to the notation $\underline{\Gamma}_Z^\dagger$ of [lS07, Definition 5.2.10].

**Example 4.2.2.** We will mainly consider the following examples of open immersions of sites.

(i) $\mathrm{AN}^\dagger U \subset \mathrm{AN}^\dagger X$ with $X$ an algebraic stack over $k$ and $U \subset X$ an open substack.

(ii) $\mathrm{AN}^\dagger(U, V) \subset \mathrm{AN}^\dagger(X, V)$ with $(X, V) \in \mathrm{AN}^\dagger \mathcal{V}$ an overconvergent variety and $U \subset X$ an open subscheme.

(iii) $\mathrm{AN}^\dagger(U_V) \subset \mathrm{AN}^\dagger(X_V)$ with $(X, V) \in \mathrm{AN}^\dagger \mathcal{V}$ an overconvergent variety and $U \subset X$ an open subscheme, where $X_V$ is the image subpresheaf of the morphism of sheaves $(X, V) \to X$ (see Definition 2.2.19).

(iv) More generally, for an overconvergent variety $(C, O)$ and an algebraic stack $X$ over $k$ with a morphism $X \to C$, we can consider the relative variants $\mathrm{AN}^\dagger X/O$ and $\mathrm{AN}^\dagger X_V/O$ (see Definition 2.2.18).

These examples are all 'representable' in the following sense.

**Definition 4.2.3.** Let $j\colon U \subset X$ be a morphism of categories fibered over $\mathrm{AN}^\dagger(\mathcal{V})$ which induces an open immersion of topoi. We say that $j$ is **representable** if for any overconvergent variety $(X', V')$ and morphism of fibered categories $(X', V') \to X$, there exists an open subscheme $U' \subset X'$ such that $(U', V')$ represents the 2-fiber product $U \times_X \mathrm{AN}^\dagger(X', V')$.

Here we rewrite the excision sequences from Subsection 4.1.

**Proposition 4.2.4** (Translation of Proposition 4.1.6)**.** *Let $j\colon U \subset X$ be a representable open immersion and let $E \in \operatorname{Mod} \mathcal{O}_X^\dagger$ be an overconvergent module. Then with the notation above, the following are true.*

*(i)* $0 \to \underline{\Gamma}_Z^\dagger E \to E \to j_* j^* E \to 0$ *is exact.*

*(ii) There is a long exact sequence*

$$\ldots \to H_Z^i(\mathrm{AN}^\dagger X, E) \to H^i(\mathrm{AN}^\dagger X, E) \to H^i(\mathrm{AN}^\dagger U, j^* E) \to H_Z^{i+1}(\mathrm{AN}^\dagger X, E) \to \ldots$$

*(iii) There is a spectral sequence*

$$\mathbb{R}^j f_{\mathrm{AN}^\dagger *} \mathbb{R}^j \underline{\Gamma}_Z^\dagger E \Rightarrow \mathbb{R}^{i+j} f_{\mathrm{AN}^\dagger, Z} E.$$

*Proof.* Most of this is Proposition 4.1.6; the only thing to check is that the map $E \to j_* j^* E$ is surjective. Since the morphism of sites of Definition 2.2.8 defines a bijection of coverings, surjectivity can be checked on realizations. Let $(Y, V)$ be an overconvergent variety over $X$. Let $j' \colon U' \subset Y$ be an open immersion such that $(U', V)$ represents the fiber product $U \times_X (Y, V)$. Then $]j'[ \colon ]U'[_V \hookrightarrow ]Y[_V$ is now a closed immersion of analytic spaces. Then, by the proof of part (i) of Proposition 4.2.5 below, there is an isomorphism

$$(j_* j^* E)_{Y,V} \cong ]j'[_* ]j'[^* E_{Y,V}$$

such that the composition

$$E_{Y,V} \to (j_* j^* E)_{Y,V} \cong ]j'[_* ]j'[^* E_{Y,V}$$

is the adjunction

$$E_{Y,V} \to ]j'[_* ]j'[^* E_{Y,V}.$$

By Proposition 4.1.2 (ii) (noting by Remark 4.1.1 that $]j'[_*$ is a closed immersion of topoi) this map is surjective.

□

The first task is to check that this agrees with the classical construction due to Berthelot of rigid cohomology supported in a closed subscheme. Most of the work is packaged into the following proposition.

**Proposition 4.2.5.** *Let $(C, O)$ be an overconvergent variety. Let $X \to \mathrm{AN}^\dagger(C, O)$ be a fibered category and let $U \subset X$ be a sub-fibered category of $X$ which is an open subtopos of $X_{\mathrm{AN}^\dagger}$ such that for all $(X', V')$, the fiber product $U \times_X (X', V')$ is isomorphic to $(U', V')$ with $U' \subset X'$ an open subscheme of $X'$. Denote the closed complement (defined in Section 4.1) of $U_{\mathrm{AN}^\dagger} \subset X_{\mathrm{AN}^\dagger}$ by $Z$. Let $E \in \mathrm{Cris}^\dagger X$ be an overconvergent module, let $(f, u) \colon (X'', V'') \to (X', V') \in \mathrm{AN}^\dagger X$ be a morphism of overconvergent varieties over $X$, and denote by $j'$ the inclusion $U' \hookrightarrow X'$, where $(U', V')$ is $U \times_X (X', V')$ (and similarly $j'' \colon U'' \hookrightarrow X''$). Then the following are true.*

*(i) The realization $(\underline{\Gamma}_Z^\dagger E)_{X',V'}$ is canonically isomorphic to the kernel of the adjunction morphism*

$$\ker\left(E_{X',V'} \to ]j'[_* ]j'[^\dagger E_{X',V'}\right).$$

*(ii) Denote by $i' \colon W' \subset ]X'[_{V'}$ the (open) complement of the closed inclusion $]U'[_{V'} \hookrightarrow ]X'[_{V'}$. Then there is an isomorphism*

$$\left(\underline{\Gamma}_Z^\dagger E\right)_{X',V'} \cong i'_! i'^{-1} E_{X',V'}.$$

*(iii) The sheaf* $\underline{\Gamma}^{\dagger}_{Z}E$ *is a crystal.*

*Proof.* It suffices to consider the case $X = (X', V')$, and $U = (U', V')$. For simplicity we drop a prime everywhere in the notation (i.e., we consider a morphism $(X', V') \to (X, V) \in \mathrm{AN}^{\dagger} X$). To avoid the potentially awkward notation $\mathrm{id}_{\mathrm{AN}^{\dagger}}$ we denote the morphisms $(U, V)_{\mathrm{AN}^{\dagger}} \to (X, V)_{\mathrm{AN}^{\dagger}}$ and $(U', V')_{\mathrm{AN}^{\dagger}} \to (X', V')_{\mathrm{AN}^{\dagger}}$ by $j_{\mathrm{AN}^{\dagger}}$ and $j'_{\mathrm{AN}^{\dagger}}$.

For (i), consider the diagram

$$\begin{array}{ccc}
\mathrm{Cris}^{\dagger}(X,V) & \xrightarrow{\varphi_{X,V*}} & \mathrm{Mod}(i_X^{-1}\mathcal{O}_V) \\
\downarrow{\scriptstyle j^*_{\mathrm{AN}^{\dagger}}} & & \downarrow{\scriptstyle ]j[^*_V} \\
\mathrm{Cris}^{\dagger}(U,V) & \xrightarrow{\varphi_{U,V*}} & \mathrm{Mod}(i_U^{-1}\mathcal{O}_V) \\
\downarrow{\scriptstyle j_{\mathrm{AN}^{\dagger}*}} & & \downarrow{\scriptstyle ]j[_{V,*}} \\
\mathrm{Cris}^{\dagger}(X,V) & \xrightarrow{\varphi_{X,V*}} & \mathrm{Mod}(i_X^{-1}\mathcal{O}_V)
\end{array}.$$

Since $E$ is a crystal the top square commutes, and the bottom square always commutes. Thus

$$(j_{\mathrm{AN}^{\dagger}*}j^*_{\mathrm{AN}^{\dagger}}E)_{X,V} \cong ]j[_*]j[^{\dagger}E_{X,V}$$

and one can check, using the explicit descriptions of all relevant morphisms of topoi given in [lS09, Section 2.3], that under this isomorphism the realization of the adjunction is the adjunction; i.e., the composition

$$E_{X,V} \to (j_{\mathrm{AN}^{\dagger}*}j^*_{\mathrm{AN}^{\dagger}}E)_{X,V} \cong ]j[_*]j[^{\dagger}E_{X,V}.$$

is the adjunction morphism (alternatively this follows from the commutative diagram of the proof of [lS09, 3.2.1]). Finally, since the realization functor $\phi_{X,V*}$ is exact we conclude that

$$\left(\underline{\Gamma}^{\dagger}_{Z}E\right)_{X,V} = \left(\ker\left(E \to j_{\mathrm{AN}^{\dagger}*}j^*_{\mathrm{AN}^{\dagger}}E\right)\right)_{X,V} \cong \ker\left(E \to j_{\mathrm{AN}^{\dagger}*}j^*_{\mathrm{AN}^{\dagger}}E\right)_{X,V}$$

and by the above isomorphism this is $\ker\left(E_{X,V} \to ]j[_*]j[^{\dagger}E_{X,V}\right)$, proving the first claim.

