The canonical ring of a stacky curve

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Let $\Gamma$ be a Fuchsian group (e.g. $\Gamma = \Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$).

**Definition**

A **modular form** for $\Gamma$ of weight $k \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

$$f(\gamma z) = (cz + d)^k f(z) \quad \text{for all } \gamma \in \Gamma$$

and such that the limit $\lim_{z \to *} f(z)$ exists for all cusps $\ast$.

**Definition**

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms for $\Gamma$ of weight $k$. 
Definition (Ring of Modular forms)

\[ M(\Gamma) := \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma) \]

Example

\[ M(\text{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6] \]

Theorem (Wagreich)

\[ M(\Gamma) \text{ is generated by two elements if and only if} \]

\[ \Gamma = \text{SL}_2(\mathbb{Z}), \Gamma_0(2), \text{or } \Gamma(2). \]
Ring of Modular forms

Definition (Ring of Modular forms)

\[ M(\Gamma) := \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma) \]

Example (LMFDB)

\[ M(\Gamma_0(11)) \cong \mathbb{C}[E_2, f_E, g_4]/(g_4^2 - F(E_2, f_E)) \]

Example (Ji, 1998)

\[ M(\Gamma_{2,3,7}) \cong \mathbb{C}[\Delta_{12}, \Delta_{16}, \Delta_{30}]/f(\Delta_{12}, \Delta_{16}, \Delta_{30}) \]
Rustom’s conjectures (2012)

Conjecture (Rustom)
The $\mathbb{C}$-algebra $M(\Gamma_0(N))$ is generated in weight at most 6 with relations in weight at most 12.

– This was proved by Wagreich in 1980/81.

Conjecture (Rustom)
The $\mathbb{Z}[1/6N]$-algebra $M(\Gamma_0(N), \mathbb{Z}[1/6N])$ is generated in weight at most 6 with relations in weight at most 12.

– $M_k(\Gamma_0(N), R)$ consists of forms with $q$-expansion in $R[[q]]$.
Conjecture (Rustom)

The $\mathbb{Z}[1/6N]$-algebra $M(\Gamma_0(N), \mathbb{Z}[1/6N])$ is generated in weight at most 6 with relations in weight at most 12.

Theorem (Voight, ZB)

*Rustom’s conjecture is true.*
Modular curves

1. \( Y = [\mathcal{H}/\Gamma] \)
2. \( X = Y \cup \Delta = [\mathcal{H}/\Gamma] \)

Kodaira-Spencer

\[
M_k(\Gamma) \cong H^0(X, \Omega^1(\Delta) \otimes k/2)
\]

\[
f(z) \mapsto f(z) \, dz \otimes k/2
\]

Log canonical ring

\[
M(\Gamma) \cong R_{X,\Delta} := \bigoplus_k H^0(X, \Omega^1(\Delta) \otimes k)
\]
Example: $X_0(11)$ (fundamental domain)
Example: $X_0(11)$, $\Delta = P + Q$

**Example (LMFDB)**

$$
\bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_0(11)) \cong \mathbb{C}[E_2, f_E, g_4]/(g_4^2 - F(E_2, f_E))
$$

**Remark (Via Kodaira Spencer)**

$$
\bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_0(11)) \cong \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(X_0(11), k(P + Q))
$$

**Remark (Riemann–Roch)**

$$
\dim H^0(X_0(11), k(P + Q)) = \max\{1, 2k\}
$$

$$
\dim \text{im} \left( H^0(X_0(11), P + Q) \otimes^2 \to H^0(X_0(11), 2(P + Q)) \right) = 3
$$
Log canonical map/ring

Definition

The **canonical map** \( \phi_K : C \to \mathbb{P}^{g-1} \) is given by \( P \mapsto [\omega_1(P) : \ldots : \omega_g(P)] \).

(An embedding iff \( C \) is not hyperelliptic.)

Facts

\[
C \cong \text{Proj} \: R_{X,1} \cong \text{Proj} \bigoplus_k H^0(X, \Omega^1(\Delta) \otimes k)
\]

Facts

The relations among \( R_{X,1} \) are the defining equations of \( \phi_K(C) \).
Let $C$ be non-hyperelliptic, non-trigonal, not a plane quintic.

**Theorem (Enriques-Noether-Baggage-Petri)**

The canonical ring $R_C$ is generated in degree 1 with relations in degree 2.

**Remark**

1. For $C$ trigonal or a plane quintic $R_C$ is generated in degree 1 with relations in degrees 2 and 3
2. (unless $g(C) = 3$, which has a single relation in degree 4)
3. For $C$ hyperelliptic, there are generators in degrees 1,2, relations in degrees up to 4.
Log Petri’s theorem

Let $C$ be a curve and $\Delta$ a log divisor.

