Progress on Mazur’s Program B

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Gratuitous picture – subgroups of $\text{GL}_2(\mathbb{Z}_2)$
Background - Image of Galois

\[ G_\mathbb{Q} := \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}) \]

\[ E[n](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/n\mathbb{Z})^2 \]

\[ \rho_{E,n} : G_\mathbb{Q} \rightarrow \text{Aut} E[n] \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \]

\[ \rho_{E,\ell\infty} : G_\mathbb{Q} \rightarrow \text{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \]

\[ \rho_E : G_\mathbb{Q} \rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim_n \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \]
Background - Galois Representations

\[ \rho_{E,n} : G_\mathbb{Q} \to H(n) \hookrightarrow GL_2(\mathbb{Z}/n\mathbb{Z}) \]

\[ G_\mathbb{Q} \twoheadrightarrow \overline{\mathbb{Q}} \to \mathbb{Q}(E[n]) \hookrightarrow H(n) \]

Problem (Mazur’s “program B”)

Classify all possibilities for \( H(n) \).
Example - torsion on an elliptic curve

If $E$ has a $K$-rational **torsion point** $P \in E(K)[n]$ (of exact order $n$) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\sigma(P) = P$$

$$\sigma(Q) = a_\sigma P + b_\sigma Q$$
Example - Isogenies

If $E$ has a $K$-rational, cyclic isogeny $\phi: E \to E'$ with $\ker \phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(K)[n]$ such that $E(K)[n] \cong \langle P, Q \rangle$,

$$\sigma(P) = a_\sigma P$$
$$\sigma(Q) = b_\sigma P + c_\sigma Q$$
Example - other maximal subgroups

Normalizer of a split Cartan:

\[ N_{sp} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \]

\( H(n) \subset N_{sp} \) and \( H(n) \not\subset C_{sp} \) iff

- there exists an unordered pair \( \{\phi_1, \phi_2\} \) of cyclic isogenies,
- neither of which is defined over \( K \)
- but which are both defined over some quadratic extension of \( K \)
- and which are Galois conjugate.
Sample subgroup (Serre)

\[ \ker \phi_2 \subset H(8) \subset \text{GL}_2(\mathbb{Z}/8\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_2 = 3 \]

\[ I + 2M_2(\mathbb{Z}/2\mathbb{Z}) \subset H(4) = \text{GL}_2(\mathbb{Z}/4\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_1 = 4 \]

\[ H(2) = \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \]

\[ \chi : \text{GL}_2(\mathbb{Z}/8\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \rightarrow \mathbb{F}_2 \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3. \]

\[ \chi = \text{sgn} \times \text{det} \]

\[ H(8) := \chi^{-1}(G), \ G \subset \mathbb{F}_2^3. \]
\[ \langle I + 2E_{1,1}, I + 2E_{2,2} \rangle \subset H(4) \subset \text{GL}_2(\mathbb{Z}/4\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_1 = 2 \]

\[ H(2) = \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \]

\[ H(2) = \left\langle \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \cong \mathbb{F}_3 \rtimes D_8. \]

\[ \text{im } \rho_{E,4} \subset H(4) \iff j(E) = -4t^3(t + 8). \]

\[ X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1). \]
A typical subgroup

\[
\begin{align*}
\ker \phi_4 & \subset H(32) \subset \text{GL}_2(\mathbb{Z}/32\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_2 = 4 \\
\ker \phi_3 & \subset H(16) \subset \text{GL}_2(\mathbb{Z}/16\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_2 = 3 \\
\ker \phi_2 & \subset H(8) \subset \text{GL}_2(\mathbb{Z}/8\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_2 = 2 \\
\ker \phi_1 & \subset H(4) \subset \text{GL}_2(\mathbb{Z}/4\mathbb{Z}) \quad \text{dim}_{\mathbb{F}_2} \ker \phi_2 = 3 \\
H(2) & = \text{GL}_2(\mathbb{Z}/2\mathbb{Z})
\end{align*}
\]
There exists a surjection \( \theta : \text{GL}_2(\mathbb{Z}/3\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \).

\[
\begin{align*}
H(6) := & \Gamma_\theta \\
\cap & \\
\text{GL}_2(\mathbb{Z}/6\mathbb{Z}) & \\
\text{GL}_2(\mathbb{Z}/2\mathbb{Z}) & \rightarrow \leftarrow \\
\text{GL}_2(\mathbb{Z}/3\mathbb{Z}) & \\
\end{align*}
\]

\[
\text{im} \rho_{E,6} \subset H(6) \Leftrightarrow K(E[2]) \subset K(E[3])
\]
Theorem

Let $E$ be an elliptic curve over $\mathbb{Q}$. Then for $\ell > 11$, $E(\mathbb{Q})[\ell] = \{0\}$.