Claim (ii) follows from (i) since exactness of

$$0 \to i_!i^{-1}E_{X,V} \to E_{X,V} \to ]j[_*]j[^{-1}E_{X,V}$$

can be checked on stalks, where it is clear. (Alternatively, this is a special case of the example of Remark 4.1.1 and Proposition 4.1.2).

Finally, by applying part (ii) twice, part (iii) amounts to showing that the natural map

$$u^*\left(\underline{\Gamma}^{\dagger}_{Z}E\right)_{X,V} \cong u^*i_!i^*E_{X,V} \to i'_!i'^*u^*E_{X,V} \cong \left(\underline{\Gamma}^{\dagger}_{Z}E\right)_{X',V'}$$

induced by the isomorphism $u^*E_{X,V} \cong E_{X',V'}$ (where $i$ is the inclusion of the complement $i\colon W \subset ]X[_V$ of $]U[$ and $i'$ is the inclusion of the complement $i'\colon W' \subset ]X'[_{V'}$ of $]U'[$) is an isomorphism, which can be checked on stalks, where again it is clear.

□

*Remark* 4.2.6. It is easy to see from the isomorphism $(\underline{\Gamma}^\dagger_Z E)_{X,V} \cong i'_! i'^{-1} E_{X,V}$ of Proposition 4.2.5 (ii) that $(\underline{\Gamma}^\dagger_Z E)_{X,V}$ is generally locally finitely presented and thus $\underline{\Gamma}^\dagger_Z E$ is not locally finitely presented. This will make the comparison Theorem 4.2.10 more subtle, since we won't be able to apply Theorem 2.3.16.

*Remark* 4.2.7. Let $X/k$ be a stack, let $U \subset X$ be an open substack, and let $i\colon W \hookrightarrow X$ be the closed complement (as stacks) of $U$ in $X$. It is important to note that the closed complement $Z$ of $U_{\mathrm{AN}^\dagger} \subset X_{\mathrm{AN}^\dagger}$ (as topoi) is *not* $W_{\mathrm{AN}^\dagger}$. In particular, for a module $E \in \operatorname{Mod} \mathcal{O}^\dagger_X$, the module $\underline{\Gamma}^\dagger_Z E$ is not isomorphic to $i_{\mathrm{AN}^\dagger !} i^*_{\mathrm{AN}^\dagger} E$ or $i_{\mathrm{AN}^\dagger *} i^*_{\mathrm{AN}^\dagger} E$ and cannot be described in terms of $W_{\mathrm{AN}^\dagger}$.

Let $(X, V)$ be a good overconvergent variety. Recall (see Appendix 5.2 or the discussion preceeding Definition 2.3.1) that the set $V_0$ of rigid points of $V$ naturally has the structure of a rigid analytic variety and that the inclusion $V_0 \hookrightarrow V$ induces an equivalence of categories

$$\operatorname{Coh} \mathcal{O}_{V_0} \cong \operatorname{Coh} \mathcal{O}_V.$$

We also defined (see Definition 2.3.1) functors $j^\dagger_X$ (resp. $j^\dagger_{X_0}$) from $\operatorname{Mod} \mathcal{O}_V$ (resp. $\operatorname{Mod} \mathcal{O}_{V_0}$) to itself, which are isomorphic to the functors given by the formula

$$E \mapsto \varinjlim j'_* j'^{-1} E$$

where the limit is taken over all neighborhoods $j'\colon V' \subset V$ of $]X[_V$ in $V$ (resp. strict neighborhoods $j'\colon V' \subset V_0$ of $]X[_{V_0}$ in $V_0$).

We define now the rigid analogue of the functor $\underline{\Gamma}^\dagger_Z$.

**Definition 4.2.8** ([lS07, Definition 5.2.10])**.** Let $(X, V)$ be a good overconvergent variety and let $Z \hookrightarrow X$ be a closed subscheme with open scheme-theoretic complement $j\colon U \subset X$. Let $E_0 \in \operatorname{Mod} j^\dagger_{X_0} \mathcal{O}_{V_0}$. We define the subsheaf $\underline{\Gamma}^{\dagger,\mathrm{Ber}}_Z E_0$ of $E_0$ of sections supported on $Z$ as the kernel

$$\ker\left(E_0 \to j^\dagger_{U_0} E_0\right).$$

**Proposition 4.2.9.** *Let $S$ be a formal scheme over $\mathcal{V}$ and suppose that $(S_k, S_K)$ is a good overconvergent variety. Let $p\colon X \to S_k$ be an algebraic variety over $S_k$ and let $X \subset P$ be an immersion of $X$ into a formal scheme $P/S$ such that $u\colon P \to S$ is smooth in a neighborhood of $X$ and proper at $X$ (see the paragraph before Definition 2.2.20). Let $i\colon Z \hookrightarrow X$ be a closed subscheme with open scheme-theoretic complement $j\colon U \subset X$. Denote by $P_0 \hookrightarrow P_K$*

*the underlying rigid analytic variety of $P_K$. Let $E \in \operatorname{Mod}_{\mathrm{fp}} \mathcal{O}^\dagger_{(X,P)}$ and $E_0 \in \operatorname{Coh} j^\dagger_{X_0} \mathcal{O}_{P_0}$ such that there is an isomorphism*

$$\phi \colon E_{X,P} \cong i_X^{-1} E_0^{\mathrm{an}}.$$

*Then $\phi$ induces an isomorphism*

$$(\underline{\Gamma}^\dagger_Z E)_{X,P} \cong i_X^{-1} (\underline{\Gamma}^{\dagger,\mathrm{Ber}}_Z E_0)^{\mathrm{an}}.$$

*Proof.* We have a sequence of isomorphisms

$$\begin{aligned} i_X^{-1} \left(\underline{\Gamma}^{\dagger,\mathrm{Ber}}_Z E_0\right)^{\mathrm{an}} &:= i_X^{-1} \left(\ker\left(E_0 \to j^\dagger_{U_0} E_0\right)\right)^{\mathrm{an}} \\ &\cong i_X^{-1} \ker\left(E_0^{\mathrm{an}} \to \left(j^\dagger_{U_0} E_0\right)^{\mathrm{an}}\right) \\ &\cong i_X^{-1} \ker\left(E_0^{\mathrm{an}} \to j^\dagger_U E_0^{\mathrm{an}}\right) \\ &\cong \ker\left(i_X^{-1} E_0^{\mathrm{an}} \to \ i_X^{-1} j^\dagger_U E_0^{\mathrm{an}}\right) \\ &\cong \ker\left(i_X^{-1} E_0^{\mathrm{an}} \to ]j[_* ]j[^\dagger i_X^{-1} E_0^{\mathrm{an}}\right) \\ &\cong \ker\left(E_{X,P} \to ]j[_* ]j[^\dagger E_{X,P}\right) \\ &\cong \left(\underline{\Gamma}^\dagger_Z E\right)_{X,P} \end{aligned}$$

where the functors $i_X^{-1}$ and $(-)^{\mathrm{an}}$ commute with ker because they are left exact, and all other justifications (e.g., that the composition $E_0^{\mathrm{an}} \to \left(j^\dagger_{U_0} E_0\right)^{\mathrm{an}} \cong j^\dagger_U E_0^{\mathrm{an}}$ is the adjunction) follow from the explicit descriptions of each functor and isomorphism.

□

Let $(C, O)$ be an overconvergent variety and let $X$ be an algebraic variety over $C$. Recall (see Definition 2.3.4) that we defined categories $\operatorname{Strat}^\dagger$, MIC, $\mathrm{MIC}^\dagger$, and $\operatorname{Isoc}^\dagger$ and constructed natural maps

$$\operatorname{Cris}^\dagger X_V/O \cong \operatorname{Strat}^\dagger i_X^{-1} \mathcal{O}_V \to \mathrm{MIC}(X, V/O)$$

which induce an equivalence of categories

$$\operatorname{Mod}^\dagger_{\mathrm{fp}}(X_V/O) \cong \mathrm{MIC}^\dagger(X, V/O).$$

Let $(X, V)$ be an overconvergent variety and let $Z \hookrightarrow X$ be a closed subscheme with open scheme-theoretic complement $U$. Let $E \in \operatorname{Cris}^\dagger X_V/O$. Then by Proposition 4.2.5 (iii), $\underline{\Gamma}^\dagger_Z E$ is also a crystal, and so the realization $(\underline{\Gamma}^\dagger_Z E)_{X,V}$ admits a stratification. In fact, $\underline{\Gamma}^\dagger_Z E$ is a subsheaf of $E$ and the stratification of $(\underline{\Gamma}^\dagger_Z E)_{X,V}$ is the restriction of the stratification on $E_{X,V}$. On the other hand, let $E_0 \in \operatorname{Strat}^\dagger i_{X_0}^{-1} \mathcal{O}_{V_0}$. Then le Stum proves in [lS07, Corollary 6.1.4] that $\underline{\Gamma}^{\dagger,\mathrm{Ber}}_Z E_0$ is stable under the stratificaiton of $E_0$. This gives the following.

