Families of Abelian Varieties with Big Monodromy

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Slides available at http://www.mathcs.emory.edu/~dzb/slides/

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Background - Galois Representations

$$\rho_{A,n} : \quad G_K \rightarrow \text{Aut} \ A[n] \cong \text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$$

$$\rho_{A,\ell\infty} : \quad G_K \rightarrow \text{GL}_{2g}(\hat{\mathbb{Z}}_{\ell}) = \lim_{\leftarrow n} \text{GL}_{2g}(\mathbb{Z}/\ell^n\mathbb{Z})$$

$$\rho_{A} : \quad G_K \rightarrow \text{GL}_{2g}(\widehat{\mathbb{Z}}) = \lim_{\leftarrow n} \text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$$
$\rho_{A,n} : G_K \to G_n \hookrightarrow \text{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$

$G_n \cong \text{Gal}(K(A[n])/K)$
If $E$ has a $K$-rational torsion point $P \in E(K)[n]$ (of exact order $n$), then the image is constrained:

$$G_n \subset \begin{pmatrix} 1 & \ast \\ 0 & \ast \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\sigma(P) = P$$

$$\sigma(Q) = a_\sigma P + b_\sigma Q$$
1. \( U \subset \mathbb{P}^N_K \) (non-empty open)
2. \( \eta \in U \) (generic point)
3. \( \mathcal{A} \to U \) (family of principally polarized abelian varieties)
4. \( \rho_{\mathcal{A}_\eta} : G_K(U) \to GSp_{2g}(\hat{\mathbb{Z}}) \)

**Definition**

The **monodromy** of \( \mathcal{A} \to U \) is the image \( H_\eta \) of \( \rho_{\mathcal{A}_\eta} \). We say that \( \mathcal{A} \to U \) has **big monodromy** if \( H_\eta \) is an open subgroup of \( GSp_{2g}(\hat{\mathbb{Z}}) \).
Monodromy of a family over a stack

1. $U$ is now a stack.

**Definition**

The **monodromy** of $\mathcal{A} \to U$ is the image $H$ of $\rho_\mathcal{A}$. We say that $\mathcal{A} \to U$ has **big monodromy** if $H$ is an open subgroup of $GSp_{2g}(\hat{\mathbb{Z}})$.

1. $\text{Spec } \Omega \xrightarrow{\eta} U$ (geometric generic point)
2. $\pi_{1,\text{et}}(U)$

1. $\mathcal{A} \to U$ (family of principally polarized abelian varieties)
2. $\rho_\mathcal{A} : \pi_{1,\text{et}}(U) \to GSp_{2g}(\hat{\mathbb{Z}})$
(Example) standard family of elliptic curves

\[ E : y^2 = x^3 + ax + b \]

\[ U = \mathbb{A}^2_K - \Delta \]

\[ H = \{ A \in \text{GL}_2(\hat{\mathbb{Z}}) : \text{det}(A) \in \chi_K(\text{Gal}(\overline{K}/K)) \} \]
(Example) elliptic curves with full two torsion

\[ E : y^2 = x(x - a)(x - b) \]

\[ U = \mathbb{A}_\mathbb{Q}^2 - \Delta \]

\[ H = \{ A \in \text{GL}_2(\hat{\mathbb{Z}}) : A \equiv I \pmod{2} \} \]
Exotic example from Zywina’s HIT paper

\[ E : y^2 + xy = x^3 - \frac{36}{j - 1728}x - \frac{1}{j - 1728} \]
Exotic example from Zywina’s HIT paper

\[ E : y^2 + xy = x^3 - \frac{36}{j - 1728} x - \frac{1}{j - 1728} \text{ over } U \subset \mathbb{A}_K^1 \]

\[ j = \frac{(T^{16} + 256 T^8 + 4096)^3}{T^{32}(T^8 + 16)} \]

\[ [\text{GL}_2(\hat{\mathbb{Z}}) : H] = 1536 \]
Exotic example from Zywina’s HIT paper

\[ E: y^2 + xy = x^3 - \frac{36}{j - 1728} x - \frac{1}{j - 1728} \quad \text{over } U \subset \mathbb{A}^1_K \]

\[ j = \frac{(T^{16} + 256 T^8 + 4096)^3}{T^{32} (T^8 + 16)} \]

\[ [\text{GL}_2(\hat{\mathbb{Z}}) : H] = 1536 \]

\( H \) is the subgroup of matrices preserving \( h(z) = \eta(z)^4 / \eta(4z) \).
(Example) Hyperelliptic

\[ E : y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0 \]

over \( U \subset \mathbb{A}^{2g+2} \)

\[ H = \{ A \in \text{GSp}_{2g}(\hat{\mathbb{Z}}) : A \pmod{2} \in S_{2g+2} \} \]
Main Theorem

Theorem (ZB-Zywina)

Let $U$ be a non-empty open subset of $\mathbb{P}^N_K$ and let $\mathcal{A} \to U$ be a family of principally polarized abelian varieties. Let $\eta$ be the generic point of $U$ and suppose moreover that $\mathcal{A}_\eta/K(\eta)$ has big monodromy. Let $H_\eta$ be the image of $\rho_{\mathcal{A}_\eta}$.