In other words, for $\ell > 11$ the mod $\ell$ image is not contained in a subgroup conjugate to

$$
\begin{pmatrix}
1 & * \\
0 & *
\end{pmatrix}.
$$
Theorem (Mazur)

Let $E$ be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$ the mod $\ell$ image is not contained in a subgroup conjugate to

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$ 

Theorem (Bilu, Parent)

Let $E$ be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 13$ the mod $\ell$ image is not contained in a subgroup conjugate to

$$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$
Main conjecture

Conjecture

Let $E$ be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.
Serre’s Open Image Theorem

Theorem (Serre, 1972)

Let $E$ be an elliptic curve over $K$ without CM. The image of $\rho_E$

$$\rho_E(G_K) \subset \text{GL}_2(\hat{\mathbb{Z}})$$

is open.

Note:

$$\text{GL}_2(\hat{\mathbb{Z}}) \cong \prod_p \text{GL}_2(\mathbb{Z}_p)$$
"Vertical" image conjecture

Conjecture
There exists a constant $N$ such that for every $E/\mathbb{Q}$ without CM

$$\left[ \rho_E(G_{\mathbb{Q}}) : \text{GL}_2(\hat{\mathbb{Z}}) \right] \leq N.$$ 

Remark
This follows from the "$\ell > 37$" conjecture.

Problem
Assume the "$\ell > 37$" conjecture and compute $N$. 

Main Theorems

Rouse, ZB (2-adic)
The index of $\rho_{E,2^\infty}(G_Q)$ divides 64 or 96; all such indices occur.

Zywina (mod $\ell$)
Classifies $\rho_{E,\ell}(G_Q)$ (modulo some conjectures).

Zywina (all possible indices)
The index of $\rho_{E,N}(G_Q)$ divides 220, 336, 360, 504, 864, 1152, 1200, 1296 or 1536.

Morrow (composite level)
Classifies $\rho_{E,2}\cdot\ell(G_Q)$.

Camacho–Li–Morrow–Petok–ZB (composite level)
Classifies $\rho_{E,\ell_1\cdot\ell_2}(G_Q)$ (partially).
Main Theorems continued

Zywina–Sutherland (stay tuned!)
Parametrizations in all **prime power** level, $g = 0$ and $g = 1, r > 0$ cases.

Gonzalez–Jimenez, Lozano–Robledo
Classify $E / \mathbb{Q}$ with $\rho_{E,n}(G_{\mathbb{Q}})$ abelian.

Brau–Jones, Jones–McMurdy (in progress)
Equations for $X_H$ for entanglement groups $H$.

Rouse–ZB for other primes (tonite’s problem session)
Partial progress; e.g. for $N = 3^n$.

Derickx–Etropolski–Morrow–van Hoejk–ZB (in progress)
Classify possibilities for cubic torsion.
Some applications and complements

**Theorem (R. Jones, Rouse, ZB)**

1. **Arithmetic dynamics**: let $P \in E(\mathbb{Q})$.
2. How often is the order of $\tilde{P} \in E(\mathbb{F}_p)$ odd?
3. Answer depends on $\rho_{E,2\infty}(G_\mathbb{Q})$.
4. Examples: $11/21$ (generic), $121/168$ (maximal), $1/28$ (minimal)

**Theorem (Various authors)**

Computation of $S_\mathbb{Q}(d)$ and $S(d)$ for particular $d$.

**Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)**

Classification of $E(\mathbb{Q}(3\infty))_{\text{tors}}$
More applications

Theorem (Sporadic points)

Najman’s example $X_1(21)^{(3)}(\mathbb{Q})$; “easy production” of other examples.