**Corollary 4.2.10.** *Under the assumptions of Proposition 4.2.9 above, let* $E \in \mathrm{Mod}_{\mathrm{fp}}\,\mathcal{O}^\dagger$ *and* $E_0 \in \mathrm{Isoc}^\dagger(X \subset \overline{X}/S)$ *be such that there is an isomorphism*

$$\phi\colon E_{X,P} \cong i_X^{-1} E_0^{\mathrm{an}}.$$

*Then the induced isomorphism*

$$(\underline{\Gamma}_Z^\dagger E)_{X,P} \cong i_X^{-1}(\underline{\Gamma}_Z^{\dagger,\mathrm{Ber}} E_0)^{\mathrm{an}}$$

*of Proposition 4.2.9 respects the stratifications (and thus the connections).*

*Proof.* This is clear from the preceding construction since everything is functorial and the stratifications on $\underline{\Gamma}_Z^\dagger E$ and $\underline{\Gamma}_Z^{\dagger,\mathrm{Ber}} E_0$ are the restrictions of the stratifications on $E$ and $E_0$, which agree by [lS09, Theorem 2.5.9].

□

**Theorem 4.2.11.** *Let the assumptions be as in Corollary 4.2.10. Let* $p_{\mathrm{AN}^\dagger}\colon \mathrm{AN}^\dagger X_{P_K}/S_K \to \mathrm{AN}^\dagger(S_k, S_K)$ *be the induced morphism of sites. Then there is a natural isomorphism*

$$\left(\mathbb{R}p_{\mathrm{rig}} \underline{\Gamma}_Z^{\dagger,\mathrm{Ber}} E_0\right)^{\mathrm{an}} \cong \left(\mathbb{R}p_{\mathrm{AN}^\dagger_*} \underline{\Gamma}_Z^\dagger E\right)_{S_k,S_K}$$

*which is compatible with the excision exact sequence of Proposition 4.2.4 (ii) (and its rigid analogue* [lS07, Proposition 6.3.9]*).*

Actually, we will use excision to deduce the isomorphism.

*Proof.* The exact sequences

$$0 \to \underline{\Gamma}_Z^{\dagger,\mathrm{Ber}} E_0 \to E \to j_{U_0}^\dagger E \to 0$$

([lS07, Lemma 5.2.9]) and

$$0 \to \underline{\Gamma}_Z^\dagger E \to E \to j_* j^* E \to 0$$

(Proposition 4.1.2 (ii)) induce a pair of morphisms of exact triangles

$$\begin{array}{ccccc}
\left(\mathbb{R}p_{\mathrm{rig}} \underline{\Gamma}_Z^{\dagger,\mathrm{Ber}} E_0\right)^{\mathrm{an}} & \longrightarrow & (\mathbb{R}p_{\mathrm{rig}} E_0)^{\mathrm{an}} & \longrightarrow & \left(\mathbb{R}p_{\mathrm{rig}} j_{U_0}^\dagger E_0\right)^{\mathrm{an}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}u_{K*} \underline{\Gamma}_Z^\dagger E_{X,P} \otimes i_X^{-1}\Omega^\bullet_{P_K/S_K} & \longrightarrow & \mathbb{R}u_{K*} E_{X,P} \otimes i_X^{-1}\Omega^\bullet_{P_K/S_K} & \longrightarrow & \mathbb{R}u_{K*}]j[_*]j[^\dagger E_{X,P} \otimes i_X^{-1}\Omega^\bullet_{P_K/S_K} \\
\uparrow & & \uparrow & & \uparrow \\
\left(\mathbb{R}p_{\mathrm{AN}^\dagger_*} \underline{\Gamma}_Z^\dagger E\right)_{S_k,S_K} & \longrightarrow & \left(\mathbb{R}p_{\mathrm{AN}^\dagger_*} E\right)_{S_k,S_K} & \longrightarrow & \left(\mathbb{R}p_{\mathrm{AN}^\dagger_*} j_* j^* E\right)_{S_k,S_K}
\end{array}$$

where the top vertical arrows are defined as in Theorem 2.3.16 and the bottom vertical arrows are defined as in 2.3.14. The vertical arrows of the middle column are isomorphisms by Theorems 2.3.16 and 2.3.14, and since the functors $j_*$, $]j[_*$, and $j^\dagger_{U_0}$ are exact, the right vertical arrows are also isomorphisms (again by Theorems 2.3.16 and 2.3.14). By the five lemma, the left column consists of quasi-isomorphisms too. The excision statement is clear since the excision long exact sequences are the long exact sequences associated to these exact triangles.

□

**Proposition 4.2.12** (Functorality)**.** *Let $(C, O)$ be an overconvergent variety. Let $f\colon X' \to X$ be a morphism of algebraic stacks over $C$, let $i\colon Z \hookrightarrow X$ be a closed substack with open stack-theoretic complement $j\colon U \subset X$, and let $Z' = Z \times_X X'$ and $U' = U \times_X X'$, where we denote the inclusions into $X'$ by $i'$ and $j'$. Let $E \in \operatorname{Mod} \mathcal{O}^\dagger_{X/O}$ be an overconvergent module. Then there is a natural map*

$$H^i_Z(X_{\mathrm{AN}^\dagger}, E) \to H^i_{Z'}(X'_{\mathrm{AN}^\dagger}, f^*_{\mathrm{AN}^\dagger} E);$$

*in particular (setting $E = \mathcal{O}^\dagger$), the assignment*

$$(Z \hookrightarrow X) \mapsto H^i_{\mathrm{rig},Z}(X_{\mathrm{AN}^\dagger})$$

*is a contravariant functor from the category of closed immersions of stacks (with morphisms cartesian diagrams) to the category of $K$-vector spaces.*

Of course, for a map $X/O \to T$, with $T$ a fibered category over $\mathrm{AN}^\dagger(C, O)$, there is a similar map of relative cohomology with supports in $Z$. Also, the same statement holds if we replace everything by its good variant.

*Proof.* This is just a translation of Proposition 4.1.7 to the overconvergent site.

□

*Remark* 4.2.13. Using the notation of Proposition 4.2.9, let $E \in \operatorname{Mod}_{\mathrm{fp}} \mathcal{O}^\dagger_{(X,P)}$ and let $E_0 \in \operatorname{Isoc}^\dagger(X \subset \overline{X}/S)$. Then the techniques used in Proposition 4.2.9 to show that $\left(\underline{\Gamma}^\dagger_Z E\right)_{X,P} \cong i_X^{-1}\left(\underline{\Gamma}^{\dagger,\mathrm{Ber}}_Z E_0\right)^{\mathrm{an}}$ also show that this functorality map agrees with the classical functorality map of Berthelot [lS07, 6.3.5] – both arise from the very general constructions of [73] and again the work is to show that the adjunctions used in Proposition 4.1.7 match up in both contexts.

We end by applying cohomological descent (Theorem 3.2.12) to prove the finiteness of rigid cohomology with support in a closed subscheme.

**Proposition 4.2.14.** *Assume Conjecture 3.0.6 is true. Let $f\colon X \to \operatorname{Spec} k$ be a separated algebraic stack of finite type over $k$ and let $Z \subset X$ be a closed substack. Then for every $i \geq 0$, $H^i_Z(\mathrm{AN}^\dagger_{\mathrm{g}} X)$ is a finite dimensional $K$-vector space.*

*Proof.* First we do the case without supports (i.e., $Z = X$). Let $p_0 \colon X' \to X$ be a projective surjection from a scheme $X'$ which is quasi-projective over $k$ (which exists by [Ols05, Theorem 1.1]), and as usual denote by $p_i \colon X'_i \to X$ the $(i+1)$-fold fiber product of $p_0$. Then by Corollary 3.2.12 and Remark 3.1.9.3 (noting that by definition $p_i^* \mathcal{O}^\dagger_{X_{\mathrm{g}}} = \mathcal{O}^\dagger_{X_{i,\mathrm{g}}}$) there is a spectral sequence

$$H^j(\mathrm{AN}^\dagger_{\mathrm{g}} X'_i) \Rightarrow H^{i+j}(\mathrm{AN}^\dagger_{\mathrm{g}} X).$$

When $X$ is an algebraic space, $X'_i$ is a scheme, so by Theorem 2.3.17 (noting that $X'_i$ is quasi-projective and thus the structure morphism is realizable), there is an isomorphism

$$H^j(\mathrm{AN}^\dagger_{\mathrm{g}} X'_i) \cong H^j_{\mathrm{rig}}(X'_i)$$

which is finite dimensional by [Ked06a, Theorem 1.2.1], so by the spectral sequence $H^i(X)$ is finite dimensional as well. Now that we know the result for an algebraic space, the case of $X$ a stack follows directly from the spectral sequence. Finally, the case with support in $Z$ follows from the excision exact sequence of Proposition 4.2.4.