**Theorem (Voight, ZB)**

The log canonical ring $R_C$ is generated in degree at most 3 with relations in degree at most 6.

**Remark**

Lots of exceptional cases if $0 < \deg \Delta \leq 2$.

**Remark (Things stabilize)**

1. Generators in degree 1 with relations in degree 2,3 if $\Delta = 3$
2. (Mumford.) Generators in degree 1 with relations in degree 2 if $\Delta \geq 4$
Let $C$ be a curve and $\Delta$ a log divisor.

**Theorem (Voight, ZB)**

The log canonical ring $R_C$ is generated in degree at most 3 with relations in degree at most 6.

**Corollary**

Rustom’s conjecture is true if $\Gamma$ acts without stabilizers.
Modular curves
1. $Y = \mathcal{H}/\Gamma$
2. $X = Y \cup \Delta = \mathcal{H}/\Gamma$

Kodaira-Spencer
$$M_k(\Gamma) \cong H^0(X, \Omega^1(\Delta) \otimes k/2)$$
$$f(z) \mapsto f(z) \, dz \otimes k/2$$

Log canonical ring
$$M(\Gamma) \cong R_{X,\Delta} := \bigoplus_k H^0(X, \Omega^1(\Delta) \otimes k)$$
Fundamental Domain for $X(1)$

**Fig. 1**
Fundamental Domain for $X(1)$

$$D = K + \Delta = -\infty$$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$dD$</th>
<th>$\dim H^0(X, \lfloor dD \rfloor)$</th>
<th>$\dim M_{2d}(\text{SL}_2(\mathbb{Z}))$</th>
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</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>$-\infty$</td>
<td>0</td>
<td>0</td>
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<tr>
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<tr>
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<td>1</td>
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<tr>
<td>6</td>
<td>$-6\infty$</td>
<td>0</td>
<td>2</td>
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Fractional divisors

Remark
1. Divisors are now fractional.
2. \( D = D_0 + \frac{n_1}{a_1} P_1 + \frac{n_2}{a_2} P_2 + \frac{n_3}{a_3} P_3 \)

Fact
\[
K_{\mathcal{X}} = K_X + \sum \frac{e_P - 1}{e_P} P
\]
Definition

The **floor** $\lfloor D \rfloor$ of a Weil divisor $D = \sum_i a_i P_i$ on $X$ is the divisor on $X$ given by

$$\lfloor D \rfloor = \sum_i \left\lfloor \frac{a_i}{\# G_{P_i}} \right\rfloor \pi(P_i).$$

Fact

$$H^0(X^\times, D) = H^0(X, \lfloor D \rfloor).$$
Example: \( X(1) \)

\[
D = K + \Delta = \frac{1}{2}P + \frac{2}{3}Q - \infty
\]

<table>
<thead>
<tr>
<th>( d )</th>
<th>([dD])</th>
<th>(\deg [dD])</th>
<th>(\dim H^0(X, [dD]))</th>
<th>(M_{2d}(\text{SL}_2(\mathbb{Z})))</th>
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<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>-\infty</td>
<td>-1</td>
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<td>0</td>
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<tr>
<td>2</td>
<td>(P + Q - 2\infty)</td>
<td>0</td>
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<td>(E_4)</td>
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<td>3</td>
<td>(P + 2Q - 3\infty)</td>
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<td>(E_6)</td>
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<td>(E_4 E_6)</td>
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<tr>
<td>6</td>
<td>(3P + 4Q - 6\infty)</td>
<td>1</td>
<td>2</td>
<td>(E_4^3, E_6^2)</td>
</tr>
</tbody>
</table>
Main theorem

**Theorem (Voight, ZB)**

Let \((\mathcal{X}, \Delta)\) be a tame log stacky curve with signature \((g; e_1, \ldots, e_r; \delta)\) over a field \(k\), and let \(e = \max(1, e_1, \ldots, e_r)\). Then the canonical ring

\[
R(\mathcal{X}, \Delta) = \bigoplus_{d=0}^{\infty} H^0(\mathcal{X}, \Omega(\Delta)\otimes^d)
\]

is generated as a \(k\)-algebra by elements of degree at most \(3e\) with relations of degree at most \(6e\).

**Remark**

Moreover, if \(2g - 2 + \delta \geq 0\), then \(R(\mathcal{X}, \Delta)\) is generated in degree at most \(\max(3, e)\) with relations in degree at most \(2 \max(3, e)\).
Final comments

Remark

1. We generalize to the relative and spin cases.
2. We give (relative) Gröbner bases, generic initial ideals.
3. Exact formulations of theorems are amenable to computation.