Theorem (Jack Thorne)

Elliptic curves over $\mathbb{Q}_\infty$ are modular.
(One step is to show $X_0(15)(\mathbb{Q}_\infty) = X_0(15)(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.)

Theorem (Zywina)

Constants in the Lang–Trotter conjecture.
<table>
<thead>
<tr>
<th>Index</th>
<th># of isogeny classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>727995</td>
</tr>
<tr>
<td>2</td>
<td>7281</td>
</tr>
<tr>
<td>3</td>
<td>175042</td>
</tr>
<tr>
<td>4</td>
<td>1769</td>
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<tr>
<td>6</td>
<td>57500</td>
</tr>
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<td>8</td>
<td>577</td>
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<tr>
<td>12</td>
<td>29900</td>
</tr>
<tr>
<td>16</td>
<td>235</td>
</tr>
<tr>
<td>24</td>
<td>5482</td>
</tr>
<tr>
<td>32</td>
<td>20</td>
</tr>
<tr>
<td>48</td>
<td>1544</td>
</tr>
<tr>
<td>64</td>
<td>0 (two examples)</td>
</tr>
<tr>
<td>96</td>
<td>241 (first example - (X_0(15)))</td>
</tr>
<tr>
<td>CM</td>
<td>1613</td>
</tr>
</tbody>
</table>
Index, # of isogeny classes

\[ j = -3 \cdot 2^{18} \cdot 5 \cdot 13^3 \cdot 41^3 \cdot 107^3 \cdot 17^{-16} \]

\[ j = -2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 \cdot 79^{-16} \]

Rational points on \( X_{ns}^+(16) \) (Heegner, Baran)
Fun 2-adic facts

1. All indicies dividing 96 occur infinitely often; 64 occurs only twice.
2. The 2-adic image is determined by the mod 32 image.
3. 1208 different images can occur for non-CM elliptic curves.
4. There are 8 “sporadic” subgroups.
If $E/\mathbb{Q}$ is a non-CM elliptic curve whose mod 2 image has index
- 1, the 2-adic image can have index as large as 64.
- 2, the 2-adic image has index 2 or 4.
- 3, the 2-adic image can have index as large as 96.
- 6, the 2-adic image can have index as large as 96;
- (although some quadratic twist of $E$ must have 2-adic image with index less than 96).
Modular curves

**Definition**

1. $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q \rangle \} \cup \{\text{cusps}\}$
2. $X(N)(K) \ni (E/K, P, Q) \iff \rho_{E,N}(G_K) = \{I\}$

**Definition**

- $\Gamma(N) \subset H \subset \text{GL}_2(\hat{\mathbb{Z}})$ (finite index)
- $X_H := X(N)/H$
- $X_H(K) \ni (E/K, \iota) \iff H(N) \subset H \mod N$

**Stacky disclaimer**

This is only true up to twist; there are some subtleties if

1. $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
2. if $-I \in H$. 

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Mazur’s program B

Compute $X_H(\mathbb{Q})$ for all $H$.

Remark

- Sometimes $X_H \cong \mathbb{P}^1$ or elliptic with rank $X_H(\mathbb{Q}) > 0$.
- Some $X_H$ have sporadic points.
- Can compute $g(X_H)$ group theoretically (via Riemann–Hurwitz).

Fact

$g(X_H), \gamma(X_H) \to \infty$ as $\left[ H : \text{GL}_2(\hat{\mathbb{Z}}) \right] \to \infty$. 

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Minimality

Definition

- \( H \subset H' \iff X_H \to X_{H'} \)
- Say that \( H \) is **minimal** if
  1. \( g(X_H) > 1 \) and
  2. \( H \subset H' \iff g(X_{H'}) \leq 1 \)
- Every modular curve maps to a minimal or genus \( \leq 1 \) curve.