□

*Remark* 4.2.15. Classically, many results only hold for the category $F\text{-Isoc}^\dagger(X \subset \overline{X})$ of isocrystals with Frobenius action (see [lS07, Definition 8.3.2]). One can define an analogue on the overconvergent site, and the same argument will show that the cohomology of an $F$-isocrystal will be finite dimensional.

# Chapter 5

# Background

In this chapter we recall basic facts about categories, topoi, analytic spaces and stacks.

## 5.1 Categorical constructions and topoi

Here we recall definitions and basic facts about categories, presheaves, sheaves, sites, topoi, localization, fibered categories, and 2-categories. We refer to [Stacks] (and its prodigious index and table of contents) for any omitted details and a more leisurely and complete discussion of these concepts, and in particular follow their convention that a left exact functor is defined to be a functor that commutes with finite limits and a right exact functor is a functor that commutes with colimits (see [Stacks, 0034]).

5.1.1. Let $C$ be a category. We denote by $\widehat{C}$ the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})$ of **presheaves** on $C$. We denote by $h\colon C \to \widehat{C}$ the Yoneda embedding which sends an object $X$ of $C$ to the presheaf $h_X := \mathrm{Hom}(-, X)$. We say that a presheaf $F \in \widehat{C}$ is **representable** if there exists an $X \in C$ and an isomorphism $h_X \to F$, and we say that $F$ is **representable by** $X$ if $F$ is isomorphic to $h_X$. The functor $h$ is fully faithful, and so when there is no confusion we will consider $C$ as a full subcategory of $\widehat{C}$; i.e., we will identify $h_X$ with the object $X$ that it represents.

Similarly, we say an object $X \in C$ **corepresents** a covariant functor $F\colon C \to \mathbf{Sets}$ if $F$ is isomorphic to the functor $Y \mapsto \mathrm{Hom}(X, Y)$.

5.1.2. Let $X \in C$ be an object. We define the **localized** (or 'comma') category $C_{/X}$ to be the category of maps $Y \to X$ whose morphisms are commuting diagrams

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array} .$$

There is a projection functor $j_X : C_{/X} \to C$ which we denote by $j$ when the context is clear.

5.1.3. Let $C \underset{R}{\overset{L}{\rightleftarrows}} D$ be a pair of functors between categories $C$ and $D$. We say that $L$ is **left adjoint** to $R$ (or equivalently that $R$ is **right adjoint** to $L$) if there is a natural isomorphism

$$\mathrm{Hom}(L(-), -) \cong \mathrm{Hom}(-, R(-))$$

of bifunctors. The natural transformation $\mathrm{id} \colon L \to L$ (resp. $\mathrm{id} \colon R \to R$) induces (via the adjunction) a functor $\mathrm{id}_C \to R \circ L$ (resp. $L \circ R \to \mathrm{id}_D$) called the **unit** (resp. **counit**) of adjunction.

**Lemma 5.1.4.** *The functor $L$ (resp. $R$) is fully faithful if and only if the unit (resp. counit) of adjunction is an isomorhpism.*

*Proof.* Let $Y \in C$. By adjunction, for any $X \in C$, the second morphism of the composition

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, R(L(Y))) \to \mathrm{Hom}(L(X), L(Y)).$$

is an isomorphism. By definition the composition is an isomorphism for all $Y$ if and only if $L$ is fully faithful, and by Yoneda's lemma, the first map is an isomorphism for all $Y$ if and only if the unit of adjunction is an isomorphism. The second claim is proved in the same way using the co-Yoneda lemma.

□

5.1.5. Let $C$ and $D$ be categories, and let $u \colon C \to D$ be a functor. Then from $u$ we can constuct a triple $\widehat{u}_!, \widehat{u}^*, \widehat{u}_*$ of functors

$$\widehat{C} \quad \begin{array}{c} \xrightarrow{\widehat{u}_!} \\ \xleftarrow{\widehat{u}^*} \\ \xrightarrow[\widehat{u}_*]{} \end{array} \quad \widehat{D}$$

with each left adjoint to the functor directly below. The functor $\widehat{u}^*$ is the easiest to define, and sends a presheaf $G \in \widehat{D}$ to the presheaf $\widehat{u}^* G := G \circ u$ on $C$ (i.e., the presheaf $X \mapsto F(u(X))$. To construct a left adjoint $\widehat{u}_!$ one first observes that for $X \in C$ one is forced by the adjunction

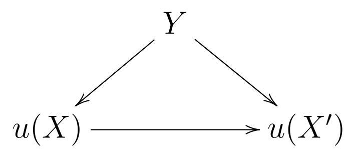
$$\mathrm{Hom}(\widehat{u}_!\, h_X, F) = \mathrm{Hom}(h_X, \widehat{u}^* F) = (\widehat{u}^* F)(X) = F(u(X))$$

to define $\widehat{u}_!(h_X) = h_{u(X)}$. Every sheaf $F \in C$ is isomorphic to a colimit of representable sheaves via the natural map $\mathrm{colim}_{h_X \to F}\, h_X \to F$, where the colimit is taken over the comma category $C_{/F}$ whose objects are maps $h_X \to F$ and whose morphisms are commuting diagrams of maps. One's hand is again forced – since a functor with a right adjoint is right exact, $\widehat{u}_!$ should commute with colimits and we are forced to define $\widehat{u}_! F$ as $\mathrm{colim}_{h_X \to F}\, h_{u(X)}$. Alternatively, a rearrangement gives the usual formula (see for instance [Stacks, 00VD])

$$Y \mapsto \mathrm{colim}_{h_X \to F}(h_{u(X)}(Y)) \cong \mathrm{colim}_{h_X \to F}\, \mathrm{colim}_{|h_{u(X)}(Y)|} * \cong$$

$$\operatorname{colim}_{X \in (I_u^Y)^{\mathrm{op}}} \operatorname{colim}_{|h_X \to F|} * \cong \operatorname{colim}_{X \in (I_u^Y)^{\mathrm{op}}} F(X),$$

where $* = \{\emptyset\}$, $|C|$ denotes the underlying set of a category $C$, $I_u^Y$ is the category whose objects are pairs $(X, Y \to u(X))$ and whose morphisms are morphisms $X \to X'$ which make the diagram




commute, and the colimit is taken in the category of sets. Later it will be important to observe that when $F(X)$ has extra algebraic structure (e.g., $F$ is a sheaf of abelian groups), we can take this colimit in a different category and construct a different left adjoint $\widehat{u}_!$. If the category $(I_u^Y)^{\mathrm{op}}$ is directed then $\widehat{u}_!$ is exact, but this does not hold in general. By construction it is left adjoint to $\widehat{u}^*$.

The functor $\widehat{u}_*$ is easier to construct – by adjunction we can define for $Y \in D$ and $F \in \widehat{C}$ value of the presheaf $\widehat{u}_* F$ on $X$ as

$$(\widehat{u}_* F)(Y) = \operatorname{Hom}(h_Y, \widehat{u}_* F) = \operatorname{Hom}(\widehat{u}^* h_Y, F);$$

and writing $\widehat{u}^* h_Y$ as a colimit of representable presheaves we deduce a description of $\widehat{u}_* F$ as the presheaf $Y \mapsto \lim_{u(X) \mapsto Y} F(X)$. Any functor with a left (resp. right) adjoint commutes with arbitrary limits (resp. colimits) when the limits exist [Stacks, 0038]. Thus, $\widehat{u}_*$ commutes with limits, and $\widehat{u}^*$ commutes with both limits and colimits .

**Example 5.1.6.** Let $X$ be a topological space, let $\operatorname{Open} X$ be the category of open subsets of $X$, and consider the inclusion $i\colon \operatorname{Open} U \hookrightarrow \operatorname{Open} X$ induced by the open inclusion of topological spaces $U \subset X$. Then the morphisms $\widehat{i}^*$ and $\widehat{i}_*$ are the usual morphisms (induced by the alternative functor $\operatorname{Open} X \to \operatorname{Open} U$ given by intersection), and $\widehat{i}_!$ is the 'extension by the empty set' functor, so that $\widehat{i}_! F$ is given by

$$U' \mapsto \begin{cases} F(U') & \text{if } U' \subset U, \\ \emptyset & \text{if } U' \not\subset U. \end{cases}$$

Finally, we note that for any category $C$, the category $\widehat{C}$ has a final object $\mathrm{e}_{\widehat{C}}$ given by the presheaf $X \mapsto \{\emptyset\}$; this is also a limit of the empty diagram. Since left exact functor $\widehat{C} \to E$ with $E$ a category must send $\mathrm{e}_{\widehat{C}}$ to a final object of $E$, we conclude that the functor $\widehat{i}_!$ is not left exact.