Let

$$B_K(N) = \{ u \in U(K) : h(u) \leq N \}.$$  

Then a random fiber has maximal monodromy, i.e. (if $K \neq \mathbb{Q}$)

$$\lim_{N \to \infty} \frac{|\{ u \in B_K(N) : \rho_{A_u}(G_K) = H_\eta \}|}{|B_K(N)|} = 1.$$
Corollary - Variant of Inverse Galois Problem

For every $g > 2$, there exists an abelian variety $A/\mathbb{Q}$ such that

$$\text{Gal}(\mathbb{Q}(A_{\text{tors}})/\mathbb{Q}) \cong \text{GSp}_{2g}(\hat{\mathbb{Z}}),$$

i.e, for every $n$,

$$\text{Gal}(\mathbb{Q}(A[n])/\mathbb{Q}) \cong \text{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}).$$
Monodromy of trigonal curves

Theorem (ZB, Zywina)

For every $g > 2$

1. the stack $T_g$ of trigonal curves has monodromy $GSp_{2g}(\hat{\mathbb{Z}})$, and

2. there is a family of trigonal curves over a nonempty rational base $U \subset \mathbb{P}^N_Q$ with monodromy $GSp_{2g}(\hat{\mathbb{Z}})$. 

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Question

For every $g$, does there exist a family $\mathcal{A} \to U$ of PP abelian varieties of dimension $g$, $U$ rational, which are not generically isogenous to Jacobians, with monodromy $\text{GSp}_{2g}(\widehat{\mathbb{Z}})$?

1. One can (probably) take $\mathcal{A} \to U$ to be a family of Prym varieties associated to tetragonal curves, or
2. (Tsimerman) one can take $\mathcal{A} \to U$ to be a family of Prym varieties associated to bielliptic curves.
Sketch of trigonal proof

**Theorem**

For every $g$ the stack $T_g$ of trigonal curves has monodromy $\text{GSp}_{2g}(\hat{\mathbb{Z}})$.

**Proof.**

1. $\mathcal{M}_{g,d-1} \subset \overline{\mathcal{M}_{g,d}}$ (suffices for $\ell > 2$)
2. $\mathcal{M}_{g-2} \subset \overline{\mathcal{M}_g}$
3. The mod 2 monodromy thus contains subgroups isomorphic to
   1. $S_{2g+2}$
   2. $\text{Sp}_{2(g-2)+2}(\mathbb{Z}/2\mathbb{Z})$
(Example) Hyperelliptic

\[ E : y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0 \]

over \( U \subset \mathbb{A}^{2g+2} \)

\[ H = \{ A \in \text{GSp}_{2g}(\hat{\mathbb{Z}}) : A \pmod{2} \in S_{2g+2} \} \]
Hyperelliptic example continued

Theorem

1. (Yu) unpublished
2. (Achter, Pries) the stack of hyperelliptic curves has maximal monodromy
3. (Hall) any 1-parameter family \( y^2 = (t - x)f(t) \) over \( K(x) \) has full monodromy
Hyperelliptic example proof

Corollary

\[ E : y^2 = x^{2g+2} + a_{2g+1}x^{2g+1} + \ldots + a_0 \]

has monodromy \( \{ A \in \text{GSp}_{2g}(\hat{\mathbb{Z}}) : A \pmod{2} \in S_{2g+2} \} \).

Proof.

1. \( U = \text{space of distinct unordered } 2g + 2\text{-tuples of points on } \mathbb{P}^1 \)
2. \( U \twoheadrightarrow \mathcal{H}_{g,2} \)
3. \( \mathcal{H}_{g,2} \cong [U / \text{Aut } \mathbb{P}^1] \)
4. fibers are irreducible, thus

\[ \pi_{1,\text{et}}(U) \twoheadrightarrow \pi_{1,\text{et}}(\mathcal{H}_{g,2}) \]

is surjective.
Sketch of trigonal proof

Theorem (ZB, Zywina)

For every \( g > 2 \) there is a family of trigonal curves over a nonempty rational base \( U \subset \mathbb{P}^N_{\mathbb{Q}} \) with monodromy \( \text{GSp}_{2g}(\widehat{\mathbb{Z}}) \).

Proof.

1. **Main issue:**

\[ f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) = 0 \]

2. The stack \( \mathcal{T}_g \) is unirational, need to make this explicit

3. (Bolognesi, Vistoli) \( \mathcal{T}_g \cong [U/G] \) where \( U \) is rational and \( G \) is a connected algebraic group.

4. **Maroni-invariant** (normal form for trigonal curves).
Sketch of trigonal proof - Maroni Invariant

Maroni-invariant

1. The image of the canonical map lands in a scroll

\[ C \hookrightarrow F_n \hookrightarrow \mathbb{P}^{g-1} \]

\[ F_n \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \]
\[ F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \]
\[ F_1 \cong \text{Bl}_P \mathbb{P}^2 \]

2. \( n \) has the same parity as \( g \)

3. generically \( n = 0 \) or \( 1 \)

4. e.g., if \( g \) even we can take \( U = \) space of bihomogenous polynomials of bi-degree \((3, d)\)

5. \[ [U/G] \cong \mathcal{T}_g^0 \subset \mathcal{T}_g. \]
\( C \to D \cong \ker_0(J_C \to J_D) \), generally not a Jacobian
Example (Tsimerman)

The space of (ramified) double covers of a fixed elliptic curve is rational, so the space of Pryms is also rational, with base isomorphic to a projective space over $X_1(2)$. The associated family of Prym’s has big monodromy.