Definition

We say that \( H \) is **arithmetically minimal** if

1. \( \det(H) = \hat{\mathbb{Z}}^* \), and
2. a few other conditions.
1. Compute all arithmetically minimal $H \subset \text{GL}_2(\mathbb{Z}_2)$
2. Compute equations for each $X_H$
3. Find (with proof) all rational points on each $X_H$. 
Gratuitous picture – subgroups of $\text{GL}_2(\mathbb{Z}_2)$
Gratuitous picture – subgroups of $\GL_2(\mathbb{Z}_3)$
Gratuitous picture – subgroups of $\text{GL}_2(\mathbb{Z}_{11})$
318 curves $X_H$ with $-I \in H$ (excluding pointless conics)

<table>
<thead>
<tr>
<th>Genus</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>175</td>
<td>52</td>
<td>57</td>
<td>18</td>
<td>20</td>
<td>4</td>
</tr>
</tbody>
</table>
1. The canonical map \( C \hookrightarrow \mathbb{P}^{g-1} \) is given by \( P \mapsto [\omega_1(P) : \cdots : \omega_g(P)] \).
2. For a general curve, this is an embedding, and the relations are quadratic.
3. For a modular curve,

\[
M_k(H) \cong H^0(X_H, \Omega^1(\Delta) \otimes k/2)
\]

given by

\[
f(z) \mapsto f(z) \, dz \otimes k/2.
\]
Equations – Example: $X_1(17) \subset \mathbb{P}^4$

\[
q - 11q^5 + 10q^7 + O(q^8) \\
q^2 - 7q^5 + 6q^7 + O(q^8) \\
q^3 - 4q^5 + 2q^7 + O(q^8) \\
q^4 - 2q^5 + O(q^8) \\
q^6 - 3q^7 + O(q^8)
\]

\[
xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 = 0 \\
xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv = 0 \\
2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv = 0
\]
1. $H' \subset H$ of index 2, $X_{H'} \to X_H$ degree 2.
2. Given equations for $X_H$, compute equations for $X_{H'}$.
3. Compute a new modular form on $H'$, compute (quadratic) relations between this and modular forms on $H$.
4. **Main technique** – if $X_{H'}$ has “new cusps”, then write down Eisenstein series which vanish at “one new cusp, not others”.
Rational points rundown, $\ell = 2$

318 curves (excluding pointless conics)

<table>
<thead>
<tr>
<th>Genus</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>175</td>
<td>52</td>
<td>56</td>
<td>18</td>
<td>20</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rank of Jacobian</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>25</td>
<td>46</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>??</td>
</tr>
<tr>
<td>1</td>
<td>27</td>
<td>3</td>
<td>9</td>
<td>10</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>–</td>
<td>–</td>
<td>??</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>–</td>
<td>??</td>
<td>??</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>??</td>
<td>??</td>
<td>??</td>
<td>??</td>
<td>??</td>
</tr>
</tbody>
</table>
More 2-adic facts

1. There are 8 “sporadic” subgroups
   1. Only one genus 2 curve has a sporadic point
   2. Six genus 3 curves each have a single sporadic point
   3. The genus 1, 5, and 7 curves have no sporadic points

2. Many accidental isomorphisms of $X_H \cong X_{H'}$.

3. There is one $H$ such that $g(X_H) = 1$ and $X_H \in X_H(\mathbb{Q})$. 
Rational Points rundown: $\ell = 3$

<table>
<thead>
<tr>
<th>$g$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\geq 3$ rational points</td>
</tr>
<tr>
<td>1</td>
<td>Handled by Sutherland-Zywina</td>
</tr>
<tr>
<td>4</td>
<td>map to $g = 1$</td>
</tr>
<tr>
<td>2</td>
<td>Chabauty works</td>
</tr>
<tr>
<td>4</td>
<td>no 3-adic points</td>
</tr>
<tr>
<td>3</td>
<td>Picard curves; descent works, try Chabauty</td>
</tr>
<tr>
<td>4</td>
<td>3 left; have models, $\geq 3$ rational points</td>
</tr>
<tr>
<td>6</td>
<td>trigonal, with model, $\geq 3$ rat pts</td>
</tr>
<tr>
<td>12</td>
<td>gonality $\leq 9$, plane model, degree 121</td>
</tr>
<tr>
<td>43</td>
<td>New ideas needed</td>
</tr>
</tbody>
</table>
\( \ell = 3 \) example

\[
X_H: \quad -x^3 y + x^2 y^2 - xy^3 + 3xz^3 + 3yz^3 = 0
\]
Rational Points rundown: $\ell = 5$