5.1.7. Let $I$ and $D$ be categories. For $Y \in D$, define $F_Y\colon I \to D$ to be the constant functor $i \mapsto Y$. Let $F\colon I \to D$ be a functor. We say that $X$ is a **limit** of $F$ if $X$ represents the functor $Y \mapsto \operatorname{Hom}(F_Y, F)$, and we say that $X$ is a **colimit** of $F$ if it corepresents the functor $Y \mapsto \operatorname{Hom}(F, F_Y)$. We will often refer to $F$ as a diagram.

When $D$ is the category of sets, limits and colimits exist. It follows that when $D$ is the category $\widehat{C}$ of presheaves on a category $C$, limits and colimits exist – indeed, the limit (resp. colimit) of a diagram $F\colon I \to \widehat{C}$ of presheaves is the presheaf sending $X$ in $C$ to the limit (resp. colimit) of the diagram $\mathrm{ev}_X \circ F$ (i.e., the functor given by $i \mapsto I(i)(X)$).

5.1.8. In particular, let $I$ be a category whose only morphisms are the identity morphisms, and let $\{X_i\}_{i\in I}$ be a collection of objects of $\widehat{C}$. Then the colimit of the diagram $i \mapsto X_i$, which we call the **disjoint union** of $\{X_i\}$ and denote by $\coprod_{i\in I} X_i$, exists in $\widehat{C}$. Moreover, coproducts commute with localization; i.e., if we define $\coprod C_{/X_i}$ to be the 2-categorical (see 5.1.13) fiber product $I \times_C \operatorname{Mor} C$ via the map $F\colon I \to C$, then the natural map

$$\coprod C_{/X_i} \to C_{/\coprod X_i},$$

is an equivalence of categories.

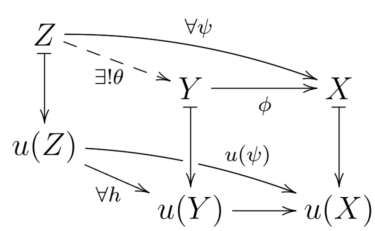
5.1.9. Let $X \in C$ be an object of a category $C$ and consider the projection morphism $j_X\colon C_{/X} \to C$ (see 5.1.2). One can make the triple of adjoint functors of 5.1.5 more explicit as follows. The collection of maps $Y \to X$ is cofinal in $(I^Y_j)^{\mathrm{op}}$, and so the functor $\widehat{j}_!$ may be concisely described as sending a presheaf $F \in \widehat{C_{/X}}$ to the presheaf

$$\widehat{j}_!F\colon Y \mapsto \coprod_{Y\to X} F(Y \to X). \tag{5.1.9.1}$$

Alternatively, the presheaf category $\widehat{C_{/X}}$ is canonically isomorphic to the localization $\widehat{C}_{/h_X}$ via the map $\widehat{C}_{/h_X} \to \widehat{C_{/X}}$ which sends $F \to h_X$ to the presheaf $(Y \to X) \mapsto \operatorname{Hom}_{h_X}(h_Y, F)$; the inverse map is $F \mapsto (\widehat{j}_!F \to h_X)$ ($\widehat{C_{/X}}$ has a final object represented by $\mathrm{id}\colon X \to X$, and the map to $h_X$ is $\widehat{j}_!$ of the map from $F$ to the final object). Via this identification the functor $\widehat{j}_!$ simply sends a presheaf $F \to h_X$ to $F$, and the map $u^*F$ sends a presheaf $F \in \widehat{C}$ to the product $h_X \times F \to h_X$ (where the map is the first projection).

For $F \in \widehat{C}$, we define the localization $C_{/F}$ similarly, by the formula $C_{/F} := C \times_{\widehat{C}} \left(\widehat{C}_{/F}\right)$.

5.1.10. Let $u\colon C \to D$ be a functor. We say that an arrow $Y \to X$ of $C$ is **cartesian** if for any $\psi : Z \to X$ and for any $h\colon u(Z) \to u(Y)$ such that $u(\psi) = u(\phi) \circ h$, there exists a unique $\theta\colon Z \to Y$ so that $\psi = \phi \circ \theta$.



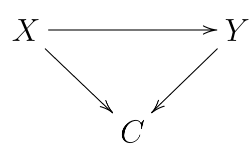
and we say that $u$ (or when the base $D$ is clear, '$C$') is a **fibered category** or a **category fibered over** $D$ if for every $X \in C$ and every arrow $Y \to u(X)$ in $D$, there exists a cartesian arrow over $Y \to u(X)$.

For $X \in D$ we define the **fiber over** $X$ to be the category $C(X) := u^{-1}(\mathrm{id}\colon X \to X)$ of all objects of $C$ which map to $X$ with morphisms which map to the identity $\mathrm{id}\colon X \to X$ under $u$. If for every $X$, the category $C(X)$ is a groupoid (i.e., a category such that every arrow is an isomorphism), then we call $C$ a **category fibered in groupoids over** $D$. In this case every arrow of $C$ is cartesian.

**Example 5.1.11.** Let $C$ be a category and let $F \in \widehat{C}$ be a presheaf. Then the comma category $j\colon C_{/F} \to C$ is a category fibered in groupoids; in fact it is fibered in sets (i.e., categories such that every arrow is the identity), and any category fibered in sets over a category $C$ is equivalent (but not necessarily isomorphic) to a fibered category $C_{/F}$ for some $F \in \widehat{C}$.

Let $C$ be a category with fiber products. Another example of a fibered category is the codomain fibration $\mathrm{Mor}\, C \to C$: objects of $\mathrm{Mor}\, C$ are morphisms of $C$ and arrows are commutative diagrams, and the map $t\colon \mathrm{Mor}\, C \to C$ sends an arrow $Y \to X$ to its target $X$. Then for $X \in C$, the comma category is equal to $(\mathrm{Mor}\, C)(X)$.

5.1.12. Categories fibered over $C$ form a 2-category, i.e., a category enriched over categories (so that $\mathrm{Hom}(X, Y)$ is not just a set, but a category). An element of $\mathrm{Mor}(\mathrm{Hom}(X, Y))$ is called a 2-morphism. Let $X, Y$ be two categories fibered over $C$. A **morphism** of categories fibered over C is a functor $F\colon X \to Y$ such that the diagram




commutes and $F$ takes cartesian arrows to cartesian arrows (if $X$ and $Y$ are fibered in groupoids, then every arrow is cartesian, so this last condition is automatic). A 2-morphism between morphisms $F, R\colon X \to Y$ is a natural transformation $t\colon F \to R$ such that for every $x \in X$, the induced map $t_x\colon F(x) \to R(x)$ in $Y$ projects to the identity morphism in $C$. One can check that when $X$ and $Y$ are fibered in groupoids, any 2-morphism is actually an isomorphism.

*Remark* 5.1.13. A main point of the use of the formalism of 2-categories is that equivalence of categories is not respected by fiber products of categories. Instead one considers 2-categorical fiber products, defined as in [Stacks, 02X9]).

5.1.14. Let $C$ be a category. We define a **pretopology** (often called a Grothendieck Topology) on $C$ to be a set $\mathrm{Cov}\, C$ of families of morphisms (which we call the coverings of $C$) such that each element of $\mathrm{Cov}\, C$ is a collection $\{X_i \to X\}_{i \in I}$ of morphisms of $C$ with a fixed target satisfying the usual axioms (see [Stacks, 00VH]):

(i) For every isomorphism $X \cong X'$, $\{X \cong X'\} \in \operatorname{Cov} C$;

(ii) Refinements of a covering by coverings form a covering;

(iii) For every $\{X_i \to X\}_{i\in I} \in \operatorname{Cov} C$ and every $Y \to X$, each of the fiber products $X_i \times_X Y$ exists and $\{X_i \times_X Y \to Y\}_{i\in I} \in \operatorname{Cov} C$.

We call a category $C$ with a pretopology $\operatorname{Cov} C$ a **site**. This generates a **topology** on $C$ in the sense of [Stacks, Definition 00Z4].

5.1.15. Let $C$ be a site whose topology is defined by a pretopology and let $F \in \widehat{C}$ be a presheaf. We say $F$ is a **sheaf** if for every covering $\{X_i \to X\}_{i\in I} \in \operatorname{Cov} C$ the diagram

$$F(X) \longrightarrow \prod_{i\in I} F(X_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(X_{i_0} \times_X X_{i_1})$$

is exact (i.e., the first arrow equalizes the rest of the diagram). We denote by $\widetilde{C}$ the category of sheaves on $C$.

The inclusion functor $i\colon \widetilde{C} \hookrightarrow \widehat{C}$ has a left adjoint, 'sheafification', which we denote by $-^a$. In particular, the inclusion $i$ commutes with limits (but not colimits!), so that the limit of a diagram of sheaves in the category of sheaves agrees with the limit considered in the category of presheaves (i.e., limits do not require sheafification in $\widetilde{C}$). We conclude that $\widetilde{C}$ has a final object $\mathrm{e}_{\widetilde{C}}$, which is the limit of the empty diagram and given (as in the case of presheaves) by the sheaf $X \mapsto \{\emptyset\}$.

5.1.16. Limits exist in $\widetilde{C}$ – indeed, given a diagram $F\colon I \to \widetilde{C}$, the limit of the diagram $i \circ F\colon I \to \widehat{C}$ is a sheaf and thus the limit of the diagram $F$. Colimits in $\widetilde{C}$ also exist – the colimit of a diagram $F$ is the *sheafication* of the diagram $i \circ F$ (an example where sheafification is required is a disjoint union of topological spaces).

5.1.17. A **topos** is a category equivalent to the category $\widetilde{C}$ of sheaves on a site $C$. A morphism $f\colon T' \to T$ of topoi is a pair $(f^*\colon T \to T', f_*\colon T' \to T)$ of functors such that $f^*$ is exact and left adjoint to $f_*$.

5.1.18. Let $C$ and $D$ be sites and let $u\colon C \to D$ be a functor. Then the functors $\widehat{u}_!, \widehat{u}^*$, and $\widehat{u}_*$ do not necessarily restrict to maps between $\widetilde{C}$ and $\widetilde{D}$ (i.e., they do not necessarily send sheaves to presheaves), and if we sheafify then they may no longer be adjoint. This motivates the following definitions.

We say that $u$ is **continuous** if $\widehat{u}^*$ of a sheaf is a sheaf, and in this case we denote the induced map $\widetilde{D} \to \widetilde{C}$ by $u^*$. If the topology on $C$ is defined by a pretopology and $u$ commutes with fiber products, then by [Stacks, 00WW], $u$ is continuous if and only if it sends coverings of $C$ to coverings of $D$. Note that we generally do not expect that $u$ commutes with *arbitrary* finite limits – consider for example an object $X \in C$ and the projection morphism $C_{/X} \to C$. If in addition $\widehat{u}_!$ is exact, we then say that $u$ is a **morphism of sites**; setting $u_! = (\widehat{u}_!)^a$ it follows that the pair $(u_!, u^*) \colon \widetilde{C} \to \widetilde{D}$ is a morphism of topoi.

Alternatively, we say that a functor $u \colon C \to D$ is **cocontinuous** if $\widehat{u}_*$ sends sheaves to sheaves, and in this case we denote the induced map $\widetilde{C} \to \widetilde{D}$ by $u_*$. The pair $(u^*, u_*) \colon \widetilde{C} \to \widetilde{D}$ is then a morphism of topoi, where $u^*$ is the sheafification $(\widehat{u}^*)^a$. If the topology on $D$ is defined by a pretopology, then by [Stacks, 00XK] $u$ is cocontinuous if and only if for every $X \in C$ and every covering $\{Y_j \to u(X)\}_{j \in J}$ of $u(X)$ in $D$ there exists a covering $\{X_i \to X\}_{i \in I}$ in $C$ such that the family of maps $\{u(X_i) \to u(X)\}_{i \in I}$ refines the covering $\{Y_j \to u(X)\}_{j \in J}$, in that the collection $\{u(X_i) \to u(X)\}_{i \in I}$ is a covering of $u(X)$ and that there is a map $\phi \colon I \to J$ such that for each $i$ there exists a factorization $u(X_i) \to Y_{\phi(i)} \to u(X)$ (note that we do not require the collections $\{u(X_i) \to Y_j\}_{\phi(i)=j}$ to be coverings).

The nicest situation is when $u \colon C \to D$ is both continuous and cocontinuous – the induced morphism $(u^*, u_*) \colon \widetilde{C} \to \widetilde{D}$ requires no sheafication and $u^*$ has a left adjoint.

5.1.19. Let $u \colon C \to D$ be a functor, and suppose that $D$ is a site. We define the **induced topology** on $C$ to be the largest topology making the map $u$ continuous. When $u$ commutes with fiber products and the topology on $D$ is defined by a pretopology, then the induced topology on $C$ is generated by the following pretopology: a collection $\{V_i \to V\}$ in $C$ is a covering if $\{u(V_i) \to u(V)\}$ is a covering in $D$.

Now suppose instead that $C$ is a site. We define the **image topology** on $D$ to be the smallest topology making the map $u$ continuous. When $u$ commutes with fiber products, the topology on $C$ is defined by a pretopology, then the image topology on $D$ is generated by the following pretopology: for every covering $\{V_i \to V\}$ in $C$, the collectin $\{u(V_i) \to u(V)\}$ is a covering in $D$ .

**Example 5.1.20.** Our main example of a cocontinuous functor is the following. Let $D$ be a site and let $u \colon C \to D$ be a fibered category such that every arrow of $C$ is cartesian, and endow $C$ with the induced topology 5.1.19. Assume further that finite limits exist in $C$ and $D$ and that the topology on $D$ is defined by a pretopology. Since $u$ is fibered in groupoids, it is an easy exercise to check that $u$ commutes with fiber products. Then it follows immediately from the definitions (using that $u$ is a fibered category) that $u$ is cocontinuous, and we get a triple of adjoints

$$\widetilde{C} \underset{u_*}{\overset{u_!}{\rightrightarrows}} \!\!\!\!\!\!\!\!\!\!\!\xleftarrow{\;u^*\;} \widetilde{D} \; .$$

We will mainly apply this when $C \cong D_{/X}$ for some $X \in D$.

5.1.21. Let $C$ be a category. We define the **canonical topology** on $C$ to be the largest topology such that representable objects are sheaves (i.e., the largest topology such that for all $x \in C$, the presheaf $h_X$ is a sheaf). We say that a topology is **subcanonical** if it is smaller than the canonical topology (in other words, for all $x \in C$, the presheaf $h_X$ is a sheaf).

**Example 5.1.22.** (i) The topology on the category of affine schemes given by jointly surjective families of flat (but not necessarily finitely presented) morphisms is subcanonical [Knu71, 3.1, 7'], and the fpqc topology is subcanonical on the category of schemes [Vis05, Theorem 2.55] (note that the flat topology is *not* subcanonical for the category of schemes).

(ii) For a site $C$ the canonical topology on $\widetilde{C}$ is given by collections $\{F_i \to F\}$ such that the map $\coprod F_i \to F$ is a surjection of sheaves [Stacks, 03A1]. The natural map $\widetilde{C} \to \widetilde{\widetilde{C}}$ is then an equivalence of categories. Thus, any topos $T$ is canonically a site.

5.1.23. For a topos $T$ we denote by $\mathrm{Ab}\, T$ the category of abelian group objects of $T$. If we view $T$ as a site with its subcanonical topology, then $\mathrm{Ab}\, T$ is equivalent to the category of sheaves of abelian groups on $T$, and when we choose a site $C$ such that $T$ is equivalent to $\widetilde{C}$, we may write $\mathrm{Ab}\, C$ instead of $\mathrm{Ab}\, T$. By [Stacks, 00YT], a morphism of $f\colon T' \to T$ topoi restricts to a pair

$$\mathrm{Ab}\, T' \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Ab}\, T$$

of adjoint functors; here the exactness of $f^*$ in the definition of a morphism of topoi is crucial (consider for example that the functor $u_!\colon \widetilde{C_{/X}} \to \widetilde{C}$ described above in 5.1.6 is not generally exact and indeed fails to send abelian sheaves to abelian sheaves).

5.1.24. Let $u\colon C \to D$ be a morphism of sites. Then $u_!$ does not necessarily take abelian sheaves to abelian sheaves. Indeed, consider the case of a localization morphism $j\colon C_{/X} \to C$ (with $X \in C$). Then for any $X' \in C$ such that $\mathrm{Hom}(X', X)$ is empty, $(j_! F)(X')$ is also empty for any abelian sheaf $F \in \mathrm{Ab}\, \widetilde{C_{/X}}$. It is nonetheless true that $u^*\colon \mathrm{Ab}\, \widetilde{D} \to \mathrm{Ab}\, \widetilde{C}$ has a left adjoint $u_!^{\mathrm{ab}}$. We will construct $u_!^{\mathrm{ab}}$ in the next few paragraphs by adapting the construction of $u_!$.