<table>
<thead>
<tr>
<th>$g$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10$ level $5$, $3$ level $25$, All level $5$ curves are genus $0$</td>
</tr>
<tr>
<td>4</td>
<td>$4$ level $25$, No $5$-adic points</td>
</tr>
<tr>
<td>2</td>
<td>$2$ level $25$, Rank $2$, $A_5 \mod 2$ image</td>
</tr>
<tr>
<td>4</td>
<td>$3$ level $25$, All isomorphic.</td>
</tr>
<tr>
<td></td>
<td>Each has $5$ rational points</td>
</tr>
<tr>
<td></td>
<td>Each admits an order $5$ aut</td>
</tr>
<tr>
<td></td>
<td>Simple Jacobian</td>
</tr>
<tr>
<td>$g = 8, 14, 22, 36$</td>
<td>$25$ and $125$ levels, No models (or ideas, yet)</td>
</tr>
</tbody>
</table>
### Rational Points rundown: $\ell = 7$

<table>
<thead>
<tr>
<th>$g$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7g = 1, 3$</td>
<td>$[Z, 4.4]$ handles these, $X_H(\mathbb{Q})$ is finite.</td>
</tr>
<tr>
<td>$g = 19, 26$, level 49</td>
<td>Maps to one of the 6 above</td>
</tr>
<tr>
<td>$g = 1$, level 49</td>
<td>$[SZ]$ handles this one (rank 0)</td>
</tr>
<tr>
<td>$g = 3, 19, 26$, level 49, 343</td>
<td>Map to curve on previous line</td>
</tr>
<tr>
<td>$g = 12$, level 49</td>
<td>Handled by</td>
</tr>
<tr>
<td></td>
<td>Greenberg–Rubin–Silverberg–Stoll</td>
</tr>
<tr>
<td>$g = 9, 12, 69, 94$</td>
<td>No models (or ideas, yet)</td>
</tr>
<tr>
<td>$\ell = 11$</td>
<td>11 all maximal are genus one</td>
</tr>
<tr>
<td>----------------</td>
<td>---------------------------------------------------------------</td>
</tr>
<tr>
<td></td>
<td>only positive rank is $X_{ns}(11)$</td>
</tr>
<tr>
<td></td>
<td>All but one are ruled out by Zywina some have sporadic points;</td>
</tr>
<tr>
<td></td>
<td>$g = 5$, level 11 [Z, Theorem 1.6]</td>
</tr>
<tr>
<td></td>
<td>$g = 5776$, level 121 [Z, Lemma 4.5]</td>
</tr>
<tr>
<td></td>
<td>Problem session</td>
</tr>
</tbody>
</table>
**Rational Points rundown: \( \ell = 13 \)**

Zywina handles all level 13 except for the cursed curve

<table>
<thead>
<tr>
<th>Level</th>
<th>Genus</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>(g = 2, 3), level 13</td>
<td>(8 total)</td>
</tr>
<tr>
<td></td>
<td>(g = 8), level 169</td>
<td>(X_0(13^2)), handled by Kenku</td>
</tr>
<tr>
<td></td>
<td>(X_{ns}(13))</td>
<td>Cursed. Genus 3, rank 3.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No torsion. Some points</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Probably has maximal mod 2 image</td>
</tr>
</tbody>
</table>
Explicit methods: highlight reel

- Local methods
- Chabauty
- Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- Equationless descent via group theory.
- New techniques for computing $\text{Aut } C$. 
Example (Genus $C = 3 \Rightarrow$ Genus $D = 5$)

- $C : Q(x, y, z) = 0$
- $Q = Q_1 Q_3 - Q_2^2$.

$D_\delta : Q_1(x, y, z) = \delta u^2$
$Q_2(x, y, z) = \delta uv$
$Q_3(x, y, z) = \delta v^2$

- $\text{Prym}(D_\delta \rightarrow C) \cong \text{Jac}_{H_\delta}$,
- $H_\delta : y^2 = -\delta \det(M_1 + 2x M_2 + x^2 M_3)$. 
Thank you!