As a first step we consider a category $C$ and construct a left adjoint $\mathbb{Z}^{\mathrm{ps}}_{-}\colon C \to \mathrm{Ab}\, C$ to the forgetful functor $\mathrm{Ab}\, C \to C$. Let $F \in \widehat{C}$ be a presheaf of sets. We define the **free abelian presheaf** on $F$ to be the presheaf $X \mapsto \bigoplus_{s \in F(X)} \mathbb{Z}$. It follows directly from this explicit formula that this is the desired left adjoint and, moreover, that the functor $F \mapsto \mathbb{Z}^{\mathrm{ps}}_F$ commutes with limits; since it has a right adjoint it also commutes with colimits and is thus exact. When $F = h_X$ for some $X \in C$, we will instead write $\mathbb{Z}^{\mathrm{ps}}_X$.

Now, suppose that $C$ is a site. Since sheafification is left adjoint to the inclusion $\widetilde{C} \hookrightarrow \widehat{C}$, the functor $\mathbb{Z}_{-}\colon \widetilde{C} \to \mathrm{Ab}\, \widetilde{C}$ given by $F \mapsto (\mathbb{Z}_F)^a$ is left adjoint to $\mathrm{Ab}\, \widetilde{C} \to \widetilde{C}$. Furthermore, since sheafification is exact, the functor $\mathbb{Z}_{-}$ also commutes with limits and colimits. When $F = (h_X)^a$ for some $X \in C$, we will instead write $\mathbb{Z}_X$.

Now we can construct $u_!^{\mathrm{ab}}$ as following the template of 5.1.5. Let $A \in \mathrm{Ab}\, C$ be a sheaf of abelian groups and let $U \in C$. Then since

$$A(U) = \mathrm{Hom}_{\widetilde{C}}(h_U, A) = \mathrm{Hom}_{\mathrm{Ab}\,\widetilde{C}}(\mathbb{Z}_U, A),$$

and since $u_!^{\mathrm{ab}}$ commutes with colimits, it follows that

$$u_!^{\mathrm{ab}} A = u_!^{\mathrm{ab}} \operatorname{colim}_{(h_U \to A) \in \widetilde{C}_{/A}} h_{h_U \to A} = \operatorname{colim}_{(h_U \to A) \in \widetilde{C}_{/A}} \mathbb{Z}_{h_U \to A}.$$

As in the case of $u_!$ for sheaves of sets, by adjunction we must have $u_!^{\mathrm{ab}} \mathbb{Z}_{h_U \to A} = \mathbb{Z}_U$, and since $u_!^{\mathrm{ab}}$ must commute with colimits we get the formula

$$u_!^{\mathrm{ab}} A = \operatorname{colim}_{(h_u \to A) \in \widetilde{C}_{/A}} \mathbb{Z}_U.$$

As before (see Equation 5.1.9.1) we get a nice formula when $u = j_X : C_{/X} \to C$ is the projection morphism associated to some object $X$ of a site $C$ (see 5.1.2); $u_!^{\mathrm{ab}} A$ is the sheafication of the presheaf

$$\widehat{j}_! A \colon Y \mapsto \bigoplus_{Y \to X} A(Y \to X). \qquad (5.1.24.1)$$

In this special case it follows from this explicit formula that $u_!^{\mathrm{ab}}$ left exact; moreover it commutes with colimits since it has a right adjoint $u^*$. Consequently, by an easy exercise we get the useful bonus that $u^*$ takes injective abelian sheaves to injective abelian sheaves.

Note that this disagrees with the functor 'extension by the empty set' $u_!$; nonetheless when there is no confusion we will write $u_!^{\mathrm{ab}}$ as $u_!$ (and if there is confusion we will refer to them by $u_!^{\mathrm{ab}}$ and $u_!^{\mathrm{set}}$).

5.1.25. A **ringed topos** is a pair $(T, \mathcal{O}_T)$ with $T$ a topos and $\mathcal{O}_T$ a ring object of $T$. Equivalently, $\mathcal{O}_T$ is a sheaf of rings on $T$, where we consider $T$ as a site with its canonical topology (see Definition 5.1.21). A morphism $f \colon (T', \mathcal{O}_{T'}) \to (T, \mathcal{O}_T)$ of ringed topoi is a morphism $f \colon T' \to T$ of topoi and a map $\mathcal{O}_T \to f_* \mathcal{O}_{T'}$. Sometimes we will write $(f^*, f_*)$ instead of $f$. Similarly, a **ringed site** is a site $C$ together with a ring object $\mathcal{O}_C$ of its topos $\widetilde{C}$, and a morphism $(C', \mathcal{O}_{C'}) \to (C, \mathcal{O}_C)$ of ringed sites is a continuous morphism $f \colon C' \to C$ of sites and a map $\mathcal{O}_C \to f_* \mathcal{O}_{C'}$.

5.1.26. Let $(T, \mathcal{O}_T)$ be a ringed topos. Then we can consider the category $\mathrm{Mod}\, \mathcal{O}_T$ of $\mathcal{O}_T$-modules (i.e., the category of abelian group objects of $T$ which admit the structure of a module object over the ring object $\mathcal{O}_T$ of $T$). Considering $T$ with its canonical topology, an $\mathcal{O}_T$-module is the same as a sheaf of $\mathcal{O}_T$-modules.

We say that $M \in \mathrm{Mod}\, \mathcal{O}_T$ is **quasi-coherent** (resp. **locally finitely presented**) if there exists a covering $F \to \mathrm{e}_T$ of the final object $e_T$ of $T$ (i.e., a covering in the canonical topology on $T$) such that, denoting by $j \colon T_{/F} \to T$ the localization with respect to $F$ and setting $\mathcal{O}_{T_{/F}} = j^*\mathcal{O}_T$, the pullback $j^*M$ admits a presentation (resp. finite presentation) – i.e., $j^*M$ is the cokernel of a map $\bigoplus_I \mathcal{O}_{T_{/F}} \to \bigoplus_J \mathcal{O}_{T_{/F}}$ of $\mathcal{O}_{T_{/F}}$-modules (resp. a map with $I$ and $J$ finite sets). If $C$ is a site such that $T$ is equivalent to $\widetilde{C}$ (which may have no final object) and the topology on $C$ is defined by a pretopology, then it is equivalent to ask that for all $X \in C$, there exists a covering $\{X_i \to X\}$ such that for each $i$ there exists a presentation (resp. finite presentation) of the restriction of $M$ to $T_{(h_{X_i})^a}$. We denote by $\operatorname{QCoh}\mathcal{O}_T$ (resp. $\operatorname{Mod}_{\mathrm{fp}}\mathcal{O}_T$) the subcategories of quasi-coherent (resp. locally finitely presented) $\mathcal{O}_T$-modules.

5.1.27. Let $(T, \mathcal{O})$ be a ringed topos, with $T = \widetilde{C}$ for some site $C$. As in the abelian case, the forgetful functor $\operatorname{Mod}\mathcal{O} \to T$ has a left adjoint $\mathcal{O}_{-}\colon T \to \operatorname{Mod}\mathcal{O}$. When $T$ is equivalent to $\widetilde{C}$ for some site $C$ and $\mathcal{O}$ is a sheaf on $C$, then for $F \in T$, $\mathcal{O}_F$ is defined in the same manner as $\mathbb{Z}_F$: $\mathcal{O}_F$ is the sheafification of the presheaf $X \mapsto \bigoplus_{s\in F(X)} \mathcal{O}(X)$. As usual, when $F = (h_U)^a$ for $U \in C$ we denote $\mathcal{O}_F$ by $\mathcal{O}_U$.

Let $u\colon (C, \mathcal{O}_C) \to (D, \mathcal{O}_D)$ be a morphism of ringed sites. Then the above template for the construction of $j_!^{\mathrm{ab}}$ admits a verbatim translation (replacing the free functor $\mathbb{Z}_{-}$ by the free functor $\mathcal{O}_{-}$) and allows one to construct a left adjoint $u_!^{\mathrm{Mod}}\colon \operatorname{Mod}\mathcal{O}_C \to \operatorname{Mod}\mathcal{O}_D$ to the functor $u^*(-) \otimes_{u^*\mathcal{O}_D} \mathcal{O}_C\colon \operatorname{Mod}\mathcal{O}_D \to \operatorname{Mod}\mathcal{O}_C$. When $u = j_X\colon C_{/X} \to C$ is the projection morphism associated to some object $X$ of a site $C$ and $C_{/X}$ has the induced topology, $u_!^{\mathrm{Mod}}$ is even defined by the same formula 5.1.24.1 as $u_!^{\mathrm{ab}}$. Again we will denote $u_!^{\mathrm{Mod}}$ by $u_!$.

## 5.2 Analytic spaces

Here we recall Berkovich's notion of an analytic space. We only make use of *strictly* analytic spaces (i.e., analytic spaces locally defined by strict affinoid algebras as in 5.2.1 below). The original article is [Ber90], but the main refence is [Ber93] which deals with more general spaces than the former. Brian Conrad's notes [Con08] also give a nice introduction to the subject (see also Berkovich's introduction [Ber08] to that volume for a pleasant and personal historical overview), and le Stum's appendix [lS09, 4.2] gives a quick review of everything used in this thesis.

Throughout, $K$ will denote a field of characteristic 0 that is complete with respect to a non-trivial non-archimedean valuation (which takes values in $\mathbb{R}$) with valuation ring $\mathcal{V}$, whose maximal ideal and residue field we denote by $\mathfrak{m}$ and $k$. (Of course one charm of the theory is that these constructions make sense for $\operatorname{char} K > 0$ and for trivial valuations.)

5.2.1. An **affinoid algebra** (or in Berkovich's notation a *strict* affinoid algebra) $A$ over $K$ is a quotient of a Tate algebra $K\{T_1, \cdots, T_n\}$ (the algebra of convergent formal power series). Denote by $\mathrm{M}(A)$ the set $\operatorname{Max Spec} A$ of maximal ideals of $A$, endowed with the Tate topology and ringed by a sheaf $\mathcal{O}_{\mathrm{M}(A)}$ which sends an admissible open $\mathrm{M}(B)$ to the ring $B$ (see [BGR84]). We call the pair $(\mathrm{M}(A), \mathcal{O}_{\mathrm{M}(A)})$ a **rigid affinoid variety**. Next we define a **rigid analytic space** to be a pair $(X, \mathcal{O}_X)$ where $X$ is a set with a Grothendieck topology (i.e., on its power set) and $\mathcal{O}_X$ is a sheaf of local rings such that locally $X$ is isomorphic to a rigid affinoid variety.

Alternatively, we denote by $\mathcal{M}(A)$ the **Gelfand spectrum** of bounded multiplicative semi-norms $|\cdot|\colon \mathbb{R}$ on $A/K$ and define an **affinoid variety** over $K$ to be a triple $(V, A; \mathcal{M}(A))$, where $V$ is a topological space, $A$ is an affinoid algebra, and $V \cong \mathcal{M}(A)$ is a homeomorphism. We define an **affinoid subdomain** $W \subset V$ of an affinoid variety $V \cong \mathcal{M}(A)$ to be a subset such that the functor

$$C \mapsto \{u\colon \mathcal{M}(C) \to V,\, u \text{ is induced by a map } A \to C \text{ such that } \operatorname{im}(u) \subset W\}$$

is represented by an affinoid algebra $A \to B$; one can then prove that $\mathcal{M}(B) \cong W$.

5.2.2. Glueing affinoid varieties along affinoid subdomains is more subtle than in the rigid case because an affinoid subdomain is generally not open. It is nonetheless necessary to allow such glueing in order to associate to a quasi-compact quasi-separated rigid space an analytic space. This inspires the following definitions.

A **quasi-net** $\tau$ on a topological space $V$ is a set $\tau$ of subsets of $V$ such that each point $x \in V$ has a neighborhood which is a finite union of elements of $\tau$ containing $x$. We say that a subset $W \subset V$ is $\tau$-**admissible** if $\tau_W := \{W' \in \tau \text{ s.t. } W' \subset W\}$ is a quasi-net on $W$ and we say that $\tau$ is a **net** if any finite intersection of elements of $\tau$ is $\tau$-admissible. A set-theoretic covering of a $\tau$-admissible subset $W$ by $\tau$-admissible subsets is said to be $\tau$-**admissible** if it defines a quasi-net on $W$. When $\tau$ is a net, the $\tau$-admissible coverings form a pre-topology [lS09, Section A.2] which is generally not a sub-topology of the given topology on $V$ (since a typical $W \in \tau$ may be closed).

An **affinoid atlas** on a locally Hausdorff topological space $V$ is a net $\tau$ consisting of affinoid varieties; i.e., each $W \in \tau$ is isomorphic as a topological space to an affinoid variety and any inclusion $W \subset W'$ of $\tau$ is induced by an inclusion of an affinoid subdomain. We define an **analytic variety** over $K$ to be a locally Hausdorff topological space with a maximal atlas $\tau$ and refer to any $W \in \tau$ as an **analytic domain** (or sometimes analytic subdomain). When we consider $V$ as a topological space we will still write $V$ and when we consider it as a site we write $V_G$ and refer to the topology on $V_G$ as the $G$-topology. An affinoid variety, together with $\tau$ equal to the collection of affinoid subdomains, is an example of an analytic variety and highlights the fact that an analytic domain is not necessarily open topologically.

5.2.3. The identity map $\pi\colon V_G \to V$ is a morphism of sites. The map sending $W \in \tau$ to $\mathcal{O}(W) := A$, where $W \cong \mathcal{M}(A)$, defines a sheaf of rings $\mathcal{O}_{V_G}$ on $V_G$. We endow $V$ with the sheaf of rings $\mathcal{O}_V := \pi_*\mathcal{O}_{V_G}$, making $\pi$ into a morphism of ringed sites. For $F \in \operatorname{Mod}\mathcal{O}_V$ we write $F_G$ for $\pi^*F$.

5.2.4. For a point $x \in V$, the local ring $\mathcal{O}_{V,x}$ has a semi-absolute value and we define a **morphism** $u\colon V \to V'$ **of analytic varieties** to be a morphism $V_G \to V'_G$ of valued locally ringed spaces. *A posteriori* the underlying morphism $V \to V'$ is continuous and induces a morphism $(V, \mathcal{O}_V) \to (V', \mathcal{O}_{V'})$ of ringed spaces.

5.2.5. The functor $\widetilde{V} \to \widetilde{V_G}$, $F \mapsto \pi^{-1}F$ is fully faithful. We say that $V$ is **good** if it has a basis of affinoid neighborhoods; note that the analytification of a rigid space may fail to be good. When $V$ is good, the functor $F \mapsto F_G$ is fully faithful and induces an equivalence of categories $\mathrm{Coh}\,\mathcal{O}_V \cong \mathrm{Coh}\,\mathcal{O}_{V_G}$ [Ber93, 1.3.4]; the functor $F \mapsto \pi_* F$ is in general not fully faithful [Ber93, Example 1.4.8]. For $F \in \mathrm{Ab}\,\widetilde{V}$ and for $p \geq 0$ the natural map $H^p(V, F) \to H^p(V_G, \pi^{-1}F)$ is an isomorphism, and if $V$ is good than the same is true for $F \in \mathrm{Mod}\,\mathcal{O}_V$ and $H^p(V, F) \to H^p(V_G, F_G)$.

A point $x \in V$ of an analytic variety $V$ is called a **rigid point** if the residue field $K(x)$ is a finite extension of $K$. We denote the set of rigid points by $V_0$. When $V$ is Hausdorff, $V_0$ is dense in $V$ and admits the structure of a rigid analytic variety. The inclusion $V_0 \hookrightarrow V$ then induces a bijection between affinoid open subsets (resp. coverings) and affinoid subdomains (resp. coverings) and thus induces an isomorphism $\widetilde{V_G} \cong \widetilde{V}_0$ which takes $\mathcal{O}_{V_G}$ to $\mathcal{O}_{V_0}$, thus inducing equivalences of categories $\mathrm{Mod}\,\mathcal{O}_{V_G} \cong \mathrm{Mod}\,\mathcal{O}_{V_0}$ and $\mathrm{Coh}\,\mathcal{O}_{V_G} \cong \mathrm{Coh}\,\mathcal{O}_{V_0}$.

5.2.6. We will make extensive use of the following **generic fiber** construction. Let $P$ be a locally topologically finitely presented formal scheme (as in Section 2.1). Then one can construct (see [Ber94, Section 1]) a $K$-analytic space $P_K$ together with an anti-continuous map $\mathrm{sp}\colon P_K \to P_k$ (i.e., the pre-image of a closed subset is open). When $P = \mathrm{Spf}\,A$ is an affine formal scheme, the analytic space $P_K$ is given by $\mathcal{M}(A \otimes_{\mathcal{V}} K)$. The specialization map sp is defined as follows: the residue field $K(x)$ of a multiplicative semi-norm $x$ has a valuation, and the point $\mathrm{sp}(x)$ is the prime ideal which is the kernel of the induced map $A/\mathfrak{m} \to k(x)$, where $k(x)$ is the kernel of the valuation ring of $K(x)$ by its residue field. One of course must check that this is well defined and has expected properties, and for general $P$ one must glue this construction.

There is an analogous functor taking $P$ to a rigid space $(P_K)_0$, which is defined similarly (locally it looks like $\mathrm{M}(A)$). When $P_K$ is good then $(P_K)_0$ is isomorphic to the underlying rigid variety of $P_K$. See [BL93] for more details.

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